1. Show that a continuous real-valued function on a sequentially compact topological space is bounded and attains its bounds. Show also that a continuous function from a compact metric space to an arbitrary metric space is uniformly continuous.

2. Show that if A and B are closed subsets of $\mathbb{R}^n$ and if A or B is bounded, then $A + B$ is closed. Give an example in $\mathbb{R}$ to show that the boundedness condition cannot be omitted.

3. Let M be a non-empty compact metric space and $f\colon M \to M$ be a function.

(a) Show that if $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$ in M, then f has a unique fixed point.

(b) Show that if f is isometric, *i.e.*, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$, then f is surjective.

4. (a) A topological space is *normal* if disjoint closed subsets can be separated by open sets: given disjoint closed subsets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Show that a compact Hausdorff space is normal.

(b) Let (C_n) be a decreasing sequence of compact connected subsets of a Hausdorff space. Show that $\bigcap_{n \in \mathbb{N}} C_n$ is connected. (Part (a) will be useful here.) Give an example in $\mathbb{R}^2$ of a decreasing sequence of closed connected sets whose intersection is disconnected.

5. Show that $C[0, 1]$ in the uniform metric D is separable. Let $B = \{f \in C[0, 1] : D(0, f) \leqslant 1\}$ and $B' = \{f \in B : f \text{ differentiable and } f' \in B\}$. Show that B is closed but not compact. On the other hand, show that every sequence in B' has a subsequence convergent in $C[0, 1]$. Deduce that $\overline{B'}$ is compact.

6. At which points is each of the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?

(i) $f(x, y) = |x||y|$;

(ii) $g(x, y) = xy \sin(1/x)$ when $x \neq 0$ and $g(0, y) = 0$;

(iii) $h(x, y) = \frac{xy}{(x^2+y^2)^{1/2}}$ when $(x, y) \neq (0, 0)$ and $h(0, 0) = 0$.

7. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ if $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}^3 \setminus \{0\}$ with

$$f'(x)(h) = \frac{h}{\|x\|} - \frac{\langle x, h\rangle x}{\|x\|^3} \ .$$

Verify that $f'(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

8. (a) Show that the set $\mathcal{G}_n$ of invertible $n \times n$ real matrices is an open subset of $\mathcal{M}_n$. By quoting appropriate results, explain why the function $f\colon \mathcal{G}_n \to \mathcal{M}_n$ given by $f(A) = A^{-1}$ is differentiable.

(b) Given an open subset U of $\mathcal{M}_n$, show that if functions $g, h\colon U \to \mathcal{M}_n$ are differentiable at $A \in U$, then so is the product gh given by $(gh)(X) = g(X)h(X)$. Hence, or otherwise, find the derivative of the function f given in part (a).

9. Show that the function $\det\colon \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $\det'(I)(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at every invertible matrix A with $\det'(A)(H) = \det(A)\operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $\det''(I)$ as a bilinear map. Is det differentiable at any non-invertible matrix?

10. Assume that all directional derivatives $D_u f(0)$ exist for $f\colon \mathbb{R}^2 \to \mathbb{R}$, and moreover the map $T\colon \mathbb{R}^2 \to \mathbb{R}$ given by $T(u) = D_u f(0)$ if $u \neq 0$ and $T(0) = 0$ is linear. Does it follow that f is differentiable at 0? What if we assume that $f \circ \gamma$ is differentiable at 0 for every differentiable curve $\gamma\colon (-1, 1) \to \mathbb{R}^2$ with $\gamma(0) = 0$?

11. Let $U \subset \mathbb{R}^2$ be an open set and $f\colon U \to \mathbb{R}$ be a function such that for each $x \in \mathbb{R}$ the map $y \mapsto f(x, y)$ is continuous, and for each $y \in \mathbb{R}$ the map $x \mapsto f(x, y)$ is continuous. Show that f need not be continuous on U. Now assume that $D_1 f$ exists and is bounded on U and that for each $x \in \mathbb{R}$ the map $y \mapsto f(x, y)$ is continuous. Show that f is continuous.

12. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I: there exist $r > 0$ and a continuous function $g\colon D_r(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in D_r(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

13. Let $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Define a function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x, y) = (x, x^3 + y^3 - 3xy)$. Show that for every $(x_0, y_0) \in C \setminus \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ there are open sets U containing (x_0, y_0) and V containing $F(x_0, y_0)$ such that $F\restriction_U$ is a bijection from U onto V with a continuously differentiable inverse. Deduce that for every such point (x_0, y_0) there is an open neighbourhood U of (x_0, y_0), an open interval I containing x_0 and a continuously differentiable function $g\colon I \to \mathbb{R}$ such that $g(x_0) = y_0$ and $C \cap U$ is the graph of g, *i.e.*, $C \cap U = \{(x, y) \in \mathbb{R}^2 : x \in I,\ y = g(x)\}$.

14. (a) Show that there exist $\varepsilon > 0$ and C^1 functions $F, G\colon D_\varepsilon(\mathbf{0}) \to \mathbb{R}$ on the open ball $D_\varepsilon(\mathbf{0}) \subset \mathbb{R}^2$ such that $(x, y, z, w) = (x, y, F(x, y), G(x, y))$ is a solution of the equations

$$\begin{aligned} \sin xz + \cos yw &= \mathrm{e}^z \\ \cos yz + \sin xw &= \mathrm{e}^w \end{aligned}$$

for all $(x, y) \in D_\varepsilon(\mathbf{0})$.

(b) Deduce that on an open interval in $\mathbb{R}$ containing 0 the system of differential equations

$$\begin{aligned} \sin y_1 y_1' + \cos y_2 y_2' &= \mathrm{e}^{y_1'} \\ \cos y_2 y_1' + \sin y_1 y_2' &= \mathrm{e}^{y_2'} \end{aligned}$$

has a unique solution $\mathbf{y}(t) = (y_1(t), y_2(t))$ with $\mathbf{y}(0) = \mathbf{0}$.