1. Let $X$ be a metric space of size $n$. Show that $R(X) > n$ is possible.

2. Compute $c_2(K_{1,3})$.

3. Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Show that there exists an ellipsoid $E$ of minimal volume containing $K$. Show further that $E$ is unique.

4. For $n \in \mathbb{N}$ let $\psi(n)$ be the least $m$ such that every metric space of size $n$ embeds into some $m$-dimensional normed space with distortion at most 2. Show that $\psi(n) \gtrsim (\log n)^2$.

5. Let $A$ and $B$ be nonempty, disjoint, convex subsets of $\mathbb{R}^n$. Show that there is a non-zero linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $f(x) \leqslant \alpha \leqslant f(y)$ for all $x \in A$ and $y \in B$.

6. Let $X = \{x_1, \dots, x_n\}$ be a metric space of size $n$. Show the following formula for the euclidean distortion of $X$:

$$c_2(X) = \max \sqrt{\frac{\sum_{i,j} s_{ij}^+ d(x_i,x_j)^2}{\sum_{i,j} s_{ij}^- d(x_i,x_j)^2}}$$

where the max is taken over all non-zero, symmetric, positive semidefinite matrices $(s_{ij})$ with zero row-sums.

7. Show that the Walsh functions are the only characters of $H_n$. Now consider the $n$-cycle $C_n$ which can be thought of as the abelian group $\mathbb{Z}/n\mathbb{Z}$. Find the characters of $C_n$. By developing Fourier analysis on $C_n$ and by finding a suitable Poincaré inequality for $L_2$-valued functions on $C_n$, compute the euclidean distortion $c_2(C_n)$.

8. The infinite Hamming cube $H_\infty$ is the set of all 0-1 sequences that are eventually zero. This is a metric space with distance

$$d(x,y) = \sum_{i=1}^{\infty} |x_i - y_i|$$

for $x = (x_i), y = (y_i) \in H_\infty$. Show that $H_\infty$ coarsely embeds into $L_2$. For a 1-Lipschitz map $f\colon H_\infty \to L_2$, define $\rho\colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\rho(t) = \inf\{\|f(x) - f(y)\|_2 : x, y \in H_\infty,\ d(x,y) \geqslant t\} .$$

Show that $\rho(t) \leqslant \sqrt{t}$.

9. The *diamond graphs* $(D_n)_{n\geqslant 0}$ are defined as follows. $D_0$ consists of two vertices joined by an edge. For $n \geqslant 1$, $D_n$ is obtained from $D_{n-1}$ by replacing each edge of $D_{n-1}$ with a diamond, *i.e.,* , a copy of $C_4$. Formally, for each edge $e$ of $D_{n-1}$, we introduce two new vertices $x_e$ and $y_e$ which are not in $V(D_{n-1})$ and so that $\{x_e, y_e\} \cap \{x_f, y_f\} = \emptyset$ for distinct edges $e$, $f$ of $D_{n-1}$. We then let $D_n$ have vertex set

$$V(D_n) = V(D_{n-1}) \cup \{x_e, y_e : e \in E(D_{n-1})\}$$

and edge set

$$E(D_n) = \bigcup_{e=ab\in E(D_{n-1})} \{ax_e, x_eb, by_e, y_ea\} .$$

Starting with the Poincaré inequality for $L_2$-valued functions on $D_1 = C_4$ proved in lectures, derive a Poincaré inequality for $L_2$-valued functions on $D_n$. Hence obtain a lower bound on the euclidean distortion of $D_n$.