1. Which of the following sequences (f_n) of functions converge uniformly on the set X?

(i) $f_n(x) = x^n$ on $X = (0,1)$; (ii) $f_n(x) = x^n(1-x)$ on $X = [0,1]$;

(iii) $f_n(x) = \mathrm{e}^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

2. Suppose functions $f_n \to f$ and $g_n \to g$ uniformly on a set S. Show that $f_n + g_n \to f + g$ uniformly on S. On the other hand, show that the pointwise product $f_n g_n$ need not converge uniformly to fg but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

3. Construct a sequence (f_n) of continuous real-valued functions on $[0,1]$ converging pointwise to the zero function but with $\int_0^1 f_n(x)\,\mathrm{d}x \not\to 0$. $^+$Is it possible to find such a sequence with $|f_n(x)| \leqslant 1$ for all x and for all n?

Construct a sequence (f_n) of differentiable real-valued functions on $[0,1]$ converging uniformly to a function f which has at least one point of non-differentiability.

4. Which of the following functions $f\colon [0,\infty) \to \mathbb{R}$ are uniformly continuous?

(i) $f(x) = \sin x^2$; (ii) $f(x) = \inf\left\{|x - n^2| : n \in \mathbb{N}\right\}$; (iii) $f(x) = (\sin x^3)/(x+1)$.

5. Show that the uniform limit of uniformly continuos real-valued functions on a metric space is uniformly continuous. Give an example of a sequence (f_n) of uniformly continuous, real-valued functions on $\mathbb{R}$ that converges pointwise to a continuous function f that is not uniformly continuous.

6. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is continuous and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Must f be uniformly continuous on $[0,\infty)$? Give a proof or counterexample as appropriate.

7. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $D_r(x)$ for $x \in X$ and $r > 0$ small.

(i) $X = \mathcal{R}[0,1]$, the space of intergrable functions on $[0,1]$; $d(f,g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$.

(ii) $X = \mathbb{Z}$; $d(x,x) = 0$ and, for $x \neq y$, $d(x,y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) $X = \mathbb{N}^{\mathbb{N}}$; $d(f,f) = 0$ and, for $f \neq g$, $d(f,g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z,w) = |z - w|$ if z and w lie on the same line through the origin, $d(z,w) = |z| + |w|$ otherwise.

8. Let f_n, $n \in \mathbb{N}$, and f be real-valued continuous functions on the closed bounded interval $[a, b]$. Show that if $f_n \to f$ uniformly on $[a, b]$ and $x_n \to x$ in $[a, b]$, then $f_n(x_n) \to f(x)$. On the other hand, show that if (f_n) does not converge uniformly to f, then there is a convergent sequence $x_n \to x$ in $[a, b]$ such that $f_n(x_n) \not\to f(x)$.

9. Show that for each $x \in X = \mathbb{R} \setminus \mathbb{N}$ the series $\sum_{n=1}^{\infty}(x-n)^{-2}$ converges. Does the series converge uniformly on X? Define $f\colon X \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty}(x-n)^{-2}$. Show that f is continuously differentiable on X and find its derivative.

10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable and assume that f' is bounded. Show that f is a Lipschitz function. Define $g\colon [-1, 1] \to \mathbb{R}$ by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable on $[-1, 1]$. Is g a Lipschitz function? Is g uniformly continuous?

11. Let $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Show that f is uniformly continuous, and hence prove the following.

(i) The function $g\colon [0, 1] \to \mathbb{R}$, defined by $g(y) = \int_0^1 f(x, y)\,\mathrm{d}x$, is continuous.

(ii) There is a sequence of step functions on $[0, 1] \times [0, 1]$ converging uniformly to f. (A step function is a function of the form $\sum_{i=1,j=1}^{m,n} c_{i,j}\mathbf{1}_{J_i \times K_j}$ where $[0, 1] = \bigcup_{i=1}^{m} J_i = \bigcup_{j=1}^{n} K_j$ are partitions of $[0, 1]$ into intervals, $\mathbf{1}_S$ is the indicator function of a set S and the $c_{i,j}$ are real numbers.)

(iii) $\int_0^1 \int_0^1 f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 f(x, y)\,\mathrm{d}y\,\mathrm{d}x$.

12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous but not uniformly continuous function. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = \sin(f(x))$ for $x \in \mathbb{R}$. Show that g is continuous but not uniformly continuous.

13. Let (f_n) be a sequence of continuous real-valued functions on $[0, 1]$ converging pointwise to a function f. Prove that there is some subinterval $[a, b]$ of $[0, 1]$ with $a < b$ on which f is bounded.