1. Let $(T_k)_{k\in\mathbb{N}}$ be a sequence in $L(\mathbb{R}^m,\mathbb{R}^n)$. Show that if $T_k\to 0$ in the euclidean metric, then $T_k\to 0$ pointwise. Is the converse true? Do your answers change if (T_k) is a sequence in $\mathrm{Bil}(\mathbb{R}^m\times\mathbb{R}^n,\mathbb{R}^p)$?

2. At which points is each of the following functions $\mathbb{R}^2\to\mathbb{R}$ differentiable?

(i) $f(x,y)=|x||y|$;

(ii) $g(x,y)=xy\sin(1/x)$ when $x\neq 0$ and $g(0,y)=0$;

(iii) $h(x,y)=\frac{xy}{(x^2+y^2)^{1/2}}$ when $(x,y)\neq(0,0)$ and $h(0,0)=0$.

3. Consider the map $f\colon\mathbb{R}^3\to\mathbb{R}^3$ given by $f(x)=x/\|x\|$ if $x\neq 0$ and $f(0)=0$. Show that f is differentiable on $\mathbb{R}^3\setminus\{0\}$ with

$$f'(x)(h)=\frac{h}{\|x\|}-\frac{\langle x,h\rangle x}{\|x\|^3}\ .$$

Verify that $f'(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

4. (a) Show that the set $\mathcal{G}_n$ of invertible $n\times n$ real matrices is an open subset of $\mathcal{M}_n$. By quoting appropriate results, explain why the function $f\colon\mathcal{G}_n\to\mathcal{M}_n$ given by $f(A)=A^{-1}$ is differentiable.

(b) Given an open subset U of $\mathcal{M}_n$, show that if functions $g,h\colon U\to\mathcal{M}_n$ are differentiable at $A\in U$, then so is the product gh given by $(gh)(X)=g(X)h(X)$. Hence, or otherwise, find the derivative of the function f given in part (a).

5. Show that the function $\det\colon\mathcal{M}_n\to\mathbb{R}$ is differentiable at the identity matrix I with $\det'(I)(H)=\mathrm{tr}(H)$. Deduce that det is differentiable at every invertible matrix A with $\det'(A)(H)=\det(A)\mathrm{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $\det''(I)$ as a bilinear map. Is det differentiable at any non-invertible matrix?

6. Assume that all directional derivatives $D_uf(0)$ exist for $f\colon\mathbb{R}^2\to\mathbb{R}$, and moreover the map $T\colon\mathbb{R}^2\to\mathbb{R}$ given by $T(u)=D_uf(0)$ if $u\neq 0$ and $T(0)=0$ is linear. Does it follow that f is differentiable at 0? What if we assume instead that $f\circ\gamma$ is differentiable at 0 for every differentiable curve $\gamma\colon(-1,1)\to\mathbb{R}^2$ with $\gamma(0)=0$?

7. Let $U\subset\mathbb{R}^2$ be an open set and $f\colon U\to\mathbb{R}$ be a function such that for each $x\in\mathbb{R}$ the map $y\mapsto f(x,y)$ is continuous, and for each $y\in\mathbb{R}$ the map $x\mapsto f(x,y)$ is continuous. Show that f need not be continuous on U. Now assume that D_1f exists and is bounded on U and that for each $x\in\mathbb{R}$ the map $y\mapsto f(x,y)$ is continuous. Show that f is continuous.

8. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I: there exist $r > 0$ and a continuous function $g\colon D_r(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in D_r(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

9. Let $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Define a function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x, y) = (x, x^3 + y^3 - 3xy)$. Show that for every $(x_0, y_0) \in C \setminus \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ there are open sets U containing (x_0, y_0) and V containing $F(x_0, y_0)$ such that $F\restriction_U$ is a bijection from U onto V with a continuously differentiable inverse. Deduce that for every such point (x_0, y_0) there is an open neighbourhood U of (x_0, y_0), an open interval I containing x_0 and a continuously differentiable function $g\colon I \to \mathbb{R}$ such that $g(x_0) = y_0$ and $C \cap U$ is the graph of g, *i.e.*, $C \cap U = \{(x, y) \in \mathbb{R}^2 : x \in I,\ y = g(x)\}$.

10. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0)\ .$$

Show that D_1D_2f and D_2D_1f exist on $\mathbb{R}^2$ and take the value 1 and -1 at $(0, 0)$, respectively. Where are D_1D_2f and D_2D_1f continuous?

11. Let $U \subset \mathbb{R}^m$ be an open set, and $f\colon U \to \mathbb{R}^n$ be differentiable on U. Show carefully that if f' has directional derivative $D_u f'(a)$ for some $a \in U$ and $u \in \mathbb{R}^m \setminus \{0\}$, then for every $v \in \mathbb{R}^m \setminus \{0\}$, the directional derivative $D_u D_v f(a)$ exists and equals $D_u f'(a)(v)$.

12. Let U be an open subset of $\mathbb{R}^2$ containing the rectangle $[a, b] \times [c, d]$. Suppose that $f\colon U \to \mathbb{R}$ is continuous and that $D_2 f$ exists and is continuous on U. Show that $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$ is differentiable on some open interval containing $[c, d]$ with $F'(y) = \int_a^b D_2 f(x, y)\,\mathrm{d}x$.

13. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ be twice differentiable functions on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Show that $g \circ f$ is twice differentiable on $\mathbb{R}^m$ and find $(g \circ f)''$.