Unordered pairs in the set theory of Bourbaki 1949

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Abstract. Working informally in ZF, we build a pair of supertransitive models of Z, of which pair the union is shown to be a supertransitive model of Bourbaki's 1949 system for set theory in which some unordered pair fails to exist even though ordered pairs are available.

0: Introduction and notation

In this note we follow French usage whereby the *pair* of a and b is the unordered pair $\{a,\, b\}$, while their *couple* is the ordered pair $(a,\, b)$.

Some authors, including Bourbaki in their earlier publications, treat the operation of coupling two sets as a primitive, for which axioms such as the following would be given:

$$\forall x, y \exists z\; z = (x, y)$$
$$\forall a, b, c, d\, [(a,\, b) = (c,\, d) \implies (a = c \;\&\; b = d)]$$

0·0 Definition The Cartesian product, $x \times y$, is the class $\{(a,\, b) \mid a \in x \text{ and } b \in y\}$.

0·1 Definition Let CPr be the statement that for all sets x and y, $x \times y$ is a set.

In many set theories, coupling, rather than being a primitive, may be introduced by definition; of the many possible definitions, we distinguish that of Kuratowski by a special notation:

0·2 Definition $\langle a, b\rangle_K = \{\{a\},\, \{a,\, b\}\}$.

Here the singleton $\{a\}$ is $\{a,\, a\}$. The Kuratowskian definition of Cartesian product will then be:

0·3 Definition $x \times_K y \;=_{\mathrm{df}}\; \{\langle a, b\rangle_K \mid a \in x \;\&\; b \in y\}$.

For the early history of these concepts, see Kanamori [K1], and for a recent proposal for the definition of ordered pair, see [Sc, M].

0·4 Definition Let AxSing be the assertion that for each set x, its singleton $\{x\}$ is a set. Let AxPair be the assertion that for all sets x and y, their pair $\{x,\, y\}$ is a set.

In the address of Nicolas Bourbaki [Bou 49], read by André Weil to the Association of Symbolic Logic at their meeting in Columbus, Ohio, on December 31, 1948, a system of set theory, which we shall call Bou49, was presented. This system was roughly the system Z which originates in a 1908 paper of Zermelo, but among the differences are these: Zermelo included AxPair among his axioms, whereas Bourbaki did not; Bourbaki treated coupling as a primitive, with axioms to match, whereas Zermelo did not mention couples explicitly, it being left to Wiener, Hausdorff, and finally Kuratowski to suggest ways of introducing them by definition.

AxSing is derivable in Bou49 using the axiom of power set and the scheme of separation. It might be asked whether AxPair can be derived in Bou49; the purpose of this note is to show that it cannot.

Theorem *There is a model of* Bou49 *in which* AxPair *is false.*

The model in question will be the union of two carefully chosen models of the set theory Z.

We now list the axioms of Bou49 and Z, and shall then outline our method for proving the theorem. In our presentation of Bou49, we reproduce exactly the notation of [Bou49], and passages in *slanted type* are (shortened versions of passages) from that paper. In our presentation of Z, $\mathcal{P}(x)$ is, as usual, the class $\{y \mid y \subseteq x\}$ of all subsets of x, and $\bigcup x$ the class $\{z \mid \exists y (y \in x \;\&\; z \in y)\}$.

The system Bou49

First, an axiom and an axiom scheme for equality:

E(1) $\forall x(x = x)$
S(1) $[(x = y) \longrightarrow (R\{x, y, x\} \longrightarrow R\{x, y, y\})]$

then three axioms for coupling, in which initially the sign $|$ is used:

E(2) $\forall x \forall y \exists z(z|xy)$
E(3) $\forall x \forall y \forall z \forall t[(z|xy)$ *and* $t|xy \longrightarrow (z = t)]$

Henceforth write "$z = (x, y)$" for "$z|xy$".

E(4) $\forall x \forall x' \forall y \forall y'[((x, y) = (x', y')) \longrightarrow (x = x'$ *and* $y = y')]$

Write $x \subset y$ for $\forall z(z \epsilon x \longrightarrow z \epsilon y)$.

Then the axiom of extensionality, given as:

E(5) $\forall x \forall y[(x \subset y$ *and* $y \subset x) \longrightarrow (x = y)]$

Then the scheme of separation:

S(2) $\forall \cdot \cdot \cdot \forall_{set} E \exists X \forall x[(x \epsilon X) \leftrightarrow (x \epsilon E$ *and* $R)]$

where X is an argument which does not occur in the relation R.

The power set axiom:

E(6) $\forall_{set} X \exists Y \forall Z[(Z \subset X) \longleftrightarrow (Z \epsilon Y)]$

The existence of the cartesian product $X \times Y$ of two sets X and Y:

E(7) $\forall_{set} X \forall_{set} Y \exists W \forall z[\exists x \exists y(x \epsilon X$ *and* $y \epsilon Y$ *and* $z = (x, y)) \leftrightarrow (z \epsilon W)]$
E(8) *Zermelo's axiom* (of choice, presumably, though it is not given explicitly).

Finally a verbal formulation of an axiom of infinity: the formulation is correct, given Bourbaki's convention (on which Rosser comments in his review [Ro]) that a set is never empty.

The system Z

0·5 Zermelo in his 1908 paper [Z] gives these axioms:

I: the axiom of extensionality,
II: the axiom of null set, AxSing and Axpair: $\varnothing \in V$, $\{a\} \in V$, $\{a, b\} \in V$,
III: the scheme of separation for "definite" properties,
IV: the power set axiom: $\mathcal{P}(x) \in V$,
V: the axiom of union: $\bigcup x \in V$,
VI: an axiom of choice: every family of pairwise disjoint non-empty sets has a selector,
VII: an axiom of infinity.

There was no axiom of foundation, nor does it seem that Zermelo then had the concept of a couple. We write ZC for the system obtained from **I** – **VII** by adding the axiom of foundation for sets, replacing axiom **VI** by the principle WO, proved by Zermelo to be equivalent, that any set admits a well-ordering, and, following Weyl and Skolem, interpreting the scheme of Separation to mean

$$a \cap \{x \mid \Phi(x, a, b)\} \in V$$

for any parameter b and formula Φ of the usual first-order language of set theory. We write Z for ZC with WO omitted.

0·6 Remark Bou49 may be seen as a subsystem of ZC: for Axiom **II** supports the Kuratowski definition of couple, and with that definition CPr becomes provable, since $\vdash_{\mathsf{Z}} x \times_K y \subseteq \mathcal{P}\mathcal{P}\bigcup\{x, y\}$; so that ZC interprets Bou49 + AxPair + Foundation + Union.

0·7 Remark The Weyl–Skolem interpretation of **III** is often called the scheme of *full* separation. Richard Pettigrew, in his talk at the Zermelo Centenary Conference held in Brussels in October 2008, speculated that Zermelo's own preference in 1908 might have been for a weaker interpretation, possibly the scheme of $\Delta_0^{\mathcal{P}}$ separation shown in theorem 6·9 of [M1] to be derivable in the system M_0 of that paper; if so, Zermelo's intended system of 1908 would be M_0 together with the axioms of infinity and choice.

Kanamori in his study [K2, p. 520] of Zermelo notes that by 1929 Zermelo's thoughts were running on a much stronger interpretation, and that Zermelo believed then that in 1908 no very specific formulation would have been possible as there was at the time no generally accepted notion of logical system to which appeal might have been made.

Our proof strategy

We shall work in ZF. In our constructions of various models of subsystems of ZC, we make no use of WO, but as the models we construct are supertransitive, WO will automatically hold in them provided it holds in the universe in which we are working.

We review some familiar definitions.

0·8 Definition $\varrho(x)$ is the set-theoretical rank of x. **HF** is the class of hereditarily finite sets. $\mathcal{S}(x)$ is the class of finite subsets of x.

0·9 Definition $V_\lambda =_{\mathrm{df}} \{x \mid \varrho(x) < \lambda\}$. Thus $V_0 = \emptyset$ and for $n \in \omega$, $V_{n+1} = \mathcal{P}(V_n)$.

0·10 Definition A set or class, $\mathbf{N}$, is said to be *transitive* if $x \in y \in \mathbf{N} \implies x \in \mathbf{N}$. $\mathbf{N}$ is said to be *supertransitive* if it is transitive and, further, $x \subseteq y \in \mathbf{N} \implies x \in \mathbf{N}$.

0·11 Remark It has long been known that $V_{\omega+\omega}$ is a supertransitive model of Z. $V_{\omega+\omega}$ and the notion of a transitive set seem to have been first defined by Bernays in Part VI, published in 1948, of his sequence of papers developing a formal system of set theory: see [Ber] and Kanamori [K3]. On the other hand, Skolem in part 4 of his 1922 paper "Einige Bemerkungen" [Sk, p.225] defines sets of the erste Stufe and zweite Stufe; working in ZF, the erste Stufe would be the sets in V_ω, and the union of the first and second Stufen would be $V_{\omega+\omega}$; Skolem gives arguments that would have shown $V_{\omega+\omega}$ to be a model of Z had that notion been available to him.

Our strategy for the rest of the paper is this: in section 1, we outline a general method for constructing models of subsystems of Z. The proofs, being straightforward, are omitted.

In section 2, we apply the method of section 1 to construct a certain model of Z which contains infinite sets but not the von Neumann ordinal ω: this construction serves to correct the proof and strengthen the statement of theorem 2.1.3 of Gandy [G], whose own proof, as shown in [M3], is incorrect.

Then in section 3, we vary slightly the construction of section 2 to obtain two supertransitive models of Z, neither included in the other—in [M2, Section 13] it is shown that the union of any such pair will be a model of Z shorn of the axiom of pairing—but both included in $V_{\omega+\omega}$. This last condition leads to a proof in section 4, by what might be called an amalgamation argument, that the said union admits an implementation of coupling, thus proving the main result of the paper.

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1: Some general constructions leading to models of subsystems of Z

1·0 PROPOSITION *Let* $(G_n)_{n\in\omega}$ *be a sequence of sets, and* G *their union.*

(1·0·0) *If* G *is non-empty,* G *will model the axiom of null set.*

(1·0·1) *If* $\forall n\ G_n \cup [G_n]^2 \subseteq G_{n+1}$, G *will model* AxPair.

(1·0·2) *If* $\forall n\ \mathcal{S}(G_n) \subseteq G_{n+1}$, *then* $\forall n\ V_n \subseteq G_n$ *and* $\mathbf{HF} \subseteq G$.

(1·0·3) *If* G_0 *is transitive, each* $\mathcal{S}(G_n) \subseteq G_{n+1}$, *and each* $G_{n+1} \subseteq G_n \cup \mathcal{P}(G_n) \cup \mathcal{P}(\mathbf{HF})$, G *will be transitive, and will therefore model the axiom of extensionality and the scheme of foundation for all classes.*

1·1 PROPOSITION *Suppose now that* $(g_n)_n$ *is a sequence of subsets of* $\mathbf{HF}$, *and that* g *is a transitive subset of* $\mathbf{HF}$. *Let* $G_0 =_{\mathrm{df}} g$; $G_{n+1} =_{\mathrm{df}} \mathcal{P}(G_n) \cup G_n \cup g_n$; $G =_{\mathrm{df}} \bigcup_{n\in\omega} G_n$.

(1·1·0) $x \subseteq G_n \implies x \in G$;

(1·1·1) G *is supertransitive;*

(1·1·2) G *models the scheme of full separation;*

(1·1·3) *if* $x \in G_n$, *then for some* k, $x \subseteq G_k$;

(1·1·4) G *models the axiom of power set;*

(1·1·5) *if* $\omega \subseteq G_0$, G *will model an axiom of infinity;*

(1·1·6) $G = \bigcup_n \mathcal{P}(G_n)$.

The axiom of union will hold if we impose a further condition:

1·2 PROPOSITION *Suppose in addition that for each* n *there are* k *and* ℓ *such that* $\bigcup g_n \subseteq V_k \cup g_\ell$.

(1·2·0) *Each* $\bigcup G_n$ *is a subset of some* G_m *and thus is in* G *by supertransitivity.*

(1·2·1) G *models the axiom of union.*

Thus under appropriate conditions G models Z.

2: A model of Z of which ω is not a member

2·0 DEFINITION $\iota^0(x) =_{\mathrm{df}} x$; $\iota^{n+1}(x) =_{\mathrm{df}} \{\iota^n(x)\}$.

2·1 LEMMA *If* $\bigcup x \subseteq y$, $\varrho(x) \leqslant \varrho(y) + 1$.

2·2 LEMMA $\varrho(\iota(x)) = \varrho(x) + 1$.

Suppose that $(t_n)_n$ is a sequence of sets, with $\bigcup t_n \subseteq t_{n+1}$ for each n.

2·3 LEMMA $\bigcup\{a\} = a$.

2·4 LEMMA *If no* t_n *is a singleton or empty, and* $\iota^n(t_\ell) = \iota^m(t_k)$ *then* $n = m$ *and* $t_\ell = t_k$.

2·5 DEFINITION $h_n =_{\mathrm{df}} \{\iota^k(t_{n+k}) \mid k \in \omega\}$.

2·6 EXAMPLE $h_0 = \{t_0, \{t_1\}, \{\{t_2\}\}, \ldots\}$, $h_1 = \{t_1, \{t_2\}, \{\{t_3\}\}, \ldots\}$.

2·7 LEMMA $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

2·8 LEMMA $\bigcup h_k = t_k \cup h_{k+1}$; $\bigcup^2 h_k = \bigcup h_{k+1}$; $\bigcup^3 h_k = \bigcup h_{k+2}$; $\bigcup^{n+1} h_k = \bigcup h_{k+n} = t_{k+n} \cup h_{k+n+1}$.

Proof : first part by inspection; that then gives

$$\bigcup{}^2 h_k = \bigcup t_k \cup \bigcup h_{k+1} = \bigcup t_k \cup t_{k+1} \cup h_{k+2} = t_{k+1} \cup h_{k+2} = \bigcup h_{k+1}.$$

The rest now follows by iteration. ⊣ (2·8)

2·9 PROPOSITION *Each* $\bigcup^m h_\ell$ *is an infinite set.*

2·10 LEMMA *Suppose that each* t_n *is finite. Then*

(i) for any m *and* ℓ, $\bigcup^m h_\ell$ *contains only finitely many non-singletons;*

(ii) if x *contains only finitely many singletons, then for any* m *and* ℓ, $x \cap \bigcup^m h_\ell$ *is finite.*

2·11 Definition Let H_0 be any transitive set; $H_{n+1} =_{\rm df} \mathcal{P}(H_n) \cup H_n \cup h_n$; $H =_{\rm df} \bigcup_{n\in\omega} H_n$.

2·12 Remark Thus each h_n, being a subset of H_{n+1} is in H_{n+2} and therefore in H.

2·13 Definition Let $z_0 = \varnothing$, $z_1 = \bigcup h_0$, $z_2 = \bigcup^2 h_0 \cup \bigcup^2 h_1$, …, $z_k = \bigcup\{\bigcup^k h_m \mid_m m < k\}$.

2·14 Lemma $\bigcup H_{n+1} = H_n \cup \bigcup H_n \cup \bigcup h_n$.

2·15 Lemma *(i)* $z_{k+1} = \bigcup z_k \cup \bigcup^{k+1} h_k$; *(ii)* $\bigcup z_k \subseteq z_{k+1}$.

2·16 Lemma $\bigcup^n H_n = H_0 \cup \bigcup\{z_k \mid k \leqslant n\}$.

Proof by induction on n: For $n = 0$, both sides equal H_0; for the inductive step use the lemmata and the transitivity of H_0. ⊣ (2·16)

2·17 Proposition *Suppose that each t_n is finite. Then*

(i) If $x \in H$, then $\exists n \bigcup^n x \setminus H_0$ contains only finitely many non-singletons.

(ii) If λ is a limit ordinal in H then $\lambda \subseteq H_0$.

Proof : (i) True if $x \in H_0$; if $x \in H_{n+1}$, then $x \subseteq \bigcup H_{n+1}$; so $\bigcup^n x \subseteq \bigcup^{n+1} H_{n+1}$; the last lemma now completes the proof since each z_k only contains finitely many non-singletons.

(ii) $\bigcup \lambda = \lambda$; λ contains one singleton; so $\lambda \smallsetminus H_0$ is finite. So there is $\zeta < \lambda$ such that $\lambda \smallsetminus \zeta \subseteq H_0$; as H_0 is transitive, $\lambda \subseteq H_0$. ⊣ (2·17)

2·18 Theorem *If $H_0 \in \mathbf{HF}$, and each $t_n \in \mathbf{HF}$, H will be a supertransitive model of* Z *of which ω is not a member.*

Proof : For the axiom of infinity, note that h_0 is necessarily an infinite set. ⊣ (2·18)

2·19 Remark Model $\mathbf{M}_{13,\omega}$ of [M2] gives another such. TCo is true in it, but not in the above H, because $\omega = \mathrm{tcl}(h_0)$.

3: Incomparable supertransitive models of Z included in $V_{\omega+\omega}$.

3·0 Lemma $\bigcup V_{n+1} = V_n$; *if V_k is a singleton then $k = 1$.*

3·1 Lemma *If $2 \leqslant \min\{\ell, m\}$ and $\iota^n(V_\ell) = \iota^m(V_k)$ then $n = m$ and $\ell = k$.*

3·2 Definition Let D and E be two almost disjoint infinite subsets of $\omega \smallsetminus 2$, and let $(d_n)_n$, $(e_n)_n$ be their respective monotonic enumerations.

3·3 Definition For each $k \in \omega$, set $p_k =_{\rm df} \{\iota^n(V_{d_{k+n}}) \mid n < \omega\}$. and $q_k =_{\rm df} \{\iota^n(V_{e_{k+n}}) \mid n < \omega\}$.

3·4 Example $p_0 = \{V_{d_0}, \{V_{d_1}\}, \{\{V_{d_2}\}\}, \ldots\}$; $p_1 = \{V_{d_1}, \{V_{d_2}\}, \{\{V_{d_3}\}\}, \ldots\}$;
$p_2 = \{V_{d_2}, \{V_{d_3}\}, \{\{V_{d_4}\}\}, \ldots\}$

3·5 Lemma $p_{n+1} \subseteq \bigcup^n p_0$.

3·6 Lemma $\bigcup p_k = V_{d_k} \cup p_{k+1}$; $\bigcup^2 p_k = \bigcup p_{k+1}$; $\bigcup^3 p_k = \bigcup p_{k+2}$; $\bigcup^{n+1} p_k = \bigcup p_{k+n} = V_{d_{k+n}} \cup p_{k+n+1}$.

Proof : by Lemma 2·8. ⊣ (3·6)

3·7 Lemma *For any n, m, k, ℓ, $\bigcup^n p_k \cap \bigcup^m q_\ell$ is finite and $\bigcup^n p_k \cap \omega$ is finite.*

3·8 Definition Let $P_0 = Q_0 = \omega$; $P_{n+1} =_{\rm df} \mathcal{P}(P_n) \cup P_n \cup p_n$; $\mathbf{P} =_{\rm df} \bigcup_{n\in\omega} P_n$.
$Q_{n+1} =_{\rm df} \mathcal{P}(Q_n) \cup Q_n \cup q_n$; $\mathbf{Q} =_{\rm df} \bigcup_{n\in\omega} Q_n$.

3·9 Lemma $\bigcup^n Q_n = Q_0 \cup [\bigcup q_0] \cup [\bigcup^2 q_0 \cup \bigcup^2 q_1] \cup \ldots \cup [\bigcup^n q_0 \cup \bigcup^n q_1 \cup \ldots \cup \bigcup^n q_{n-1}]$.

Proof : by Lemma 2·16 ⊣ (3·9)

3·10 Proposition $p_0 \notin \mathbf{Q}$; $q_0 \in \mathbf{Q}$.

Proof : Plainly $p_0 \notin Q_0$. If $p_0 \in Q_{n+1}$; then $p_0 \subseteq \bigcup Q_{n+1}$; so $\bigcup^n p_0 \subseteq \bigcup^{n+1} Q_{n+1}$; Lemmata 3·9 and 3·7 now yield a contradiction.

$q_0 \in \mathbf{Q}$ by Remark 2·12. ⊣ (3·10)

3·11 Corollary TCo *fails in* $\mathbf{Q}$.

Proof : $p_0 \subseteq \mathbf{HF}$, so $\mathbf{HF} \notin \mathbf{Q}$ as $\mathbf{Q}$ is supertransitive. But $\mathbf{HF} = \mathrm{tcl}\ q_0$ and $q_0 \in \mathbf{Q}$. ⊣ (3·11)

3·12 Theorem $\mathbf{P}$ *and* $\mathbf{Q}$ *are two supertransitive models of* Z, *each of rank $\omega + \omega$, neither a subset of the other.*

4: A model omitting some unordered pair

Let T^- be the theory $\mathsf{Bou49}$ + $\mathsf{Foundation}$ + $\exists x \exists y\, \{x, y\} \notin V$ + $\forall x \bigcup x \in V$. We use the models $\mathbf{P}$ and $\mathbf{Q}$ to build a model for T^- by setting $\mathbf{R} =_{\text{df}} \mathbf{P} \cap \mathbf{Q}$ and $\mathbf{M} =_{\text{df}} \mathbf{P} \cup \mathbf{Q}$.

$\mathbf{R}$ will be a supertransitive model of ZC, being the intersection of two such, and therefore if x and y are in $\mathbf{R}$, so are $x \times_K y$ and $x \cup y$. $\mathbf{M}$ will be our desired supertransitive model of T^-. In verifying that, our chief tasks will be to define couples in $\mathbf{M}$ and to show that Cartesian products exist in that model.

$\mathbf{R} \supseteq V_\omega \notin \mathbf{M}$. $\mathbf{R}$ is of cardinal $\beth_\omega$, since it includes $\omega \cup \mathcal{P}(\omega) \cup \mathcal{PP}(\omega) \cup \ldots$. For $x \in \mathbf{R}$, $\{g_0, x\}$ will be in $\mathbf{P} \smallsetminus \mathbf{R}$ and $\{h_0, x\}$ will be in $\mathbf{Q} \smallsetminus \mathbf{R}$, so that $\mathbf{P} \smallsetminus \mathbf{R}$ and $\mathbf{Q} \smallsetminus \mathbf{R}$ will also be of cardinal $\beth_\omega$.

Let $r = \{0, 3\}$, $p = \{1, 3\}$, $q = \{2, 3\}$. Let $f : \mathbf{R} \xrightarrow{1-1} \mathbf{R} \times_K \{r\}$, $g : \mathbf{P} \smallsetminus \mathbf{R} \xrightarrow{1-1} \mathbf{R} \times_K \{p\}$ and $h : \mathbf{Q} \smallsetminus \mathbf{R} \xrightarrow{1-1} \mathbf{R} \times_K \{q\}$; let f_x be f if $x \in \mathbf{R}$, g if $x \in \mathbf{P} \smallsetminus \mathbf{R}$ and h if $x \in \mathbf{Q} \smallsetminus \mathbf{R}$; we define the couple (x, y) of x and y to be $\langle f_x(x), f_y(y) \rangle_K$, and verify, easily, that if $(x, y) = (u, v)$ then $x = u$ and $y = v$.

A little care is needed in the selection of the functions f, g and h, to ensure that cartesian products exist. It will suffice to arrange that if a is in $\mathbf{M}$ then each of $f``(a \cap \mathbf{R})$, $g``(a \cap (\mathbf{P} \smallsetminus \mathbf{R}))$ and $h``(a \cap (\mathbf{Q} \smallsetminus \mathbf{R}))$ is in $\mathbf{R}$, for then $a \times b$ will be the union of nine sets in $\mathbf{R}$, each of the form $s \times_K t$ for some s and t in $\mathbf{R}$.

To do that, define

$$\mathbf{I}^{(0)} = \omega; \qquad \mathbf{I}^{(n+1)} = \mathcal{P}^{n+1}(\omega) \smallsetminus \bigcup_{k \leqslant n} \mathcal{P}^k(\omega);$$

and for $\mathbf{X} = \mathbf{P}$, $\mathbf{Q}$ or $\mathbf{R}$, and $n \in \omega$

$$\mathbf{X}^{(0)} = V_\omega; \qquad \mathbf{X}^{(n+1)} = \mathbf{X} \cap (V_{\omega+n+1} \smallsetminus V_{\omega+n}).$$

We seek f, g and h such that for each n

$$f \restriction \mathbf{R}^{(n)} : \mathbf{R}^{(n)} \xrightarrow{1-1} \mathbf{I}^{(n)} \times_K \{r\};$$
$$g \restriction \mathbf{P}^{(n+1)} : \mathbf{P}^{(n+1)} \xrightarrow{1-1} \mathbf{I}^{(n+1)} \times_K \{p\};$$
$$h \restriction \mathbf{Q}^{(n+1)} : \mathbf{Q}^{(n+1)} \xrightarrow{1-1} \mathbf{I}^{(n+1)} \times_K \{q\};$$

which is *prima facie* possible as for each n the cardinality of $\mathbf{I}^{(n)}$ is $\beth_n$, the cardinal of $V_{\omega+n}$.

To find such f, g and h without appeal to an axiom of choice, start from the injection $a_0 : V_\omega \xrightarrow{1-1} \omega$ obtainable from Ackermann's relation E such that $(\omega, E) \cong (V_\omega, \in)$. Define successively

$$a_1(x) = \begin{cases} a_0(x) & \text{if } x \in V_\omega \\ a_0``x & \text{if } x \in V_{\omega+1} \smallsetminus V_\omega \end{cases}, \ldots\ a_{n+1}(x) = \begin{cases} a_n(x) & \text{if } x \in V_{\omega+n} \\ a_n``x & \text{if } x \in V_{\omega+n+1} \smallsetminus V_{\omega+n} \end{cases}, \ldots$$

To check that each a_{n+1} maps $V_{\omega+n+1} \smallsetminus V_{\omega+n}$ into $\mathbf{I}^{(n+1)}$, note first that if $x \in V_{\omega+1} \smallsetminus V_\omega$, then $a_1(x)$ is an infinite set and therefore in $\mathbf{I}^{(1)}$; and then remark that if $x \in V_{\omega+n+2} \smallsetminus V_{\omega+n+1}$, then some $y \in x$ is in $V_{\omega+n+1} \smallsetminus V_{\omega+n}$, so that if, inductively, $a_{n+1}(y) \in \mathbf{I}^{(n+1)}$, then $a_{n+2}(x) \in \mathbf{I}^{(n+2)}$.

The desired f, g and h may now be found by composing restrictions of these injections with the natural bijections between $\mathbf{I}^{(n)}$ and $\mathbf{I}^{(n)} \times_K \{s\}$ for $n \in \omega$ and $s \in \{p, q, r\}$, as appropriate.

4·0 THEOREM $\mathbf{M}$ *is a model of* T^-; *further, there will be a failure in this model of the principle that the union of two sets is a set.*

Proof: $\mathbf{M}$ is supertransitive, and hence absolute for most of the set-theoretical concepts used in the axioms; therefore it will be a model of the axioms of extensionality, union , power set, foundation, the full scheme of separation, TCo (whence also the full scheme of foundation), and WO.

Pairing fails, for if we put $c = h_0$ and $d = g_0$, then $c \in \mathbf{P} \smallsetminus \mathbf{Q}$ and $d \in \mathbf{Q} \smallsetminus \mathbf{P}$; hence $\{c, d\} \notin \mathbf{M}$. As AxSing holds, $\{c\}$ and $\{d\}$ are in the model, though their union $\{c\} \cup \{d\}$, which equals $\{c, d\}$, is not.

We have shown above that the axioms for couples will hold: and our careful choice of f, g, and h will ensure that the corresponding Cartesian product of two sets exists. ⊣ (4·0)

4·1 REMARK We have worked in ZF, but our argument could be re-arranged to show in arithmetic that if Consis(Z) then Consis(T^-), either directly or by first appealing to the result, proved in section 5 of [M1], that if Consis(Z) then Consis($\mathsf{Z} + \mathsf{KP} + V = L$) and then adapting our construction, taking D and E to be the easily definable sets $\{2n \mid n \in \omega\}$ and $\{2n+1 \mid n \in \omega\}$, to show that T^- can be interpreted in that latter theory: the global form of choice that is provided by $\mathsf{KP} + V = L$ could then be used to choose the various $f \restriction \mathbf{R}^{(n)}$, $g \restriction \mathbf{P}^{(n+1)}$ and $h \restriction \mathbf{Q}^{(n+1)}$.

4·2 Remark The failure in $\mathbf{M}$ of the principle that the union of two sets is a set confirms the misgivings expressed by Rosser in his review [Ro], where he writes *"As far as the reviewer can see, Bourbaki's axioms permit one to form the union of two sets only when both are subsets of some known set."*

4·3 Remark The above model refutes the contention at the end of [Bou 49] that Bou49 suffices for all the mathematics "of the present day"—even in 1949 one would have wished to prove that for each c and d, the set $\{c, d\}$ exists.

4·4 Remark Essentially the same system as Bou49, though in visually different notation, is used in the text *Algèbre* by J. Lelong-Ferrand, J. and J.-M. Arnaudiès [L-F, A]. On page 18, they discuss the formation of the union of a family of sets but only when the sets in question are subsets of a previously known set; on page 10, they state that the existence of the union of two sets follows from "an axiom of set theory"; but the model $\mathbf{M}$ shows that it does not follow from the axioms that they state.

4·5 Remark That the pairing axiom is not redundant in Z was first shown by Boffa [Bof]. Perhaps as a result of Rosser's warning in [Ro], the pairing axiom is included in the system, which we shall call Bou54, of Bourbaki's book [Bou 54], but there it is indeed redundant: Wyler [W] noted that it could be weakened to AxSing and Sonner [So] showed that it could be dropped altogether. There is no contradiction with the results of the present paper, for Bou54 includes a version of the scheme of replacement, and once one has replacement, one can use the existence of one two-element set (for example the power set of the power set of the empty set) to establish the existence of all other two-element sets.

Coupling is retained as a primitive in Bou54 but dropped in favour of Kuratowski's definition in the 1970 edition of that text.

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