$^+$1. Suppose $f_n \to f$ pointwise on $[0,1]$ and f_n is continuous for all $n \in \mathbb{N}$. Show that f has a point of continuity.

2. Show that the metrics d_1, d_2 and d_∞ defined in lectures for $\mathbb{R}^n$ are Lipschitz equivalent and find the best Lipschitz constants.

3. Let d be a metric on a set M. Prove that $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric on M. Determine whether or not $d' \sim d$, $d' \sim_{\mathrm{u}} d$, $d' \sim_{\mathrm{Lip}} d$.

4. Let $f\colon M \to M'$ be a function between metric spaces with the property that $(f(x_n))$ is convergent in M' whenever (x_n) is convergent in M. Must f be continuous?

5. Show that a composite of uniformly continuous (or Lipschitz) functions is uniformly continuous (respectively, Lipschitz).

6. Show that a uniformly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz for large distances in the following sense: there exists $C \geqslant 0$ such that whenever $x, y \in \mathbb{R}^n$ satisfy $d(x,y) \geqslant 1$, we have $|f(x) - f(y)| \leqslant Cd(x,y)$.

7. Let A be a dense subset of a metric space M, and let N be a complete metric space. Assume $f\colon A \to N$ is uniformly continuous. Show that there is a unique continuous function $\bar{f}\colon M \to N$ such that $\bar{f}\restriction_A = f$. Show further that this unique extension of f is uniformly continuous. If we merely assume that f is continous, must there be a continuous extension to M?

8. Let M be a metric space and $A \subset M$. Assume that $f\colon A \to \mathbb{R}$ is Lipschitz with constant C. Show that for $x \in M \setminus A$ there is a Lipschitz function $\tilde{f}\colon A \cup \{x\} \to \mathbb{R}$ with constant C such that $\tilde{f}\restriction_A = f$. Deduce that there is a Lipschitz extension of f to M with constant C.

9. Let M be a metric space with a distinguished point 0. Let $\mathrm{Lip}_0(M) = \{f\colon M \to \mathbb{R} : f(0) = 0 \text{ and } f \text{ is Lipschitz}\}$. For $f \in \mathrm{Lip}_0(M)$ let $\|f\|$ denote that least $L \geqslant 0$ such that f is L-Lipschitz. Show that $d(f,g) = \|f - g\|$ defines a complete metric on $\mathrm{Lip}_0(M)$.

10. Suppose $p \neq 2$ is a prime number. Choose $a \in \mathbb{Z}$ which is not a square and not divisible by p. Suppose $x^2 \equiv a \pmod{p}$ has a solution x_0. Show that there is a sequence (x_n) such that $x_n \equiv x_{n-1} \pmod{p^n}$ and $x_n^2 \equiv a \pmod{p^{n+1}}$. Show that (x_n) is Cauchy in $\mathbb{Q}$ in the p-adic metric d_p, and deduce that $(\mathbb{Q}, d_p)$ is not complete.

11. A *completion* of a metric space M is a complete metric space $\widehat{M}$ together with an isometric map $\iota\colon M \to \widehat{M}$ such that $\iota(M)$ is dense in $\widehat{M}$. Show that completions are unique up to isometry: if $(\widehat{M}, \iota)$ and $(\widetilde{M}, \kappa)$ are completions, then there is a (necessarily unique) isometry $\varphi\colon \widehat{M} \to \widetilde{M}$ such that $\kappa = \varphi \circ \iota$. The following two methods show the existence of completions.

(a) Define a relation on the set of Cauchy sequences in M by $(x_n) \sim (y_n)$ if and only if $d(x_n, y_n) \to 0$ as $n \to \infty$. Show that $\sim$ is an equivalence relation. Let $\widehat{M}$ be the set of all equivalence classes of Cauchy seuences in M and denote by $[(x_n)]$ the quivalence class of (x_n). Show that $\hat{d}\big([(x_n)], [(y_n)]\big) = \lim_{n\to\infty} d(x_n, y_n)$ defines a metric on $\widehat{M}$. Show that $(\widehat{M}, \hat{d})$ is a completion of M with a suitable isometric map $\iota\colon M \to \widehat{M}$.

(b) Fix $x_0 \in M$. For $x \in M$ define $f_x\colon M \to \mathbb{R}$ by $f_x(y) = d(y, x) - d(y, x_0)$. Show that f_x is a bounded map and that the map $M \to \ell_\infty(M)$, $x \mapsto f_x$, is isometric. Deduce that M has a completion.

12. Let (M, d) be a nonempty complete metric space and let $f\colon M \to M$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$, then $d(f(x), f(y)) < n$, and
(ii) if $d(x, y) < 1/n$, then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?

13. Let (M, d) be a nonempty complete metric space, let $f\colon M \to M$ be a continuous function and let $\lambda \in \mathbb{R}$ with $0 \leqslant \lambda < 1$.

(a) Assume that for all $x, y \in M$ we have either $d(f(x), f(y)) \leqslant \lambda d(x, y)$ or $d(f(f(x)), f(f(y))) \leqslant \lambda d(x, y)$. Show that f has a fixed point.

$^+$(b) Suppose we assume only that for all $x, y \in \mathbb{R}$ at least one of the three distances $d(f(x), f(y))$, $d(f(f(x)), f(f(y)))$ and $d(f(f(f(x))), f(f(f(y))))$ is less than or equal to $\lambda d(x, y)$. Must f have a fixed point?

14. Let X be an infinite set with the cofinite topology. Which sequences are convergent in X? What are the continuous functions $X \to \mathbb{R}$?

15. Assume X is a topological space and $Z \subset Y \subset X$. If Y is dense in X and Z is dense in Y, must Z be dense in X?

16. Let A be a subset of a topological space X. Show that $(X \setminus A)^\circ = X \setminus \overline{A}$ and that if A is the closure of an open set, then $\overline{A^\circ} = A$. Show that starting with the set A, successive applications of taking complement, closure and interior result in at most 14 sets (including A). Show that there is a subset A of $\mathbb{R}$ for which 14 different sets can be so constructed.

17. Show that if the topology of a space has a countable base, then each base for the topology contains a countable subfamily which is also a base.

18. Show that every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals.

19. Consider the subspace $Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$. Show that for a topological space X, continuous functions $X \to Y$ correspond precisely to convergent sequences in X.

20. Show that a function $f\colon X \to Y$ between topological spaces is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset X$. Deduce that if f is continuous and surjective, then the image of a dense subset of X is dense in Y.

21. Let $f\colon X \to Y$ be a function between topological spaces. For $x \in X$, we say f *is continuous at* x if for every neighbourhood V of $f(x)$ in Y there is a neighbourhood U of x in X such that $f(U) \subset V$. Show the following.

(a) If $x_n \to x$ in X and f is continuous at x, then $f(x_n) \to f(x)$ in Y. If $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X, then f need not be continuous at x.

(b) The following are equivalent.
   (i) f is continuous at x.
   (ii) For all $A \subset X$, if $x \in \overline{A}$ then $f(x) \in \overline{f(A)}$.

(c) The following are equivalent.
   (i) f is continuous.
   (ii) f is continuous at x for every $x \in X$.

22. Which of the following properties pass to subspaces, product spaces and quotient spaces:
   (i) Hausdorff (ii) separable (iii) second countable?

23. Let (M, d) and (M', d') be metric spaces and $N \subset M$. Show carefully that the metric topology on N induced by the restriction of d to N is the same as the subspace topology on N induced by the metric topology of M. Show also that the metric topology on $M \times M'$ induced by any of the natural product metrics d_1, d_2 or d_∞ is the same as the product topology on $M \times M'$ induced by the metric topologies of M and M'.

24. Let X_1 and X_2 be topological spaces and $Y_j \subset X_j$ for $j = 1, 2$. Identify two natural topologies on $Y_1 \times Y_2$ and show that these are the same.

25. Let X and Y be topological spaces. Show that the product topology is the coarsest topology on $X \times Y$ so that the coordinate projections π_X and π_Y are continuous. Now let τ be a topology on $X \times Y$ satisfying the following: for every space Z and function $f\colon Z \to X \times Y$, we have f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous. Show that τ is the product topology.

26. Let $q\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ denote the quotient map. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let L be the line $L = \{(x, y) \in \mathbb{R}^2 : y = \alpha x\}$ in $\mathbb{R}^2$. Show that $q(L)$ is dense in $\mathbb{R}^2/\mathbb{Z}^2$.

27. Assume that $A \subset \mathbb{R}^2$ satisfies the following conditions:
(a) if $x \in \mathbb{Q}$ then $(x, y) \in A$ for all $y \in \mathbb{R}$;
(b) if $x \notin \mathbb{Q}$, then $(x, y) \in A$ for at least one $y \in \mathbb{R}$.
Show that A is connected.

28. Find the connected components of the subspace $X = \{(0, 0), (0, 1)\} \cup \bigcup_{n \in \mathbb{N}} \{1/n\} \times [0, 1]$ of $\mathbb{R}^2$. Show that there exist $x, y \in X$ that belong to different components but there are no open sets U and V disconnecting X with $x \in U$ and $y \in V$.

29. A topological space is *totally disconnected* if its connected components are singletons. Show that $\mathbb{Q}$ is totally disconnected as is any ultrametric space.

30. A *topological group* is a group G with a topology for which multiplication $G \times G \to G$, $(x, y) \mapsto xy$, and inversion $G \to G$, $x \mapsto x^{-1}$, are continuous.
(a) Show that if $\{e\}$ is closed, then G is Hausdorff.
(b) If H is a subgroup of G with $H^\circ \neq \emptyset$, then H is closed.
(c) Show that if $H \lhd G$, then the quotient map $q\colon G \to G/H$ is open. Give an example when the quotient map is not closed, *i.e.*, when there is a closed

set A in G such that $q(A)$ is not closed in G/H.
(d) A metric d on G is *translation-invariant* if $d(xz, yz) = d(x, y) = d(zx, zy)$ for all $x, y, z \in G$. Show that if the topology of G is induced by a translation-invariant metric, and H is a closed normal subgroup of G, then G/H is metrizable. (*E.g.*, $\mathbb{R}^2/\mathbb{Z}^2$.)
(d) Show that a totally disconnected normal subgroup of a connected topological group is a subset of the centre. (See previous question for the definition of totally disconnected.)

$^+$31. Construct a compact connected subset K of $\mathbb{R}^2$ (of at least two elements) such that the only homeomorphism $K \to K$ is the identity.

32. Is there an infinite compact subset of $\mathbb{Q}$?

33. A family $\mathcal{F}$ of sets is said to have the *finite intersection property* if $\bigcap\{A : A \in \mathcal{G}\} \neq \emptyset$ for any finite, nonempty subset $\mathcal{G}$ of $\mathcal{F}$. Show that a topological space X is compact if and only if any nonempty family of closed subsets of X with the finite intersection property has nonempty intersection.

34. Does there exist a Hausdorff topology on $[0, 1]$ that is strictly coarser than the usual topology?

35. A topological space is *regular* if for any closed subset A and point $x \notin A$, there exist disjoint open sets U and V with $x \in U$ and $A \subset V$. A topological space is *Lindelöf* if every open cover has a countable subcover. Show that a regular Lindelöf space is normal.

36. Let $\mathcal{H}$ be the set of all nonempty, closed and bounded subsets of $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$ and $\delta > 0$ let $A_\delta = \bigcup_{x \in A} D_\delta(x)$. For $A, B \in \mathcal{H}$ let

$$d(A, B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\} .$$

Show that d is a complete metric on $\mathcal{H}$. For $A \in \mathcal{H}$, show that $\mathcal{H}(A) = \{B \in \mathcal{H} : B \subset A\}$ is closed and bounded; is $\mathcal{H}(A)$ compact?

37. We say that a topological space X is *locally compact* if for every $x \in X$, every neighbourhood U of x contains a compact neighbourhood of x. Show that if X is Hausdorff, then X is locally compact if and only if every point in X has a compact neighbourhood. Show that $\mathbb{R}^n$ is locally compact and that $C[0, 1]$ is not.

38. The *one-point compactification* of a topological space X is the space $X^+ = X \cup \{\infty\}$ (where ∞ is a symbol not in X, the *point at infinity*) equipped with the following topology: $U \subset X^+$ is open in X^+ if and only if either U is an open subset of X, or $\infty \in U$ and $X \setminus U$ is a closed and compact subset of X. Verify that X^+ is a compact topological space. Show that X^+ is Hausdorff if and only if X is locally compact and Hausdorff. Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| \leqslant 1\}$.

39. For real-valued functions on a topological space X, we have $f_n \to f$ locally uniformly implies $f_n \to f$ uniformly on K for every compact subset K of X. Show that the converse holds in a locally compact space X.

40. Let M be a sequentially compact metric space. Show that for an open cover for M there exists $\delta > 0$ so that every ball of radius δ is contained in some member of the cover. Deduce that M is compact.

41. Let p be a prime number, and $\mathbb{Z}_p$ be the set of all sequences $(x_n)_{n\geqslant 0}$ in $\mathbb{Z}/p\mathbb{Z}$ equipped with the metric $d((x_n),(y_n)) = p^{-k}$ where k is the least n with $x_n \neq y_n$.

(a) Show that $\mathbb{Z}_p$ is compact and totally disconnected.

(b) Show that the maps $a, m\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $a(x,y) = x + y$ and $m(x,y) = xy$ extend to continuous functions $a', m'\colon \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p$.

(c) Show that $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with the p-adic metric. Deduce that $\mathbb{Q}$ with the p-adic metric is not complete.

(d) Let b be an integer coprime to p, and assume $p > 2$. Show that the equation $x^2 = b$ has a solution in $\mathbb{Z}_p$ if and only if it has a solution in $\mathbb{Z}/p\mathbb{Z}$.

42. Let U be an open subset of $\mathbb{R}^m$, let $f\colon U \to \mathbb{R}$ be a function, and let $a \in U$. Show that if f has a local maximum or minimum at a, and is differentiable at a, then $f'(a) = 0$.

43. Let $U \subset \mathbb{R}^m$ be an open set, and $f\colon U \to \mathbb{R}^n$ be differentiable on U. Show carefully that if f' has directional derivative $D_u f'(a)$ for some $a \in U$ and $u \in \mathbb{R}^m \setminus \{0\}$, then for every $v \in \mathbb{R}^m \setminus \{0\}$, the directional derivative $D_u D_v f(a)$ exists and equals $D_u f'(a)(v)$.

44. Let U be an open subset of $\mathbb{R}^2$ containing the rectangle $[a,b] \times [c,d]$. Suppose that $f\colon U \to \mathbb{R}$ is continuous and that $D_2 f$ exists and is continuous

on U. Show that $F(y) = \int_a^b f(x, y)\,\mathrm{d}x$ is differentiable on some open interval containing $[c, d]$ with $F'(y) = \int_a^b D_2 f(x, y)\,\mathrm{d}x$.