1. A family (M_α) of metric spaces uniformly bilipschitzly embeds into another family (N_β) if there exists $C > 0$ such that for all α there is a β and a bilipschitz embedding $M_\alpha \to N_\beta$ with distortion at most C. Investigate this notion for the classes (H_n), (K_n), (B_n) and (C_n).

2. Show that for $1 \leqslant p < q < \infty$ there is a map $T\colon \ell_p \to \ell_q$ which is simultaneously uniform and coarse. (Hint: consider the infinite sum $\sum_{n=-\infty}^{\infty} \frac{(1-\cos(2^n x))^\beta}{2^{n\alpha}}$ for $0 < \alpha < 2\beta$.)

3. Let E be a finite-dimensional normed space. Show that E almost isometrically embeds into c_0. More precisely, show that for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that every n-dimensional normed space $(1+\varepsilon)$-embeds into ℓ_∞^N.

4. Show that there is a separable reflexive Banach space Z such that every finite-dimensional normed space almost isometrically embeds into X.

5. Show that $m_\infty(n) \leqslant n - 2$ for $n \geqslant 4$.

6. Show that K_n does not embed isometrically into ℓ_2^m for $m < n-1$. Deduce that $m_2(n) = n - 1$.

7. Let X be a finite metric space. Show that there is bilipschitz embedding $f\colon X \to L_p$ with $\mathrm{dist}(f) = c_p(X)$.

8. Show that the following inequalities hold for all x, y in ℓ_p.

$$\|x+y\|_p^p + \|x-y\|_p^p \leqslant 2\|x\|_p^p + 2\|y\|_p^p \qquad \text{if } 1 \leqslant p \leqslant 2$$

and

$$\|x+y\|_p^p + \|x-y\|_p^p \geqslant 2\|x\|_p^p + 2\|y\|_p^p \qquad \text{if } 2 \leqslant p < \infty.$$

(Note that for $p = 2$ this gives the parallelogram identity.) Find a necessary and sufficient condition for equality when $p \neq 2$. By considering a suitable subset of ℓ_p, show that $m_p(2n+1) \geqslant n$.

9. Let X and Y be distinct members of the family $\{\ell_p : 1 \leqslant p < \infty\} \cup \{\mathrm{c}_0\}$ of Banach spaces. Show that X does not embed isomorphically into Y.

10. Show that ℓ_2 embeds isometrically into L_1.

11. The lamplighter graph $\mathrm{La}(G)$ of a graph G has vertices (A, x) where x is a vertex of G and A is a finite set of vertices of G. Two vertices (A, x) and (B, y) form an edge in $\mathrm{La}(G)$ if and only if either $A = B$ and xy is an edge in G or $x = y$ and $A \bigtriangleup B = \{x\}$. Show that if G is connected, then so is $\mathrm{La}(G)$ and identify the graph distance of $\mathrm{La}(G)$.
Let $f\colon G \to H$ be a bilipschitz embedding between connected graphs. Show that there is an embedding $\bar{f}\colon \mathrm{La}(G) \to \mathrm{La}(H)$ with $\mathrm{dist}(\bar{f}) \leqslant C\mathrm{dist}(f)$ for some universal constant C.