1. Prove Bessel's inequality: if (e_n) is an orthonormal sequence in an inner product space, then $\sum_n |\langle x, e_n\rangle|^2 \leq \|x\|^2$ for every x.

2. Let Y be a closed subspace of a Hilbert space H. Show that $Y^{\perp\perp} = Y$. Deduce that if S is a subset of H then $S^{\perp\perp} = \overline{\text{span}} S$. Show that S has dense linear span in H if and only if $S^{\perp} = \{0\}$.

3. Let $(a_n) \in \ell_\infty$. Define $T \colon \ell_2 \to \ell_2$ by $T(x_n) = (a_n x_n)$. Show that T is in $\mathcal{B}(\ell_2)$ and $\|T\| = \|a\|_\infty$. Find the eigenvalues, approximate eigenvalues and the spectrum of T. Show that T is compact if and only if $(a_n) \in \mathrm{c}_0$.

4. Show that $T \in \mathcal{B}(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

5. Let U be a unitary operator on a complex Hilbert space. Show that $\sigma(U) \subset \mathbb{T}$.

6. Let H be a complex Hilbert space with orthonormal basis $(e_n)_{n=-\infty}^{\infty}$. The *bilateral shift* is the operator T on H defined by $Te_n = e_{n+1}$ $(n \in \mathbb{Z})$. Find the spectrum of T.

7. Prove that if T is a normal operator then $\sigma(T) = \sigma_{\text{ap}}(T)$.

8. Let S and T be compact normal operators on a complex Hilbert space with $\dim E_S(\lambda) = \dim E_T(\lambda)$ for all λ. Show that $T = USU^*$ for some unitary U.

9. Show that $\mathcal{F}(\ell_2)$ is dense in $\mathcal{K}(\ell_2)$. Is $\mathcal{G}(\ell_2)$ dense in $\mathcal{B}(\ell_2)$?

10. Let X be a complex Banach space and $T \in \mathcal{B}(X)$. Define $r(T)$ for a rational function r with no poles in $\sigma(T)$. Prove that $\sigma(r(T)) = \{r(\lambda) : \lambda \in \sigma(T)\}$.

11. Let T be an operator on a Banach space X with $\|T\| < 1$. Show that $I - T$ has a square root, *i.e.,* there exists $S \in \mathcal{B}(X)$ such that $S^2 = I - T$.

12. Let $A = (a_{ij})_{i,j=1}^{\infty}$ be a complex matrix whose rows and columns form a bounded set in ℓ_2. Must A be the matrix of a bounded linear operator on ℓ_2?

13. Construct a hermitian operator T on a (non-zero) Hilbert space H so that T has no eigenvalues.

14^+. Let X be a separable infinite-dimensional Banach space. Suppose that there is a constant C such that every finite-dimensional subspace E of X satisfies $d(E, \ell_2^n) \leq C$, where $n = \dim E$. Prove that X is isomorphic to ℓ_2.