In lectures we defined the product topology on the product of finitely many topological spaces. Here we are going to extend this to arbitrary products. Note that **this is non-examinable material and is not part of the course**.

Before we begin, let us define the notion of base. A *base* for a topology on a set X is a family $\mathcal{B}$ of open subsets of X such that every open set is a union of some members of $\mathcal{B}$; equivalently, for $U \subset X$, we have that U is open in X if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Examples The family of open balls in a metric space is a base for the metric topology. For topological spaces X and Y, the family of open box sets $U \times V$ with U open in X and V open in Y, is a base for the product topology of $X \times Y$.

We now turn to products of an arbitrary number of spaces. Fix a set Γ, and assume that for each $\gamma \in \Gamma$, a topological space (X_γ, τ_γ) is given. We consider the product space

$$X = \prod_{\gamma\in\Gamma} X_\gamma = \{x : x \text{ is a function with domain } \Gamma \text{ and } x_\gamma \in X_\gamma \text{ for all } \gamma \in \Gamma\} .$$

We often write x_γ instead of $x(\gamma)$ for the value of the function x at γ. For $x \in X$ we write $x = (x_\gamma)_{\gamma\in\Gamma}$, and think of x as a "Γ-tuple" in analogy with finite products. For each $\gamma \in \Gamma$, we let $q_\gamma \colon X \to X_\gamma$ denote the coordinate-projection defined by $q_\gamma\big((x_\delta)_{\delta\in\Gamma}\big) = x_\gamma$.

Guided by the definition of the product topology for finite products, it is natural to guess that the product topology on X should be the one with base

$$\left\{ \prod_{\gamma\in\Gamma} U_\gamma : U_\gamma \in \tau_\gamma \text{ for all } \gamma \in \Gamma \right\} .$$

This is indeed a base for a topology, called the *box topology*, for X. However, this has too many open sets. For example, the product of connected or compact spaces is not necessarily connected or, respectively, compact in the box topology. Instead, we shall define the product topology τ on X to be the smallest topology that makes the coordinate-projections continuous. So τ must contain all sets of the form $q_\gamma^{-1}(U)$ where $\gamma \in \Gamma$ and U is an open set in X_γ. Since a topology is closed under finite intersections, τ must contain the family

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} q_{\gamma_i}^{-1}(U_i) : n \in \mathbb{N},\ \gamma_i \in \Gamma,\ U_i \in \tau_{\gamma_i} \text{ for } i = 1, \dots, n \right\} .$$

Then τ must also contain the family σ of arbitrary unions of members of $\mathcal{B}$. It is easy to check that σ is a topology on X, and hence $\tau = \sigma$. Thus, a subset W of X is open in the product topology τ if and only for all $x \in W$ there exist $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and open sets U_i in X_{γ_i} for $i = 1, \dots, n$ such that

$$x \in \bigcap_{i=1}^{n} q_{\gamma_i}^{-1}(U_i) \subset W .$$

To put it in another way, W is open in the product topology if and only if for all $x = (x_\gamma)_{\gamma\in\Gamma} \in W$ there exist $n \in \mathbb{N}$, $\gamma_1, \dots, \gamma_n \in \Gamma$ and neighbourhoods U_i of x_{γ_i} in X_{γ_i} for $i = 1, \dots, n$ such that

$$\{y = (y_\gamma)_{\gamma\in\Gamma} \in X : y_{\gamma_i} \in U_i \text{ for } i = 1, \dots, n\} \subset W .$$

So a typical neighbourhood of a point places restrictions on finitely many coordinates only whereas all the other coordinates can vary arbitrarily. So open sets are in some sense quite big in the product topology.

The following result is an extension to arbitrary products of a similar result in lectures. The proof is essentially the same.

Theorem 1. *With the above notation, the coordinate-projections q_γ are continuous for all $\gamma \in \Gamma$. Moreover, for any space Z and function $f\colon Z \to X$ the following holds: f is continuous if and only if the composite functions $q_\gamma \circ f\colon Z \to X_\gamma$ are continuous for all $\gamma \in \Gamma$.*

By far the most important result concerning the product topology is Tychonov's theorem.

Theorem 2 (Tychonov)**.** *The product of compact topological spaces is compact in the product topology.*

This was stated and proved in lectures for finite products. A proof for the general case can be found in most books on topology, *e.g.,* Kelley's *General Topology.* Showing that the product of connected spaces is connected is easier and is left as an exercise.

We conclude by considering a special case. Assume that $X_\gamma = Y$ for all $\gamma \in \Gamma$. In this case $X = Y^\Gamma$ is the set of all functions $\Gamma \to Y$. The coordinate-projection q_γ maps $f \in X$ to $f(\gamma) \in Y$ and is referred to as the evaluation map at γ and is sometimes denoted ϵ_γ. Given a sequence (f_n) in X and $f \in X$, it is easy to check that $f_n \to f$ as $n \to \infty$ in the product topology if and only if $f_n \to f$ pointwise on Γ. For this reason the product topology on X is also called the topology of pointwise convergence.

Exercises (not for supervisions!)

1. Let σ be a topology on $X = \prod_{\gamma\in\Gamma} X_\gamma$ such that for any space Z and function $f\colon Z \to X$ the following holds: f is continuous if and only if the composites $q_\gamma \circ f\colon Z \to X_\gamma$ are continous for all $\gamma \in \Gamma$. Show that σ is the product topology.

2. Show that the product of Hausdorff spaces is Hausdorff in the product topology.

3. For each $n \in \mathbb{N}$ we are given a metric space (X_n, d_n). Show that

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, d_n(x_n, y_n)\}$$

defines a metric on $X = \prod_{n\in\mathbb{N}} X_n$ and that the metric topology induced by d is the product topology. Thus, a countable product of metrizable spaces is metrizable.

4. Let $X = [0,1]^{[0,1]}$ be equipped with the topology of pointwise convergence. Let Y be the subset of X consisting of all functions $f\colon [0,1] \to [0,1]$ for which $\{x \in [0,1] : f(x) \neq 0\}$ is countable. Show that if $f_n \in Y$ for all $n \in \mathbb{N}$ and $f_n \to f$ in X, then $f \in Y$. Show also that Y is dense in X. Deduce that X is not metrizable.

5. For $n \in \mathbb{N}$ and $A \subset \mathbb{N}$ set $x_n(A) = 1$ if $n \in A$ and $x_n(A) = 0$ otherwise. Show that the sequence (x_n) in $X = \{0,1\}^{\mathcal{P}(\mathbb{N})}$ has no convergent subsequence in the product topology ($\{0,1\}$ is given the discrete topology). Deduce that X is not metrizable.

6. Prove that the product of connected spaces is connected. [Hint: fix $x \in X = \prod_{\gamma\in\Gamma} X_\gamma$ and for a finite $F \subset \Gamma$ consider $C_F = \{z \in X : z_\gamma = x_\gamma \text{ for all } \gamma \notin F\}$.]

7. Show that a countable product of sequentially compact topological spaces is sequentially compact. Deduce the following special case of Tychonov's theorem: a countable product of compact metric spaces is compact.

8. Let A be an infinite set and consider $Y = \mathbb{R}^A$ with the box topology. Show that the connected component of $x \in Y$ is

$$\big\{ y \in Y : \{a \in A : y_a \neq x_a\} \text{ is finite} \big\} .$$