1. Which of the following subsets of $\mathbb{R}^2$ are (a) connected, (b) path-connected?
(i) $D_1((-1,0)) \cup D_1((1,0))$ (ii) $D_1((-1,0)) \cup B_1((1,0))$
(iii) $\{(x,y) : x = 0 \text{ or } y/x \in \mathbb{Q}\}$ (iv) $\{(x,y) : x = 0 \text{ or } y/x \in \mathbb{Q}\} \setminus \{(0,0)\}$.

2. Let $f\colon X \to S$ be a function from a connected space X to a set S. Assume f is *locally constant*: every $x \in X$ has a neighbourhood on which f is constant. Show that f is constant.

3. Show that homeomorphic spaces have the same number of connected components. Show that no two of $[0,1]$, $[0,1)$ and $(0,1)$ are homeomorphic. Show also that the letters A and H drawn in the plane are not homeomorphic.

4. Find the connected components of the subspace $X = \{(0,0),(0,1)\} \cup \bigcup_{n\in\mathbb{N}}\{1/n\} \times [0,1]$ of $\mathbb{R}^2$. Show that there exist $x, y \in X$ that belong to different components but there are no open sets U and V disconnecting X with $x \in U$ and $y \in V$.

5. Let $A \subset \mathbb{R}^n$ be such that every continuous function $f\colon A \to \mathbb{R}$ is bounded. Show that A is compact.

6. Show that if A and B are closed subsets of $\mathbb{R}^n$ and if A or B is bounded, then $A + B$ is closed. Give an example in $\mathbb{R}$ to show that the boundedness condition cannot be omitted.

7. (a) Let R be the equivalence relation on $Q = [0,1]^2$ defined as follows: $(x_1,y_1) \sim (x_2,y_2)$ if and only if EITHER $(x_1,y_1) = (x_2,y_2)$ OR $\{x_1,x_2\} = \{0,1\}$ and $y_1 = y_2$ OR $\{y_1,y_2\} = \{0,1\}$ and $x_1 = x_2$ OR $x_1,x_2,y_1,y_2 \in \{0,1\}$. Show that any two of the following spaces (in their natural topologies) are homeomorphic: Q/R, $\mathbb{R}^2/\mathbb{Z}^2$, $S^1 \times S^1$ and the subspace

$$T^2 = \Big\{\big((2+\cos\theta)\cos\varphi, (2+\cos\theta)\sin\varphi, \sin\theta\big) : \theta,\varphi \in [0,2\pi]\Big\}$$

of $\mathbb{R}^3$.

(b) Let R be the equivalence relation on Q defined as follows: $(x_1,y_1) \sim (x_2,y_2)$ if and only if EITHER $(x_1,y_1) = (x_2,y_2)$ OR $\{x_1,x_2\} = \{0,1\}$ and $y_1 = y_2$ OR $y_1 = y_2 = 0$ OR $y_1 = y_2 = 1$. Show that Q/R is homeomorphic to the sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.

8. Show that a continuous real-valued function on a sequentially compact topological space is bounded and attains its bounds. Show also that a continuous function from a compact metric space to an arbitrary metric space is uniformly continuous.

9. Let M be a non-empty compact metric space and $f\colon M \to M$ be a function.
(a) Show that if $d(f(x),f(y)) < d(x,y)$ for all $x \neq y$ in M, then f has a unique fixed point.
(b) Show that if f is isometric, *i.e.*, $d(f(x),f(y)) = d(x,y)$ for all $x,y \in M$, then f is surjective.

10. (a) Show that the coordinate projections π_X and π_Y on a product space $X \times Y$ are open maps. Show that if Y is compact, then π_X is a *closed map*: for a closed subset F of $X \times Y$, its projection $\pi_X(F)$ is closed in X. Give an example of a closed set in $\mathbb{R}^2$ whose projections are not closed in $\mathbb{R}$.

(b) Let $f\colon X \to Y$ be a function between topological spaces. The *graph of* f is the set $\Gamma = \{(x, y) \in X \times Y : y = f(x)\}$. Show that if f is continuous and Y is Hausdorff, then Γ is closed in the product topology. Conversely, show that if Γ is closed and Y is compact, then f is continuous.

$^+$11. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function under which the image of any path-connected set is path-connected and the image of any compact set is compact. Show that f must be continuous.

Some more questions

12. (a) A topological space is *normal* if disjoint closed subsets can be separated by open sets: given disjoint closed subsets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Show that a compact Hausdorff space is normal.

(b) Let (C_n) be a decreasing sequence of compact connected subsets of a Hausdorff space. Show that $\bigcap_{n\in\mathbb{N}} C_n$ is connected. (Part (a) will be useful here.) Give an example in $\mathbb{R}^2$ of a decreasing sequence of closed connected sets whose intersection is disconnected.

13. (a) Let R_1 be an equivalence relation on a topological space X and let R_2 be an equivalence relation on the quotient space X/R_1. Define

$$R = \{(x, y) \in X \times X : (q(x), q(y)) \in R_2\}$$

where $q\colon X \to X/R_1$ is the quotient map. Show that R is an equivalence relation on X and that X/R is homeomorphic to $\big(X/R_1\big)/R_2$.

(b) For a topological space X and for $A \subset X$, we let X/A denote the quotient space of X by the relation identifying the points of A: $x \sim y$ if and only if either $x = y$ or $x, y \in A$. Now consider the subset $A = \{(0, 0, 1), (0, 0, -1)\}$ of the two-dimensional sphere S^2, and the subset $B = \{(2 + \cos\theta, 0, \sin\theta) : \theta \in [0, 2\pi]\}$ of T^2. Show that S^2/A and T^2/B are homeomorphic.

14. Show that $C[0, 1]$ in the uniform metric D is separable. Let $B = \{f \in C[0, 1] : D(0, f) \leqslant 1\}$ and $B' = \{f \in B : f \text{ differentiable and } f' \in B\}$. Show that B is closed but not compact. On the other hand, show that every sequence in B' has a subsequence convergent in $C[0, 1]$. Deduce that $\overline{B'}$ is compact.

15. Show that there exist topological spaces X and Y with continuous bijections $f\colon X \to Y$ and $g\colon Y \to X$ such that X and Y are not homeomorphic.