1. Show that the dependence on n in the Johnson–Lindenstrauss lemma is sharp: There exists a constant $c > 0$ such that if every n-element subset of ℓ_2 embeds into ℓ_2^k with distortion at most 2, say, then $k \geqslant c \log n$.

2. Show that for $2 \leqslant p < \infty$ and for $f, g \in L_p(\mu)$ we have

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leqslant \frac{\|f\|_p^p + \|g\|_p^p}{2} .$$

Show that $L_p(\mu)$ is uniformly convex for $2 \leqslant p < \infty$. Deduce that $L_p(\mu)^* \cong L_q(\mu)$ for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

3. Let $1 \leqslant p \leqslant \infty$. Show that L_p is finitely representable in ℓ_p.

4. Show that if a Banach space X is uniformly homeomorphic to ℓ_2, then X is isomorphic to ℓ_2.

5. Let X be a Banach space, E and F be finite-dimensional subspaces of X^* and X^{**}, respectively. Show that for every $\varepsilon > 0$, there is a linear map $T\colon F \to X$ such that

(i) $\widehat{T\varphi}\restriction_E = \varphi\restriction_E$ for all $\varphi \in F$;

(ii) $\|T\|\|T^{-1}\| < 1 + \varepsilon$;

(iii) $T\restriction_{F\cap X}$ is the inclusion map.

6. Let I be a nonempty set and $\mathcal{U}$ be an ultrafilter on I. Given Banach spaces X_i, $i \in I$, let $X = \left(\bigoplus_{i\in I} X_i\right)_\infty$. Show that

$$\mathcal{N}_\mathcal{U} = \{(x_i) \in X : \lim_\mathcal{U} \|x_i\| = 0\}$$

is a closed subspace of X and that

$$\|x + \mathcal{N}_\mathcal{U}\| = \inf\{\|x + n\|_\infty : n \in \mathcal{N}_\mathcal{U}\} = \lim_\mathcal{U} \|x_i\| .$$

for $x \in X$.

7. Prove that every uniformly convex Banach space is superreflexive.

8. Show that if X is finitely representable in Y, then X is isometrically isomorphic to a subspace of $Y^\mathcal{U}$ for some free ultrafilter $\mathcal{U}$.