1. Let X be a Hausdorff space. Show that $\{x\}$ is closed for all $x \in X$ and that A' is closed for every $A \subset X$. Show that in any topological space, the derived set A' of a closed set A is closed.

2. Let X be an uncountable set. Show that the family τ consisting of the emptyset and all subsets of X with countable complement is a topology on X. Fix $x \in X$ and let $Y = X \setminus \{x\}$. Show that Y is dense in X but no sequence in Y converges to x. Is the space (X, τ) metrizable? Identify the convergent sequences in X and the continuous functions $X \to \mathbb{R}$.

3. The diagonal of a set Y is the set $\Delta_Y = \{(x, y) \in Y \times Y : x = y\}$. Show that a topological space Y is Hausdorff if and only if Δ_Y is closed in $Y \times Y$. Deduce or otherwise show that if $f, g \colon X \to Y$ are continuous functions from a space X to a Hausdorff space Y, then $\{x \in X : f(x) = g(x)\}$ is closed in X; in particular, if f and g agree on a dense subset of X, then $f = g$ on X.

4. (a) Show that a space with a countable base is separable and that every separable *metric* space has a countable base. Deduce that a subspace of a separable metric space is separable.

(b) Prove that the family of all half-open intervals $[a, b)$ in $\mathbb{R}$ is a base for a topology τ on $\mathbb{R}$. Let $X = (\mathbb{R}, \tau)$. Show that X is separable but has no countable base. Show that $X \times X$ with the product topology is separable. Identify the subspace topology on $Y = \{(x, y) \in \mathbb{R}^2 : x+y = 0\}$. Is Y separable?

5. Let X be a topological space. Show that the set $C(X) = \{f \colon X \to \mathbb{R} : f \text{ is continuous}\}$ is a linear subspace of the real vector space of all functions $X \to \mathbb{R}$. Show that if $f_n \to f$ uniformly on X and $f_n \in C(X)$ for all n, then $f \in C(X)$. Deduce that $C_b(X) = C(X) \cap \ell_\infty(X)$ is complete in the uniform metric.

6. Let X be a topological space, R be an equivalence relation on X and $q \colon X \to X/R$ be the quotient map. Show that

(a) if X/R is Hausdorff, then R is closed in $X \times X$, and

(b) if R is closed in $X \times X$ and q is an open map, then X/R is Hausdorff.

7. Which of the following subsets of $\mathbb{R}^2$ are (a) connected, (b) path-connected?

(i) $D_1((-1, 0)) \cup D_1((1, 0))$ (ii) $D_1((-1, 0)) \cup B_1((1, 0))$

(iii) $\{(x, y) : x = 0 \text{ or } y/x \in \mathbb{Q}\}$ (iv) $\{(x, y) : x = 0 \text{ or } y/x \in \mathbb{Q}\} \setminus \{(0, 0)\}$.

8. Let $f \colon X \to S$ be a function from a connected space X to a set S. Assume f is *locally constant*: every $x \in X$ has a neighbourhood on which f is constant. Show that f is constant.

9. Show that homeomorphic spaces have the same number of connected components. Show that no two of $[0, 1]$, $[0, 1)$ and $(0, 1)$ are homeomorphic. Show also that the letters A and H drawn in the plane are not homeomorphic.

10. Let $A \subset \mathbb{R}^n$ be such that every continuous function $f\colon A \to \mathbb{R}$ is bounded. Show that A is compact.

11. Let X be a Hausdorff space. Let $\tau = \{U \subset X : U = \emptyset \text{ or } X \setminus U \text{ is compact}\}$. Show that τ is a topology on X. When is τ Hausdorff?

12. Let A be an infinite subset of a compact topological space X. Show that $A' \neq \emptyset$.

13. (a) Let R be the equivalence relation on the unit square $Q = [0,1]^2$ defined as follows: $(x_1, y_1) \sim (x_2, y_2)$ if and only if either $(x_1, y_1) = (x_2, y_2)$, or $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = y_2$, or $x_1 = x_2$ and $\{y_1, y_2\} = \{0, 1\}$. Show that any two of the following spaces (in their natural topologies) are homeomorphic: Q/R, $\mathbb{R}^2/\mathbb{Z}^2$, $S^1 \times S^1$ and the subspace

$$T^2 = \big\{\big((2+\cos\theta)\cos\varphi, (2+\cos\theta)\sin\varphi, \sin\theta\big) : \theta, \varphi \in [0, 2\pi]\big\}$$

of $\mathbb{R}^3$.

(b) Let R be the equivalence relation on $Q = [0,1]^2$ defined as follows: $(x_1, y_1) \sim (x_2, y_2)$ if and only if either $(x_1, y_1) = (x_2, y_2)$, or $\{x_1, x_2\} = \{0, 1\}$ and $y_1 = y_2$, or $y_1 = y_2 = 0$, or $y_1 = y_2 = 1$. Show that Q/R is homeomorphic to the sphere $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.

14. (a) Let R_1 be an equivalence relation on a topological space X and R_2 be an equivalence relation on the quotient space X/R_1. Define

$$R = \{(x, y) \in X \times X : (q(x), q(y)) \in R_2\}$$

where $q\colon X \to X/R_1$ is the quotient map. Show that R is an equivalence relation on X and that X/R is homeomorphic to $\big(X/R_1\big)/R_2$.

(b) For a topological space X and for $A \subset X$, we let X/A denote the quotient space of X by the relation identifying the points of A: $x \sim y$ if and only if either $x = y$ or $x, y \in A$. Now consider the subset $A = \{(0,0,1), (0,0,-1)\}$ of the two-dimensional sphere S^2, and the subset $B = \{(2+\cos\theta, 0, \sin\theta) : \theta \in [0, 2\pi]\}$ of T^2 (see previous question). Show that S^2/A and T^2/B are homeomorphic.

15. (a) Show that the coordinate projections π_X and π_Y on a product space $X \times Y$ are open maps. Show that if Y is compact, then π_X is a *closed map*: for a closed subset F of $X \times Y$, its projection $\pi_X(F)$ is closed in X. Give an example of a closed set in $\mathbb{R}^2$ whose projections are not closed in $\mathbb{R}$.

(b) Let $f\colon X \to Y$ be a function between topological space. The *graph of f* is the set $\Gamma = \{(x, y) \in X \times Y : y = f(x)\}$. Show that if f is continuous and Y is Hausdorff, then Γ is closed in the product topology. Conversely, show that if Γ is closed and Y is compact, then f is continuous.

16. Given topological spaces X and Y and continuous bijections $f\colon X \to Y$ and $g\colon Y \to X$, show that X and Y need not be homeomorphic.

$^+$17. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a function under which the image of any path-connected set is path-connected and the image of any compact set is compact. Show that f must be continuous.