1. Which of the following sequences (f_n) of functions converge uniformly on the set X?

(i) $f_n(x) = x^n$ on $X = (0, 1)$; (ii) $f_n(x) = x^n(1 - x)$ on $X = [0, 1]$;

(iii) $f_n(x) = \mathrm{e}^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

2. Suppose functions $f_n \to f$ and $g_n \to g$ uniformly on a set S. Show that $f_n + g_n \to f + g$ uniformly on S. On the other hand, show that the pointwise product $f_n g_n$ need not converge uniformly to fg but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

3. Construct a sequence (f_n) of continuous real-valued functions on $[0, 1]$ converging pointwise to the zero function but with $\int_0^1 f_n(x)\,\mathrm{d}x \not\to 0$. $^+$Is it possible to find such a sequence with $|f_n(x)| \leqslant 1$ for all x and for all n?

Construct a sequence (f_n) of differentiable real-valued functions on $[0, 1]$ converging uniformly to a function f which is not differentiable on the whole of $[0, 1]$.

4. Which of the following functions $f\colon [0, \infty) \to \mathbb{R}$ are uniformly continuous?

(i) $f(x) = \sin x^2$; (ii) $f(x) = \inf\big\{|x - n^2| : n \in \mathbb{N}\big\}$; (iii) $f(x) = (\sin x^3)/(x + 1)$.

5. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $D_r(x)$ for $x \in X$ and $r > 0$ small.

(i) $X = \mathcal{R}[0, 1]$, the space of intergrable functions on $[0, 1]$; $d(f, g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$ and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) $X = \mathbb{N}^{\mathbb{N}}$; $d(f, f) = 0$ and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

6. Let $\big(x^{(m)}\big)$ and $\big(y^{(m)}\big)$ be sequences in $\mathbb{R}^n$ converging to x and y, respectively. Show that the scalar product $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are continuous at x, then so is the pointwise scalar product $f \cdot g\colon \mathbb{R}^n \to \mathbb{R}$.

7. Show that the uniform limit of uniformly continuous scalar functions on a metric space is uniformly continuous. Give an example of uniformly continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ converging pointwise to a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous.

8. Let (f_n) be a sequence of scalar functions on a set S. In each of the following two cases, write out in symbols statement (i) and compare it to (ii). Prove that (i) implies (ii) if (f_n) is uniformly Cauchy.

(a) (i) Each f_n is bounded.

(ii) $\exists\, M \in \mathbb{R} \quad \forall\, n \in \mathbb{N} \quad \forall\, x \in S \quad |f_n(x)| \leqslant M$

(b) In this case assume that S is a metric space.

(i) Each f_n is continuous.

(ii) $\forall\, a \in S \quad \forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, n \in \mathbb{N} \quad \forall\, x \in S \quad d(x,a) < \delta \implies |f_n(x) - f_n(a)| < \varepsilon$

9. Let f_n, $n \in \mathbb{N}$, and f be continuous scalar functions on a metric space M. Show that if $f_n \to f$ uniformly on M and $x_n \to x$ in M, then $f_n(x_n) \to f(x)$. On the other hand, show that if $M = [a,b]$ is a closed bounded interval and (f_n) does not converge uniformly to f, then there is a convergent sequence $x_n \to x$ in M such that $f_n(x_n) \not\to f(x)$.

10. Show that for each $x \in X = \mathbb{R} \setminus \mathbb{N}$ the series $\sum_{n=1}^{\infty} (x-n)^{-2}$ converges. Does the series converge uniformly on X? Define $f\colon X \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} (x-n)^{-2}$. Show that f is continuously differentiable on X and find its derivative.

11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable and assume that f' is bounded. Show that f is a Lipschitz function. Define $g\colon [-1,1] \to \mathbb{R}$ by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable on $[-1,1]$. Is g a Lipschitz function? Is g uniformly continuous?

12. Let (f_n) be a sequence of continuous real-valued functions on $[0,1]$ converging pointwise to a function f. Prove that there is some subinterval $[a,b]$ of $[0,1]$ with $a < b$ on which f is bounded.