Notes on subsequent developments

I have not in these notes sought to be exhaustive, but have largely confined myself to mentioning general areas of development, with suggestions for further reading.

A reference such as Lévy [6] is to paper [6] in the original list of references, and one such as Jensen [a] to paper a by Jensen in the supplementary bibliography. Accounts of many of the results marked [*] in the original text have appeared in the Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society held at the University of California, Los Angeles California from July 10th to August 5th, 1967, published as Parts I and II of Volume XIII of the series of Proceedings of Symposia of the Society under the title "Axiomatic Set Theory". The two parts, the first edited by Scott and the second by Jech, are called here [*,I] and [*,II] respectively.

Of the original list of references, [10] has been published as Solovay [b]; [11] is to be found in the Symposium Papers commemorating the sixtieth birthday of Kurt Gödel, edited by J.J.Bulloff, T.C.Holyoke and S.W.Hahn, and published by the Springer-Verlag in 1969; and [15] will now appear as Kanamori, Reinhardt and Solovay [a]. [16] has not been published, but much of the material therein may be found in Drake [a].

¶1 Many results in this paragraph are proved in Jech [a].

D 1104 The properties "generic over L" and "random over L" are Π^1_2.

T 1106 For a refinement of this theorem, see Jensen [a] who exhibits a notion of forcing in L such that the property of being Jensen-generic over L is Π^1_2, and if a is Jensen generic over L, a is the only such real in $L[a]$, and is accordingly Δ^1_3 in $L[a]$. In contrast, if there is one real generic over L there is a perfect set of them.

T 1108 is false as stated: add the condition that for all $f:\omega\to 2$ $\bigcap_n T_{f\restriction n}$ is a singleton.

D 1111 Silver reals are a special case of reals studied by Grigorieff [a], who amends D 1110 by fixing a non-principal ideal I on ω, and requiring that $A\cup B\in I$. He shows that the resulting generic real is of minimal degree iff I is the ideal dual to a Ramsey ultrafilter. Silver's extension is the result of first adding a Ramsey ultrafilter generically (without adding new reals: the partial ordering used is that of the infinite subsets of ω modulo finite differences) and then using it to make a Grigorieff extension.

T 1113 follows easily from Grigorieff's work: if a is Silver over L then $\{x \mid x\in L\cap S(\omega)\wedge x\cap a\notin L\}$ is a Ramsey ultrafilter F such that $a\notin L[F]$, $F\notin L$, $S(\omega)\cap L[F]\subseteq L$ and a is Grigorieff with respect to F over $L[F]$.

D 1117 In Mathias [a] a notion of generic real is studied which has the feature that if x is generic then every infinite subset of x is also generic; moreover two disjoint infinite subsets of x will always be of incomparable L-degrees. In this connection, see Solovay [a]

P 1128 Yes (Kunen; Baumgartner)

T 1130 See Jensen and Solovay [a] for applications of Solovay's trick, which has also been developed by Jensen in an important paper [b] from which we quote Theorem 1:

Let $\langle M,A\rangle$ be a transitive model of $ZF + GCH + \neg 0^\#$, (the last being the statement that $0^\#$ does not exist (cf D 2037)) where $A\subseteq On\cap M$. Then there is an $\langle M,A\rangle$ definable class of conditions $\mathbb{P}$ such that if G is $\mathbb{P}$-generic over $\langle M,A\rangle$ and $N = M[G]$, then (a) N is a model of $ZF + GCH + \neg 0^\#$; (b) cardinals and cofinalities are preserved, as are the large cardinal properties: Mahlo, weakly compact, ineffable; (c) there is a real $a \subseteq \omega$ such that $a \in N$, $N \models V = L[a]$ and A and G are N-definable from the parameter a.

The paper also answers in a negative sense made precise therein a problem of Solovay: if b is a real, or more generally a set of ordinals, from which $0^{\#}$ is not constructible, is b generic over L with respect to a <u>set</u> of conditions ?

¶ 2.

P 1203 Great inroads have been made recently into the singular cardinals problem, starting with Silver's proof (Silver [a]; for an elementary proof, see Baumgartner and Prikry [a]) that if $\{\xi < \omega_1 \mid 2^{\aleph_\xi} = \aleph_{\xi+1}\}$ is stationary in ω_1, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. Similar results hold for every singular cardinal of uncountable cofinality. This theme has been developed by Silver; Jech and Prikry [a]; Galvin and Hajnal [a]; Jensen [c]; Magidor and Shelah. We quote three results:

(Silver) If κ is a singular strong limit cardinal with $2^\kappa > \kappa^+$ and $\lambda = cf(\lambda) > \kappa$ then the filter F of closed unbounded subsets of λ is such that λ is measurable in L^F (cf Theorem 5.8 of Kunen [a])

(Jensen) Assume $\neg 0^{\#}$. Let X be an uncountable set of ordinals. Then there is $Y \in L$ such that $\overline{\overline{X}} = \overline{\overline{Y}}$ and $X \subseteq Y$. ("The covering lemma")

(Prikry) If 2^ω is RVM then for all $\nu < 2^\omega$, $2^\nu = 2^\omega$.

As for the possibility of achieving a model in which $2^{\aleph_n} = \aleph_{n+1}$ for each $n < \omega$ but $2^{\aleph_\omega} = \aleph_{\omega+2}$, it follows from Jensen's covering lemma that that cannot be done in a Boolean extension of L, whereas Magidor [a] can do so starting from a model containing two very large cardinals.

T 1205 is due to Gödel.

P 1207 Magidor has shown that if κ is 2^κ-supercompact and λ is a regular cardinal less than κ then in some cardinal-preserving Boolean extension, $cf(\kappa) = \lambda$; which improves Prikry's T 2118.

T 1212 See Jensen and Solovay [a].

T 1214 See McAloon [b].

T 1215 See McAloon [b].

D 1216 For proofs of (improved versions of) TT 1217–1220 see Vopenka and Hajek [a] Balcar [a] Bukovský [a] and later papers by Hajek; and for further investigations of the sequence HOD, $(\mathrm{HOD})^{\mathrm{HOD}}$, ... see McAloon [a], Jech [b] and Grigorieff [b].

D 1221 Souslin's problem has been extremely stimulating: see Devlin [a] and Devlin and Johnsbråten [α]. Souslin trees also play a part in the papers by Jech [b] and Jensen and Johnsbråten [a]. For more on trees, see Jech [d].

P 1228 Yes (Jensen; see Devlin and Johnsbråten [a])

P 1229 No (Jensen; see Devlin [a])

In axiomatising and generalising his proof, Jensen formulated several combinatorial principles, of which we give two examples:

$\Diamond_\kappa$: There is a family $\langle S_\zeta \mid \zeta < \kappa \rangle$ such that $S_\zeta \subseteq \zeta$, and for each $X \subseteq \kappa$, $\{\zeta \mid X \cap \zeta = S_\zeta\}$ is stationary in κ.

$\Box_\kappa$: There is a sequence $\langle C_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda) \rangle$ such that

(i) C_λ is closed and unbounded in λ;

(ii) $\mathrm{cf}(\lambda) < \kappa \rightarrow |C_\lambda| < \kappa$;

(iii) if γ is a limit point of C_λ, then $C_\gamma = \gamma \cap C_\lambda$.

If $V = L$, then $\Box_\kappa$ holds for all infinite cardinals κ, and $\Diamond_\kappa$ for all regular κ. (Jensen). Jensen's discussion of $\Box$ in Devlin [a] and Jensen [d] turns on his <u>fine structure theory</u> of L; Silver has devised some <u>machines</u> which enable the principles $\Box_\kappa$ to be established in L without it.

P 1230 Yes.

P 1232 Mrs Rudin has constructed an example of a normal Hausdorff space which is not countably paracompact in ZF + AC alone. See Rudin [a, Chapter IX, (2)].

T 1234 Using forms of $\Diamond$ and $\Box$ Jensen has proved that if $V = L$, κ is weakly compact iff there is no Souslin κ-tree. See Devlin [a].

D 1236 The version of (*) with "at most $\aleph_1$" replaced by "fewer than $2^{\aleph_0}$" has become known as Martin's axiom, and is fully discussed in Martin and Solovay [a] and its consistency proved in Solovay and Tennenbaum [a]. For its use in general topology see Rudin [a] and Tall [a,b].

T 1237 See Solovay and Tennenbaum [a].

T 1238 See Solovay and Tennenbaum [a].

T 1240 is stated incorrectly: the hypothesis that for some real X, $\aleph_1 = \aleph_1^{L[X]}$ should be added. The conjunction of the two statements "every subset of is constructible from a real" and "for each real X, $X^{\#}$ (D 2037 relativised) exists", both of which incidentally are consequences of the axiom of determinacy, is incompatible with the axiom of choice; so using a Ramsey cardinal one may easily find a model in which (*) and the negation of T 1240 both hold.

A refined version of T 1240, due to Solovay, is stated as Theorem 3.2 of Martin and Solovay [a], and is this: suppose that (*) holds and that for some real X, $\aleph_1 = \aleph_1^{L[X]}$. Then every set of reals of power $\aleph_1$ is $\underset{\sim}{\Pi}{}^1_1$.

T 1241 Pleissner [a] has proved, using a principle similar to $\Diamond$, that if $V = L$ then every locally compact normal Moore space is metrizable.

T 1242 For more on sequential compactness, see Booth [a].

For an application of $\Diamond$ to group theory, see Shelah [a,b], parts of which are expounded in Eklof [a]; and for applications of Jensen's combinatorial principles to general topology, see Rudin [a, Chapter VI], and Ostaszewski [a].

For a study of gaps in the constructible hierarchy, see Marek and Srebrny [a].

¶ 3. For this whole paragraph see the books of Jech [c] and Felgner [a].

T 1303 Proved in Felgner [a].

T 1304 Proved in Felgner [a].

P 1309 No (Gauntt).

For an application of Thompson's list of minimal simple groups to these independence questions, see Conway [a].

T 1319 See Mathias [*,II] or Felgner [a].

P 1321 Yes (Felgner [b]).

P 1324 Yes (Pincus).

T 1325 is false, the error in Felgner's proof being found by Morris, who has shown that if FP is the statement that every total ordering has a cofinal sub-well-ordering then $ZF \vdash OE + FP \rightarrow AC$, but if Con(ZF) then $\mathrm{Con}(ZF + FP + \neg OE)$.

P 1327 No (Pincus [c]).

P 1330 Such a model has been found by Morris [a].

P 1331 By the covering lemma of Jensen [c], if ω_1 and ω_2 are both cofinal with ω, $0^{\#}$ exists.

Magidor has, using ζ supercompact cardinals and a measurable cardinal found a model in which the first ζ uncountable alephs are cofinal with ω.

P 1332 No (Sageev [a]).

P 1354 If every cardinal has a strong successor, then for each infinite cardinal p, 2p = p. (Tarski; Truss [a].)

Further papers in this area are Ash [a], Bell, Fremlin [a]; Edwards [a]; Gardiner [a]; Howard [b]; Monro [a]; and Truss [b,c].

§2.

For this whole section, see Drake's book [α], and the expository paper by Kanamori, Reinhardt and Solovay [α].

¶ 0.

D 2001 There are no Moschovakis cardinals (Martin; Kunen; Neil Williams)

T 2003 Menas [α] has proved that a measurable cardinal which is a limit of strongly compact cardinals is strongly compact, and the first such cardinal is not supercompact.

T 2007 S. Thomson has computed $\kappa(\alpha+1)$ explicitly from $\kappa(\alpha)$. See Drake [α].

T 2010 The following strong form of weak compactness is important in studying L: call κ <u>ineffable</u> if given any partition $f:[\kappa]^2 \longrightarrow 2$, there is a stationary set X with f constant on $[X]^2$. See Devlin's book [α]. If κ is ineffable then $\diamondsuit_\kappa$ holds (Kunen – see Jensen-Kunen [α]). The notion is studied further in Baumgartner [α].

T 2011 is now vacuous.

T 2013 is also due, independently, to Paris: see Kunen-Paris [α].

P 2014 Yes if V = L, by T 2018. Mitchell [α] has constructed a model in which a certain measurable cardinal has precisely two normal measures; the set of smaller measurable cardinals has measure 1 in one measure and measure 0 in the other, so both measures are definable.

T 2016 See Kunen [α].

For safety assume AC in the following.

T 2018 See Kunen [α], and for another study of iterations, see Gaifman [*, II].

Let U_0 be a normal measure on κ_0, and let $V_{(0)} = V$. Define $V_{(1)}$ to be the transitive collapse of the ultrapower V^{κ_0}/U_0: in short, $V_{(1)} = \mathrm{Ult}(V,U_0)$; and let i_{01} be the canonical elementary embedding $V_{(0)} \longrightarrow V_{(1)}$. Let $\kappa_1 = i_{01}(\kappa_0)$ and $U_1 = i_{01}(U_0)$. Let $V_{(2)} = \mathrm{Ult}(V_{(1)},U_1)$, constructed using only those functions $f: \kappa_1 \longrightarrow V_{(1)}$ which are in $V_{(1)}$; define i_{12}, κ_2, U_2 in analogy to the above, and put $i_{02} = i_{01} \cdot i_{12}$. We obtain a sequence of inner models $V_{(n)}$. Let $V_{(\omega)}$ be the direct limit of those with respect to the maps i_{nm}; Gaifman proved that $V_{(\omega)}$ is well-founded and (identifying it with its transitisation) that the κ_n's go under the canonical embeddings to $\kappa_\omega =_{df} \sup_{n<\omega} \kappa_n$, and the U_n's to an ultrafilter U_ω in $V_{(\omega)}$ on κ_ω. Let N be the intersection of the $V_{(n)}$, $n < \omega$. $N \supseteq V_{(\omega)}$. Bukovsky showed that N is a model of $ZF + AC$, and further that it is a generic extension of $V_{(\omega)}$. It follows from Mathias [b] that the sequence $\langle \kappa_n \mid n < \omega \rangle$ is Prikry generic over $V_{(\omega)}$ with respect to U_ω. Dehornoy [a] has proved that in fact

$$N = V_{(\omega)}[\langle \kappa_n \mid n < \omega \rangle] \quad .$$

In fact iterations $V_{(\omega)}$ can be defined for all ordinals ζ. Call (by T 2016) M <u>the</u> κ-<u>model</u> if M is an inner model of $ZF + AC + V = L$ in which κ is the measurable cardinal. Kunen [a] showed that if the κ- and λ-models exist and $\kappa < \lambda$ the λ-model is an iterated ultrapower of the κ-model. Let Q be the intersection of all κ-models as κ runs through all appropriate ordinals. If κ-models exist, Q coincides with the <u>core model</u> defined and studied by Dodd and Jensen in [a]. The definition of the core model is via fragments, called <u>mice</u>, of which well-founded ultrapowers can be formed and iterated; and "works" whether there are κ-models or not. The following results are quoted from the manuscript of Dodd and Jensen. If $0^\#$ (D 2037) does not exist, the core model is equal to L. If there is no κ-model for any κ, the covering lemma holds over the core model: that is, given an uncountable set X of ordinals, there is a set $Y \supseteq X$ with $\overline{\overline{X}} = \overline{\overline{Y}}$ and Y in the core model. If $0^\dagger$ (D 2040) does not exist but there is a κ-model, let κ be minimal such and U normal on κ in L^U: then, still assuming AC, either the covering lemma holds for L^U or there is a Prikry generic sequence C over L^U such that the covering lemma holds for $L^C = L^{U,C}$.

Kunen showed that if there is a non-trivial embedding of L into itself which is elementary, or at least preserves Σ_1 predicates, $0^\#$ exists. Dodd and Jensen have also shown that if there is a non-trivial Σ_1 embedding of the core model into itself, then there is a κ-model, though no such κ need be equal to the first ordinal moved by the embedding.

T 2021 has been strengthened by Solovay to "has 2^{2^κ} normal measures."

P 2022 is now known to be impossible: Magidor [b] has shown that if Con(ZFC + SCC) then Con(ZFC + SCC + the first measurable cardinal is strongly compact).

T 2025 should read "If there is a uniform κ-complete ultrafilter on λ^+, then..."

P 2028 False: see Kunen [b] who shows that the existence of a κ-complete uniform ultrafilter on κ^+ implies the conclusion of T 2030.

P 2031 is answered by Magidor [b] : if Con(ZFC + SuCC) then Con(ZFC + SuCC + the first strongly compact cardinal is supercompact). Thus from Magidor's work on P 2022 and P 2031, the first strongly compact may be either equal to the first measurable cardinal or to the first supercompact cardinal, which by T 2020 is much larger than the first measurable. For further indecision in the ordering of large cardinals, see Morgenstern [a] where it is shown that the first huge cardinal may be either larger or smaller than the first strongly compact cardinal.

T 2033 See Silver [b].

T 2036 See Rowbottom [a].

P 2047 Kunen [c] has shown that if there are ω_1 measurable cardinals, AC is false in Chang's model $M^{On}_{\omega_1\omega_1}$.

T 2050 See Solovay [*,I]. Kunen has shown in [a] that if I is a normal κ-complete κ^+-saturated ideal on κ then $I \cap L^I$ is prime in L^I. That may fail if I is not normal: see Wagon [a].

D 2052 The following table is taken from Kunen [a].

Let $S^*(\kappa,\lambda)$ mean that there is a κ-complete ideal on κ which is λ-saturated but not λ'-saturated for any $\lambda' < \lambda$. Then for κ not measurable, the following is known:

	$\omega < \lambda < \kappa$	$\lambda = \kappa$	$\lambda = \kappa^+$
κ a successor cardinal	FALSE Ulam [a]	FALSE Ulam [a]	Consistent Kunen [d]
κ weakly but not strongly inaccessible	Consistent Prikry [a]	Consistent Kunen, Paris [a]	Consistent
κ strongly inaccessible but not weakly compact	FALSE Tarski [α]	Consistent Kunen [d]	Consistent Boos [a]
κ strongly inaccessible and weakly compact	FALSE Tarski [α]	FALSE Lévy, Silver	Consistent Kunen, Paris [a]

P 2055 A consistency proof is implicit in Kunen [d].

P 2057 Done by Kunen [d] assuming the consistency of a huge cardinal. Can it be done with less ?

T 2059 For a discussion of possible refinements of this result, see Baumgartner, Hajnal and Maté [α].

T 2061 No (Williams [a, Chapter II, Theorem 3.4]).

Let us note here the theorem of Solovay [c] that if κ and λ are cardinals with $\kappa \leq \lambda$ and there is a normal measure on $S_\kappa(\lambda^+)$, then $\Box_\lambda$ fails.

For partition theorems in the presence of the axiom of choice, see the forthcoming encyclopaedic book of Erdős, Hajnal, Mate and Rado [a]. For partitions in L, see Shore [a]. For generalisations of $\diamondsuit$ to large cardinals, see Jech [e]; for results relating weak compactness in various languages, see Bell [a]; and for more on compactness, see G.V.Choodnovsky [a].

§ 3.

The state of knowledge of descriptive set theory has advanced greatly since this Landscape was written: see in particular the Notes on Scales by Kechris and Moschovakis [a], and Martin's forthcoming book.

¶ 0.

T 3015 From work of Martin and Kunen it is now known that, assuming AD plus DC, each $\aleph_n$ for $n > 2$ is of cofinality $\aleph_2$, $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ are measurable, and each $\aleph_{\omega+n}$ for $n > 2$ is of cofinality $\aleph_{\omega+2}$.

P 3021 Yes (Martin [b])

T 3022 See Martin [a]. Harrington has improved this to: $0^{\#}$ exists iff every Π^1_1 game is determined.

T 3024 is proved in Friedman [a]; in fact Friedman had only obtained Π^0_5 instead of Δ^0_5 but Martin has subsequently improved that to Π^0_4.

T 3025 Now vacuous, as there are no Moschovakis cardinals.

P 3026 Moschovakis has proved $\bigwedge n$ Unif $(\underset{\sim}{\Pi}^1_{2n+1})$ from ZF + AD + DC: more exactly, he has shown that DC + determinacy for all $\underset{\sim}{\Delta}^1_{2n}$ games implies Unif $(\underset{\sim}{\Pi}^1_{2n+1})$.

Friedman has proved the following 'non-uniform uniformisation theorem':

if V = L then every non-empty Π^1_2 set of reals contains a Π^1_2 singleton.

Harrington has proved the converse. See Harrington [a] for a proof of both.

P 3030 Harrington has shown that <u>Prewellordering</u>(Σ^1_3) (and hence red(Σ^1_3)) holds in Solovay's model of T 3306. See the comment on P 3110.

Let us record here Silver's interesting result that the set of equivalence classes of a $\underset{\sim}{\Pi}^1_1$ equivalence relation on the reals is either countable or of cardinality $2^{\aleph_0}$.

For more on the axiom of determinacy see Fenstad [a], Kechris [c], Martin [c], Martin and Paris [a], and Moschovakis [a].

¶1.

T 2109 See Silver [c] and Jensen [*,II].

P 2114 Menas in his thesis [b] showed that from an appropriate assumption the consistency of the existence of a supercompact cardinal with the GCH and the statement that all sets are ordinal definable may be proved.

T 2115 The proto-argument for Silver's work mentioned in the note to P 1203.

P 2117 No (Silver, assuming Con(ZF + AC + there is an extendible cardinal).)

T 2118 Mathias [b] has proved that a monotone function $f: \omega \to \kappa$ is Prikry generic over an inner model M with respect to a normal measure μ on κ in M iff for each A with $\mu(A) = 1$, $\bigvee n \bigwedge m > n \, (f(m) \in A)$.

P 2121 Solovay has proved that if κ is strongly compact and λ is a singular strong limit cardinal greater than κ then $2^\lambda = \lambda^+$. See Solovay [c] which was partly based on the theorem of Ketonen [a] that if for each λ greater than κ there is a κ-complete uniform ultrafilter on λ, then κ is strongly compact.

T 2123 though true, is not due to Kunen but follows from T 2115 and Silver's proof that if Con(ZF + AC + there is an extendible cardinal) then Con(ZF + AC + $\bigvee \kappa (\kappa$ 2vm + $2^\kappa > \kappa^+$)) and not by the faulty argument given, which is due to the author's misunderstanding of Kunen's work.

T 2124 See Solovay [*,I]. One direction of this can also be proved by the Kunen-Paris method [a]: let κ be a measurable cardinal, and $\mathbb{B}$ the measure algebra of 2^κ. Then $j(\mathbb{B})$ is complete and is the measure algebra of $2^{j(\kappa)}$, and, (by Fubini's theorem, not by general facts about iterated forcing) the algebra $\mathbb{C}$ in $V^{\mathbb{B}}$ such that $(V^{\mathbb{B}})^{\mathbb{C}} = V^{j(\mathbb{B})}$ is the measure algebra of $2^{j(\kappa)-\kappa}$; let ν be the measure on it. By the main theorem of Kunen-Paris [a] there is in $V^{j(\mathbb{B})}$ a $V^{\mathbb{B}}$-ultrafilter U on κ. Define, in $V^{\mathbb{B}}$, for $A \subseteq \kappa$,

$$\mu(A) = \nu([\![\hat{A} \in U]\!]^{\mathbb{C}}).$$

Then in $V^{\mathbb{B}}$, $\kappa = 2^\omega$ and μ is a real-valued measure on κ.

In that argument j is any elementary embedding of the universe into an inner model for which κ is the first ordinal moved. As a special case of a general fact about the application of the Kunen-Paris method for obtaining saturated ideals to algebras satisfying the κ-chain condition, the filter of sets in V of μ-measure 1 is generated by the normal ultrafilter $\{ X \subseteq \kappa \mid \kappa \in j(X) \}$ which lies in V.

The Kunen-Paris method for forming saturated ideals is in some sense the converse of the Boolean-valued analysis of saturated ideals in Kunen [a] and Solovay [*,I].

¶ 1.

For amplification of the remarks following T 3103, see Kechris and Moschovakis [a].

T 3105 See Jensen and Solovay [a]; for other proofs of this result, see Jensen [a] and Jensen and Johnsbråten [a].

P 3108 Solved affirmatively with $n = 3$ by Jensen [a].

P 3110 Harrington [c] has constructed models of ZF + AC in which the set of constructible reals is, respectively, exactly the following set of reals: Δ^1_3, Δ^1_4, ... Δ^1_ω = projective, Δ^m_n for $1 \leqslant n \leqslant \omega$, $2 \leqslant m \leqslant \omega$. Kechris has observed that in the above model for Δ^1_n, Prewellordering(Σ^1_n) and Prewellordering(Π^1_n) both fail. By a further argument Harrington can find models in which Sep($\underset{\sim}{\Sigma}^1_n$) and Sep($\underset{\sim}{\Pi}^1_n$) both fail.

¶ 2.

T 3200 is due to Harrison [a].

T 3202 See Guaspari [a,b] Kechris [a] and Mansfield [b] for a discussion of this. Call a set <u>thin</u> if it has no perfect subset. Then there is a largest thin Π^1_1 set, which may be defined as $\{x \mid x \in L_{\omega_1^x}\}$ or as $\{x \mid \bigwedge y\,(\omega_1^x \leq \omega_1^y \rightarrow x \leq_h y)\}$ or in at least five other ways. See also the note to T 1240.

T 3205 Truss has constructed, without using an inaccessible, a model of set theory in which every set of reals either is of cardinality $\leq \aleph_1$ or has a perfect subset, and in which the constructible reals form a set of cardinality $\aleph_1$.

T 3207 cf Guaspari [a].

T 3215 The first proof contained a mistake, but Mansfield [a] now has a correct one. For some intermediate stages see references 2 and 4 in Mansfield [a]. For a discussion of related questions see Harrington [b] and Harrington and Jech [a].

In their paper, Harrington and Jech prove that it is consistent to have a Σ_1 well-ordering of the universe, in the sense of Lévy [6], without V being equal to L. They ask whether the existence of a Σ_1 well-ordering of V implies the existence of such a well-ordering of length On. The answer is yes: let $<_1$ be a Σ_1 well-ordering of V, and rn(x) the rank of x. Obtain a Σ_1 well-ordering of V of length On by setting

$$x <_2 y \underset{df}{\longleftrightarrow} rn(x) < rn(y) \;\vee\; (rn(x) = rn(y) \;\wedge\; x <_1 y).$$

It follows from the unpublished theorem of Jensen they cite that V = L iff there is a Σ_1 well-ordering of V of length On such that the function which assigns to each set its set of predecessors in the well-ordering is also Σ_1.

T 3315 See the comment to T 3302.

P 3318 Such a model has been found by Laver [a].

T 3321 See Martin and Solovay [a].

T 3322 For extensions of this result to higher levels of the projective hierarchy using determinacy for projective games, see Kechris [b].

T 3325 Note that (iii) follows from (i) by the argument of Rothberger [a].

A propos of RVMs, Kunen has proved that if 2^{ω} is RVM then it has the Π^1_1 reflection property, whereas in his thesis (Theorem 16.8) he showed that if Con(ZF + AC + MC) then Con(ZF + AC + 2^{ω} carries an ω_1-saturated 2^{ω}-complete ideal but is Π^1_1-characterisable and hence fails to have the Π^1_1 reflection property).

For more on RVMs, see D.V.Choodnovsky [a].

¶ 4.

P 3411 The answer to this and related questions is now known to be intimately connected with the relation of L to V. In particular, if V = L, the answer to P 3411 for the case $\kappa = \omega_1$ (Prikry [b]) or $\kappa = \omega_n, n \geq 2$, (Jensen, Kunen [a]) is No. See Benda, Ketonen [a], Kanamori [a], Ketonen [b], and Jensen, Koppelberg [a]. In the last mentioned paper, it is hsown as an interesting corollary that if $\neg 0^{\#}$, then $\Box_\beta$ holds for every singular β. It was a lemma of Kanamori in [a,b] that started the wave of results mentioned in the note to P 1203. Did anyone think in 1968 that these two problems were connected ?

¶3.

T 3302 The proof of this and the next theorem has been illuminated by the application of some results of Rothberger [a], the principal one being

ZF $\vdash$ If there is a set of reals of power κ which is not first category then the real line is the union of κ sets of measure 0; and similarly with "first category" and ""measure 0" interchanged.

T 3306 Mathias [a] has shown that in the model of T 3306, which is described in Solovay [b] the partition relation $\omega \to (\omega)^{\omega}$ holds. See also Lévy and Solovay [a].

From AD + DC it has been shown (see Martin, Paris [a]) that $\omega_1 \to (\omega_1)^{\omega_1}$ and $\bigwedge \zeta < \omega_2 \;\; \omega_2 \to (\omega_2)^{\zeta}$ whereas $\omega_2 \to (\omega_2)^{\omega_2}$ is false. Prikry has proved usin DC + ADR – the axiom of determinacy for games played with reals, that $\omega \to (\omega)^{\omega}$. See Prikry [c], or for related results see Mathias [a].

There is a short proof, due to Martin, of $\bigwedge \zeta < \omega_1 \;\; \omega_1 \to (\omega_1)^{\zeta}$ from AD + DC using Jensen's theorem that if $\xi < \omega_1$ and $\langle \zeta_\nu \mid \nu < \xi \rangle$ is a sequence of countable ordinals such that each is admissible in the set of its predecessors then for some real x and all $\nu < \xi$, ζ_ν is the $(1+\nu)^{th}$ ordinal admissible in x. Jensen's theorem was proved by him using forcing in some unpublished botes on admissible sets. A proof has recently been found by Sacks and Harrington which avoids forcing.

P 3309 An affirmative answer to the following question would assist in solving P 3309, according to recent work of Truss:

if a generic or a random real is added to a model of Martin's axiom, is Martin's axiom still true ? For generic reals, the answer is No. (J. Roitman)

17 3.

¶ 0.

D 4000 Solovay has proved that if $V = L$ then $K(\aleph_\alpha)$ holds for all α. See Devlin [a].

T 4004 Solovay's theorem quoted in the last comment generalises to the result that if $X \subseteq \aleph_{\alpha+1}$ and $V = L[X]$, then $K(\aleph_\alpha)$. It follows that if $K(\aleph_\alpha)$ is false then $\aleph_{\alpha+2}$ is inaccessible in L.

T 4006 Jensen has shown that if $V = L$, or at least if $0^\#$ does not exist, then T 4006 holds for the case $\aleph_\beta$ singular as well. Furthermore he has shown that if $V = L$, then for any two κ, λ,

$$\langle \kappa^{++}, \kappa \rangle \Rightarrow_0 \langle \lambda^{++}, \lambda \rangle \quad , \quad \langle \kappa^{+++}, \kappa \rangle \Rightarrow_0 \langle \lambda^{+++}, \lambda \rangle \qquad \text{et cetera.}$$

See Devlin [a], and for related independence results, see Mitchell [b].

D 4007 For more on Rowbottom cardinals and the related notion of Jónsson cardinals see Devlin [b] and Kleinberg [a].

Kleinberg [b] has proved that if AD then the $\aleph_n$ are all Jónsson cardinals and $\aleph_\omega$ is a Rowbottom cardinal.

Silver has proved that if $\aleph_\omega$ is a Jónsson cardinal then it is measurable in an inner model.

T 4013 The hypothesis may be weakened to Con(ZF + AC + there is a Ramsey cardinal). For an account of Silver's proof, see Devlin [c].

Kunen has shown that if CC holds then $0^\#$ exists: see Devlin [a].

¶ 1. See Keisler, Silver [*, I].

§5.

¶0. For Ackermann's set theory, see Lake [a] and Reinhardt [a].

T 5010 This theorem has been considerably extended by Pincus [a,b]. See also Jech [c].

For limitations to the Fraenkel-Mostowski method, see Howard [a].

¶1. For a recent review of the state of knowledge concerning the consistency problem for NF, see Boffa [a].

T 5100 See Jensen [e].

Finally, the reader is encouraged to consult Friedman's list [b] for ideas for further areas of research.

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