1. Let X, Y be isomorphic normed spaces. Show that if X is complete then so is Y.

2. Give an example of two non-equivalent norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X such that $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are isometrically isomorphic.

3. Let C be a convex set in a normed space X such that $C^\circ \neq \emptyset$. Show that $\overline{C^\circ} = \overline{C}$ and $(\overline{C})^\circ = C^\circ$.

4. Show that there exists a vector space with two non-equivalent complete norms.

5. Construct continuum many dense subspaces of ℓ_1 with pairwise trivial intersections.

6. Assume that the vector space X is the algebraic direct sum of subspaces Y and Z. If two norms $\|\cdot\|$ and $|||\cdot|||$ on X are equivalent on Y and Z, must they be equivalent on X?

7. Let Y be a closed subspace of a normed space X. Show that $\|x + Y\| = d(x, Y)$ defines a norm on the quotient space X/Y. Show further that the quotient map $q\colon X \to X/Y$ is a continuous, open linear map.

8. Let Y be a closed subspace of a normed space X. Show that if X is complete, then so is X/Y. Show that if Y and X/Y are complete, then so is X.

9. Let X be one of the spaces ℓ_p, $1 \leqslant p < \infty$ or c_0. Denote by (e_n) the unit vector basis of X. For $N \in \mathbb{N}$ and $x = \sum x_n e_n \in X$ set $P_N(x) = \sum_{n=1}^{N} x_n e_n$. Prove the following characterization of relative compact sets in X (*cf.* Arzelà-Ascoli theorem). For $K \subset X$ TFAE.

(i) K is relatively compact.

(ii) K is bounded, *i.e.,* there exists $C \geqslant 0$ such that $\|x\| \leqslant C$ for all $x \in K$, and for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x - P_N x\| < \varepsilon$ for all $x \in K$.

10. Let (x_n) be a sequence in ℓ_1 such that $f(x_n) \to 0$ for all $f \in \ell_1^*$. Show that $x_n \to 0$ in norm.

11. Show that no two of the spaces ℓ_p, $1 \leqslant p \leqslant \infty$, and c_0 are isomorphic.

12. Let X be one of the spaces ℓ_p, $1 \leqslant p < \infty$ or c_0. Let Y be a closed, infinite-dimensional subspace of X. Show that there is a subspace Z of Y such that $Z \sim X$ and Z is *complemented* in X, *i.e.,* there is a bounded projection $P\colon X \to X$ with $P(X) = Z$.

13. Prove that $C[0,1]$ is isomorphic to the closed, one-codimensional subspace $X = \{f \in C[0,1] : f(1) = 0\}$.

14. Let F be a Hamel basis of an infinite-dimensional Banach space X. For $x \in F$ let ϵ_x be the corresponding coordinate functional: $\epsilon_x(\sum_{y\in F}\lambda_y y) = \lambda_x$. Show that all but finitely many of ϵ_x are discontinuous.

15. Fix $0 < p < 1$. Let ℓ_p be the set of sequences $x = (x_n)$ such that $\|x\|_p = \left(\sum |x_n|^p\right)^{1/p} < \infty$. Show that ℓ_p is a vector space. Show that $\|\cdot\|_p$ is not a norm on ℓ_p but that $d(x, y) = \|x - y\|_p$ is a metric. Identify the dual space ℓ_p^* of all continuous (with respect to d) linear functionals.

16. Are ℓ_1^n and ℓ_∞^n isometrically isomorphic?

17. Let X be an n-dimensional normed space, and $0 < \varepsilon < 1$. Show that if $S \subset B_X$ is an ε-net for B_X then $|S| \geqslant \varepsilon^{-n}$. On the other hand, show that $|S| \leqslant 3 \cdot \varepsilon^{-n}$ is possible.

18. Let X be an infinite-dimensional normed space. Show that there is no translation-invariant Borel measure μ on X such that $\mu(U) > 0$ for *every* non-empty open set U, and $\mu(U) < \infty$ for *some* non-empty open set U.

19. Let Y be a closed subspace of a normed space X and let F be a finite-dimensional subspace of X. Show that $Y + F$ is closed.

20. Let Y and Z be closed subspaces of X of the same finite codimension. Show that there is an isomorphism $T\colon X \to X$ with $T(Y) = Z$.

21. Let X be a normed space. Show that if X^* is separable, then so is X.

22. Show that every separable Banach space embeds isometrically into ℓ_∞.

23. Let X be a separable Banach space. Show that there is a bounded linear map from ℓ_1 onto X. Show that there is a closed subspace Y of ℓ_1 such that $X \cong \ell_1/Y$.

24. Let Y be a proper closed subspace of a normed space X. Given $x \in X$, must there be a $y \in Y$ such that $d(x, Y) = \|x - y\|$? Also, must there exist $x \in B_X$ such that $d(x, Y) = 1$?

25. Let M_n be the set of all normed spaces $(\mathbb{R}^n, \|\cdot\|)$. Let d denote the Banach-Mazur distance. Show that $\log d$ is a metric on M_n after identifying isometric normed spaces. Show that M_n is compact in this metric.

26. Let X be a finite-dimensional normed space. Show that X embeds into c_0 *almost isometrically*: for all $\varepsilon > 0$ there is a subspace Y of c_0 and a linear bijection $T\colon X \to Y$ with $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$.

27. Construct a separable reflexive Banach space X that contains every finite-dimensional space almost isometrically: for all F with $\dim F < \infty$ and for all $\varepsilon > 0$ there is an isomorphism T of F into X with $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$.

28. Give a direct, elementary proof of the Principle of Uniform Boundedness (*i.e.*, one that does not use the Baire Category Theorem).

29. Deduce the Baire Category Theorem from the fact that a non-empty complete metric space is of second category in itself. (This is more subtle than it looks!)

30. A topological space K is *locally compact* if every point of K has a compact neighbourhood. Show that if K is Hausdorff or regular, then every point has a local base consisting of closed, compact sets, *i.e.,* for every neighbourhood U of a point x there is a closed compact neighbourhood V of x such that $V \subset U$. Show further that the one-point compactification of a Hausdorff space K is Hausdorff if and only if K is locally compact.

31. Prove the Baire Category Theorem for a locally compact Hausdorff space: let X be such a space and let (U_n) be a sequence of dense open sets in X. Then $\bigcap U_n$ is dense in X.

32. Let $y_1, y_2, \dots$ be a linearly independent sequence in a Banach space X, and let Y denote its linear span. Assume that Z is a subspace of X such that $Z + Y = X$. Show that $\overline{Z}$ is finite-codimensional in X.

33. Is there a continuous surjective map $f\colon \mathbb{R} \to \ell_2$?

34. Write $[0,1]$ as $A \cup B$ where A is a null set and B is meagre in $[0,1]$.

35. Given two non-equivalent norms on a vector space X, show that there is linear functional on X which is continuous with respect to one of the norms and discontinuous with respect to the other.

36. Assume that X is a closed subspace of $L_2[0,1]$ such that every element of X is also in $L_\infty[0,1]$. Show that X is finite-dimensional.

37. Let $X = \mathbb{R}$ with the half-open interval topology. Show that X is normal but $X \times X$ is not.

38. Let K be a compact Hausdorff space. Show that $C(K)$ is finite-dimensional if and only if K is finite.

39. Let K be a compact Hausdorff space and consider the *real* space $C(K)$. Show that every element of the dual space $C(K)^*$ is the difference of two positive linear functionals.

40. For a scalar-valued continuous function f on the topological space K we say f *has compact support* if the closure of $\{x \in K : f(x) \neq 0\}$ is compact. The collection of such functions is denoted by $C_C(K)$. We say f *vanishes at infinity* if for all $\varepsilon > 0$ the set $\{x \in K : |f(x)| \geqslant \varepsilon\}$ is compact. The collection of all such functions is denoted by $C_0(K)$. Show that $C_0(K)$ is a closed subspace of $\ell_\infty(K)$ in the uniform norm, and hence it is a Banach space. Show further that if K is Hausdorff, then $C_C(K)$ is a dense subspace of $C_0(K)$.

41. Prove the Stone-Weierstrass theorem for locally compact spaces: if K is a locally compact Hausdorff space, A is a subalgebra of $C_0(K)$ strongly separating the points of K (and closed under complex conjugation in the complex case), then A is dense in $C_0(K)$.

42. Suppose that the family $\mathcal{C}$ of clopen subsets of K is a base for the topology. Show that the indicator functions of members of $\mathcal{C}$ have dense linear span in $C(K)$. [*A set is clopen if it is both open and closed.*]

43. Show that ℓ_4 is not isomorphic to a subspace of ℓ_2.

44. Let Y be a closed subspace of a Hilbert space H. Using bases, show that if Y is separable, then $H = Y \oplus Y^{\perp}$.

45. Let (e_n) be an orthonormal sequence in an inner product space. Assume that for every x there is equality in Bessel's inequality, *i.e.*, $\sum |\langle x, e_n\rangle|^2 = \|x\|^2$. Deduce that (e_n) is an orthonormal basis.

46. Let $\big(x^{(n)}\big)$ be an orthonormal sequence in ℓ_2, where $x^{(n)} = \big(x_i^{(n)}\big)_{i=1}^{\infty}$. Show that $x_i^{(n)} \to 0$ as $n \to \infty$ for each $i \in \mathbb{N}$.

47. Let $T \in \mathcal{F}(X,Y)$ with $\mathrm{rk}T = n$. Show that there exist $f_1, \dots, f_n \in X^*$ and $y_1, \dots, y_n \in Y$ such that $T = \sum_{i=1}^{n} y_i \otimes f_i$.

48. Let T be a bounded linear operator on a Hilbert space. Show that one of the maps T, T^*, TT^*, T^*T is compact, then they all are.

49. Let $T \in \mathcal{B}(X,Y)$ be a non-compact operator. Prove that there exist a bounded sequence (x_n) in X and an $\varepsilon > 0$ such that $\|Tx_m - Tx_n\| > \varepsilon$ for all $m \neq n$.

50. Show that every operator $\ell_q \to \ell_p$ and $\mathrm{c}_0 \to \ell_p$ is compact, where $1 \leqslant p < q < \infty$.

51. Let $K \subset \mathbb{C}$ be non-empty, compact. Show that K is the spectrum of some operator on ℓ_2.

52. Let T be a compact operator on a Banach space X. Show that there exist closed subspaces Y, Z, E, F of X such that $X = Y \oplus E = Z \oplus F$, $\dim E = \dim F < \infty$ and $I - T$ restricted to Y is an isomorphism from Y onto Z.

53. Using the previous question, show that the spectrum of T is either finite, or consists of zero and a sequence converging to zero. Moreover, if $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then λ is an eigenvalue of T with finite-dimensional eigenspace.

54. Construct a linear map T on a (non-zero, complex) vector space V such that $\lambda I - T$ is invertible for every $\lambda \in \mathbb{C}$.

55. Let $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \mathbb{R}$. Must T be hermitian?

56. Let X be a Banach space. Let (x_n) and (f_n) be bounded sequences in X and X^*, respectively, such that $f_i(x_j) = \delta_{ij}$ for all i, j. Let (y_n) be another sequence in X with $\varepsilon = \sum \|x_n - y_n\| < \infty$. Prove that if ε is sufficiently small then there is an invertible operator $T\colon X \to X$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

57$^+$. Let S be the set of all $(x_n) \in \ell_\infty$ such that $\{x_n : n \in \mathbb{N}\}$ is finite. Show that if $\mathcal{F} \subset \ell_\infty^*$ is pointwise bounded on S, then it is uniformly bounded.

58. Let X, Y be Banach spaces and $S, T \in \mathcal{B}(X, Y)$. Assume that T is invertible and for some $\delta < 1$ we have $\|Sx - Tx\| \leqslant \delta \|Tx\|$ for all $x \in X$. Prove that S is invertible.

59. Let K be a compact Hausdorff space. Show that the open F_σ subsets of K form a base of the topology. (Recall that a set is F_σ if it is a countable union of closed sets.) Show further that for every open set $U \subset K$ there is an open F_σ set $V \subset U$ such that $\overline{V} = \overline{U}$.