[*Throughout, K is a compact Hausdorff space.*]

1. Let $f \in C(K)$. Show that there exists $\phi \in C(K)^*$ with $\|\phi\| = 1$ and $\phi(f) = \|f\|$.

2. A linear map $\phi\colon C(K) \to \mathbb{R}$ is said to be *positive* if $\phi(f) \geq 0$ whenever $f \geq 0$. Show that a positive linear map is continuous. What is the norm of ϕ?

3. Let X be an inner product space, and let $T\colon X \to X$ be a linear map. Show that $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in X$ if and only if $\|Tx\| = \|x\|$ for all $x \in X$.

4. Let X be a complex inner product space, and let $T\colon X \to X$ be a linear map. Show that if $\langle Tx, x\rangle = 0$ for all $x \in X$, then $T = 0$. Does the same conclusion hold in the real case?

5. Let A be a subalgebra of $C^{\mathbb{R}}(K)$ that separates the points of K. Show that either $\overline{A} = C^{\mathbb{R}}(K)$ or there is a point $x_0 \in K$ such that $\overline{A} = \{f \in C^{\mathbb{R}}(K) : f(x_0) = 0\}$.

6. Let M_r, $0 < r < 1$, be real numbers with $0 < M_r < M_s$ whenever $0 < r < s < 1$. Let $\mathcal{F}$ be the space of all analytic functions $f\colon \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ satisfying $|f(z)| \leq M_r$ if $|z| \leq r < 1$. Show that for each $0 < r < 1$, the set of restrictions to $\Delta_r = \{z \in \mathbb{C} : |z| \leq r\}$ of elements of $\mathcal{F}$ is relatively compact in $C(\Delta_r)$ [*Hint: Cauchy's Integral Formula*]. Deduce that every sequence in $\mathcal{F}$ has a subsequence that converges locally uniformly on D.

7. Show that $C(\mathbb{T})$ with the convolution product and the uniform norm is a Banach algebra. Show further that this algebra is non-unital.

8. Let S be a subset of a normal topological space X. Show that there is a continuous function $f\colon X \to \mathbb{R}$ such that $S = f^{-1}(0)$ if and only if S is a closed $\mathcal{G}_\delta$ set.

9. A topological space is *Lindelöf* if every open cover has a countable subcover, and is *regular* if for every closed subset F and point $x \notin F$, there are disjoint open sets U and V with $x \in U$ and $F \subset V$. Prove that a regular Lindelöf space is normal.

10. Show that if $C(K)$ is separable, then K is metrizable.

11. Show that there is a sequence (x_n) in the unit ball of ℓ_2 such that $\|x_m - x_n\| > \sqrt{2}$ for all $m \neq n$ in $\mathbb{N}$. Can the constant $\sqrt{2}$ be improved?

12. Construct an inner product space X and a proper closed subspace Y of X such that $Y^\perp = \{0\}$.

13. A series $\sum x_i$ in a Banach space converges *unconditionally* if $\sum \varepsilon_i x_i$ converges for all choices of signs $\varepsilon_i = \pm 1$. Show that if this happens in a Hilbert space then $\sum \|x_i\|^2 < \infty$.

14. Let H be a Hilbert space, and let $f\colon [0,1] \to H$ be a continuous function. Suppose that for every $x < y < z$ we have $f(x) - f(y) \perp f(y) - f(z)$. Must f be constant?