1. Let $p < \infty$ and $x \in \ell_p$. Show that there is an $f \in \ell_p^*$ with $\|f\| = 1$ and $f(x) = \|x\|$.

2. Let f be a linear functional on a normed space X. Prove that f is continuous if and only if $\ker f$ is closed.

3. Let $A \subset Y \subset X$ with A nowhere dense in Y. Show that A is nowhere dense in X.

4. Prove Osgood's theorem: if (f_n) is a sequence of continuous functions $[0,1] \to \mathbb{R}$ such that $\big(f_n(t)\big)$ is bounded for every $t \in [0,1]$, then there is an interval $[a,b]$ with $a < b$ on which the f_n are uniformly bounded.

5. Let X be a closed subspace of ℓ_1. Assume that every $y = (x_{2n}) \in \ell_1$ extends to a sequence $x = (x_n) \in X$. Show that there is a constant C such that x can always be chosen to satisfy $\|x\| \leq C\|y\|$.

6. Assume that X is a closed subspace of $\big(C[0,1], \|\cdot\|_\infty\big)$ such that every element of X is continuously differentiable. Show that X is finite-dimensional.

7. Suppose that $T\colon X \to Y$ satisfies the conditions in the Open Mapping Lemma. Show that Y is complete.

8. Let Y be a proper subspace of a Banach space X. Can Y be dense $\mathcal{G}_\delta$, *i.e.,* can Y be the intersection of a sequence of dense open sets in X?

9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $x > 0$ we have $f(nx) \to 0$ as $n \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.

10. Let $1 \leq p < q$. Consider the *subset* $Y = \ell_p$ of the Banach space $X = (\ell_q, \|\cdot\|_q)$. Show that Y is meagre in X.

11. Does there exist a function $f\colon [0,1] \to \mathbb{R}$ which is continuous at every rational and discontinuous at every irrational?

12. Let $f\colon [0,1] \to \mathbb{R}$ be a pointwise limit of a sequence of continuous functions. Show that f has a point of continuity.

$^+$13. Let X be a normed space that is homeomorphic to a complete metric space. Prove that X is complete.

$^+$14. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ with $f^{(m)}(x) = 0$ for all $m \geq n$. Prove that f is a polynomial.