1. Let $\left(x^{(m)}\right)$ and $\left(y^{(m)}\right)$ be sequences in $\mathbb{R}^n$ converging to x and y, respectively. Show that the scalar product $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are continuous at x, then so is the pointwise scalar product $f \cdot g\colon \mathbb{R}^n \to \mathbb{R}$.

2. Show that a Cauchy sequence with a convergent subsequence is convergent.

3. Show that in an ultrametric space every triangle is isosceles and every open ball is closed.

4. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? (And why?)

(i) $\{(x, 0) : 0 \leqslant x \leqslant 1\}$;

(ii) $\{(x, 0) : 0 < x < 1\}$;

(iii) $\{(x, y) : y \neq 0\}$;

(iv) $\bigcup_{n\in\mathbb{N}}\{(x, y) : y = nx\} \cup \{(0, y) : y \in \mathbb{R}\}$;

(v) $\bigcup_{q\in\mathbb{Q}}\{(x, y) : y = qx\} \cup \{(0, y) : y \in \mathbb{R}\}$;

(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

5. Is the set $(1, 2]$ an open subset of $\mathbb{R}$? Is it closed? What if we replace $\mathbb{R}$ with the subspace $(1, 3)$, $[0, 2]$ or $(1, 2]$?

6. Let M be a metric space and $A \subset M$. Define the *closure* of A (in M) to be the set

$$\overline{A} = \{x \in M : \exists\, (x_n) \text{ in } A \text{ such that } x_n \to x\}\ ,$$

Define the *interior* of A (in M) to be the set

$$A^\circ = \{x \in M : A \text{ is a neighbourhood of } x \text{ in } M\}\ .$$

Show that $A \subset \overline{A}$ and $\overline{A}$ is closed in M and that for any closed subset F of M with $A \subset F$ we have $\overline{A} \subset F$. Show also that $A^\circ \subset A$ and A° is open in M and that for any open subset U of M with $U \subset A$ we have $U \subset A^\circ$

Show that the inclusions $D_r(x) \subset B_r(x)^\circ$ and $\overline{D_r(x)} \subset B_r(x)$ hold in every metric space and can be strict in general but that they are always equalities in $\mathbb{R}^n$.

7. Let A be a non-empty subset of a metric space M. For $x \in M$ let

$$d(x, A) = \inf\{d(x, a) : a \in A\}\ .$$

Show that the map $M \to \mathbb{R}$, $x \mapsto d(x, A)$, is 1-Lipschitz and that $d(x, A) = 0$ if and only if $x \in \overline{A}$. Deduce that if A and B are disjoint closed subsets of M, then there exist disjoint open subsets U and V of M such that $A \subset U$ and $B \subset V$.

8. Which of the following metric spaces are complete?

(i) $C^1[0,1] = \{f\colon [0,1] \to \mathbb{R} : f \text{ continuously differentiable}\}$ in the uniform metric D;

(ii) $C^1[0,1]$ in the metric $D_1(f,g) = D(f,g) + D(f',g')$;

(iii) $C[0,1]$ in the L_1-metric.

(iv) $\mathbb{Z}$ in the 2-adic metric.

9. (a) Show that there is a metric on $\mathbb{R}$ which is equivalent to the usual metric but in which $\mathbb{R}$ is not complete.

(b) Let d and d' be equivalent metrics on a set M. Show that if d and d' are uniformly equivalent, then (M,d) and (M,d') have the same Cauchy sequences (and hence one is complete if and only if the other is complete). If (M,d) and (M,d') have the same Cauchy sequences, does it follow that d and d' are uniformly equivalent?

10. Use the contraction mapping theorem to show that $\cos x = x$ has a unique solution in $\mathbb{R}$. Using a calculator, find a good approximation to this solution and justify the claimed accuracy of your approximation.

11. Let $f\colon M \to M$ be a function on a non-empty complete metric space M. Assume that for some $k \geqslant 1$, the k-fold composition $f \circ \cdots \circ f$ of f with itself is a contraction mapping. Show that f has a unique fixed point. Deduce that the initial value problem

$$f'(t) = f(t^2)\ , \qquad f(0) = y$$

has a unique solution on the interval $[0,1]$.

12. Let $a < b$ and $R > 0$ be real numbers, let $y_0 \in \mathbb{R}^n$ and let $\varphi\colon [a,b] \times B_R(y_0) \to \mathbb{R}^n$ be a continuous function. Assume that for some $K \geqslant 0$ we have $\|\varphi(t,x) - \varphi(t,y)\| \leqslant K\|x-y\|$ for all $t \in [a,b]$ and all $x,y \in B_R(y_0)$. Assume further that

$$\sup\big\{\|\varphi(t,x)\| : t \in [a,b],\ x \in B_R(y_0)\big\} \leqslant \frac{R}{b-a}\ .$$

Using the first part of the previous question, show that for any $t_0 \in [a,b]$, the initial value problem

$$f'(t) = \varphi(t, f(t))\ , \qquad f(t_0) = y_0$$

has a unique solution on $[a,b]$.

13. We are given a nested sequence $A_1 \supset A_2 \supset \dots$ of non-empty closed subsets of a complete metric space. Assume that the diameter $\operatorname{diam}(A_n) = \sup\{d(x,y) : x,y \in A_n\}$ converges to zero. Show that the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is non-empty. Is it true that a nested sequence of closed balls in a complete metric space has non-empty intersection?