Surrealist Landscape with Figures

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This is the text of the hitherto unpublished article written at Stanford University and circulated in typescript in 1968 with the title page given below. "Recent" being no longer a true description of its contents, I publish it now under a timeless title by which it has been cited by other writers. In doing so, I have corrected the typographical errors in the original text but not the mathematical ones, to which attention is drawn instead in the notes on subsequent developments which will be found together with a supplementary bibliography at the end. The sign + in the text marks points on which those notes comment.

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A SURVEY OF RECENT RESULTS IN SET THEORY

by

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§0. Introduction.

The present work unfolds the development of Zermelo-Fraenkel set theory in the years following the invention of forcing. Theorems, definitions and unsolved problems are indicated by the appropriate letter: In order to avoid decimal points, statements are numbered in thousands; thus T 1308 indicates the ninth theorem of paragraph 3 of Section 1. Proofs are few, and I should state that the observations called theorems vary from trivial remarks to the deepest results in the subject. I think that only the detailed proofs of the theorems would convey adequately their relative importance, and accordingly I have not tried.

The names of those who have worked on the same problem are separated by a comma if they worked together, and by a semicolon if independently. (See, for example, T 1308, which was proved by two independent groups.)

There are three styles of reference: those of the form [10] are found at the end, those like [H44] are to be found in the revised Bibliography of the Prague Seminar compiled by Hájek, and an occurrence of [*] shortly after people's names refers to the account of their lectures at the Summer Institute expected to appear in the Proceedings.

I assume familiarity with the basic properties of forcing and Boolean-valued models, for which I use the notation of Scott and Solovay [*]: general lemmata on forcing are given in Boolean terminology but individual constructions are described in terms I think most vivid. Independence results are stated usually in the form

"Con(A) implies Con(B)", meaning that the consistency of the system B can be deduced from that of A in elementary number theory. Model theoretic forms are preferred when they give finer information. The systems in which other results are proved are either stated, e.g., "ZF + AC ⊢ ..." or are to be inferred from the immediate context. I write "c.s.m." for "countable standard model"; and I use "a real" always to mean "a subset of ω." AC is the axiom of choice in the form "Every set has a well-ordering."

Each section is divided into paragraphs. The first section starts from Gödel's L: in ¶1, GCH is true; in ¶2, GCH is false but AC is preserved; in ¶3, AC too succumbs. The second section is devoted to large cardinals: no mention is made of those of Reinhardt [*], but it is hoped that [15] will appear soon. Proofs of many of the theorems of §2, in particular of TT 2012, 2017, 2018 and 2033, are contained in the lecture notes [16] of Jensen on large cardinals which are expected to become accessible this year. §3 is on the real line, and includes some results on real-valued measurable cardinals and on the axiom of determinacy. §§4 and 5 are brief summaries of work on problems arising in model theory, and on systems other than ZF.

§1. The Axiom of Choice and Continuum Hypothesis.

¶ 0. Before Forcing.

T 1000 (Gödel) Con(ZF) implies Con(ZF + V = L).

T 1001 (Gödel) $ZF \vdash V = L \rightarrow GCH$.

T 1002 (Lindenbaum-Tarski) $ZF \vdash GCH \rightarrow AC$.

* ¶ 1. After Forcing; Non-constructibility.

T 1100 (Cohen) Con(ZF) implies Con(ZF + GCH + there is a non-constructible subset of ω).

Let $2 = \{0, 1\}$ be given the discrete topology and the measure defined by $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$; let 2^ω be given the product topology and measure.

D 1101 $\mathbb{B}_1$ = algebra of Borel subsets of 2^ω modulo sets of first category.

D 1102 $\mathbb{B}_2$ = algebra of Borel subsets of 2^ω modulo sets of ~~first category~~. measure zero.

Define an element of $V^{(\mathbb{B}_1)}$ $(V^{(\mathbb{B}_2)})$ by

$$[\![\check{n} \in \dot{a}]\!]^{\mathbb{B}_1} = \{f \in 2^\omega | f(n) = 1\}/\text{sets of first category}$$

$$([\![\check{n} \in \dot{a}]\!]^{\mathbb{B}_2} = \{f \in 2^\omega | f(n) = 1\}/\text{sets of measure } 0).$$

Let M be a c.s.m. of ZF + AC. Then

D 1103 A real a_1 is generic over M iff there is a M-complete ultrahomomorphism F_1 of $\mathbb{B}_1^M$ with

$$a_1 = \{n | F_1([\![\check{n} \in \dot{a}]\!]^{\mathbb{B}_1^M}) = 1\} ;$$

* D 1104 A real a_2 is random over M iff there is an M-complete ultrahomomorphism F_2 of $\mathbb{B}_2^M$ with

$$a_2 = \{n | F_2([\![\check{n} \in \dot{a}]\!]^{\mathbb{B}_2^M}) = 1\} ;$$

we write $M^{(\mathbb{B}_1^M)}/F_1 = M[a_1]$, etc. For details, see the paper of Scott and Solovay [*].

Cohen proved T 1100 by showing that $M[a_1] \models \dot{a}_1 \notin L$, where M is a c.s.m. of ZF + V = L, and a_1 is generic over M. The concept of "random real" is due to Solovay, who also showed that D 1103 is equivalent to the notion of generic real implicit in Cohen's proof. We shall see later that $M[a_1]$ and $M[a_2]$, where a_2 is random over M, differ in many details; for the present, let us record Solovay's observation that $M[a_1]$ contains no reals random and $M[a_2]$ no reals generic over M.

We shall write simply "random" or "generic," omitting "over M", during discussions in which the ground model M is fixed.

Now that we know that non-constructible reals are possible, we may consider "degrees of constructibility," or L-degrees, for real numbers:

D 1105 $a \leq_L b$ iff a is constructible in b; that is, $a \in L[b]$.

Sacks invented a method of forcing which yielded a model having precisely two L-degrees. Thus

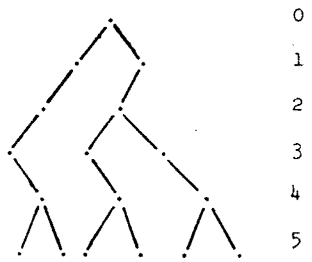
* T 1106 (Sacks) Con(ZF) implies Con(ZF + GCH + there is a non-constructible real + $\bigwedge xy(x \subseteq \omega \wedge x \notin L \wedge y \subseteq \omega \,.\!\rightarrow y \leq_L x)$).

Sacks' conditions are perfect subsets of 2^{ω} with the above topology: it is fruitful to think of a condition T as the set of maximal branches (paths) of a tree with one root and no tree tops such that along any one path are infinitely many forks. The forcing relation is then given by

D 1107 $T \Vdash \check{n} \in \dot{a}$ iff for all paths P in T, $P(n) = 1$;

$T \Vdash \check{n} \notin \dot{a}$ iff for all paths P in T, $P(n) = 0$.

Example Let the initial portion of T be



Then $T \Vdash \check{1} \notin \dot{a}$ and $T \Vdash \check{3} \in \dot{a}$.

The principal property of Sacks forcing is

* T 1108 The fusion lemma:

Let S be the set of finite sequences of 0's and 1's. Let $\{T_s | s \in S\}$ be a family of Sacks conditions such that for all $s \in S$, $T_{s0} \cap T_{s1} = \emptyset$; $T_{s0} \subseteq T_s$ and $T_{s1} \subseteq T_s$. Then $\bigcap_{n \in \omega} \bigcup_{|s|=n} T_s$ is a Sacks condition.

D 1109 Let M be a c.s.m. of $ZF + V = L$. A real a generic over M with respect to Sacks forcing will be termed a Sacks real (over M); the model $M[a]$ a Sacks model.

For further details, consult Sacks [*].

Another method of obtaining reals of L-degree minimal over L^M is to take as conditions pair $\langle A,B \rangle$ with $A \subseteq \omega \wedge B \subseteq \omega \wedge A \cap B = \emptyset \wedge \omega$ $(A \cup B) = \omega$; where

D 1110 $\langle A,B \rangle \Vdash \check{n} \in \dot{a}$ iff $n \in A$

$\langle A,B \rangle \Vdash \check{n} \notin \dot{a}$ iff $n \in B$.

* D 1111 We shall call a real a generic with respect to this notion of forcing a Silver real (over M), and the model $M[a]$ a Silver model, in honour of the man who first established that this forcing relation gives reals of minimal L-degree.

Let us note some properties of Silver and Sacks reals over a fixed $M(\vdash V = L)$.

T 1112 Every non-constructible real in a Sacks model is a Sacks real;

* T 1113 (Kunen) No Silver real occurs in a Sacks model.

By the minimal degree property, it follows that

T 1114 (Kunen) No Sacks real occurs in a Silver model.

T 1115 (Kunen) Not every non-constructible real in a Silver model is a Silver real.

T 1116 (Příkrý) Sacks forcing is equivalent to taking as truth algebra the completion of the algebra of Borel subsets of 2^{ω} modulo the countable sets.

$\mathbb{B}$ has no automorphisms, and moreover, if b_1, b_2 are two distinct elements of $\mathbb{B}$, the complete Boolean algebras $\{b|b \leq b_1\}$ and $\{b|b \leq b_2\}$ are not isomorphic.

T 1125 (McAloon) Con(ZF) implies Con(ZF + GCH + V $\neq$ K + K $\neq$ L).

Some brief comments on the proof of T 1124. Suppose that N is a c.s.m. of ZF + GCH + V $\neq$ L, and that $t \subseteq \omega$, $t \notin L$, $t \in N$. McAloon's original way of making t an ordinal definable was to expand N to an N' in which $t = \{n | 2^{\aleph_n} = \aleph_{n+1}\}$; but then N' does not satisfy GCH. A more subtle method had to be found: the answer is now described.

D 1126 Write $F(\aleph_\alpha)$ for "For every subset A of $\aleph_\alpha$ of cardinality $\aleph_\alpha$ there is a constructible $B \subseteq \aleph_\alpha$, of cardinality $\aleph_\alpha$ with either $B \subseteq A$ or $B \cap A = \emptyset$."

T 1127 (McAloon) Let M be a c.s.m. of $ZF + GCH + \Lambda\alpha > OF(\aleph_a)$; let $t \in M$, $t \subseteq \omega$, $M \models t \notin L$. Then there is an extension N of M such that $On^N = On^M$; $N \models GCH \wedge \Lambda\alpha \geq \omega F(\aleph_a)$; and in N,

$$t = \{n | F(\aleph_{n+1})\} .$$

We could code t higher up the ordinals: for if β is any ordinal, the set $\{n | F(\aleph_{\beta+n})\}$ is also ordinal definable. By a judicious iteration of the coding process T 1127, we get a model of ZF + GCH + V = K. See McAloon [*].

We end this part with some remarks on the property **F** and an extremely fruitful coding device due to Solovay.

If M is a c.s.m. of ZF + V = L, and t is generic, then in M[t], $F(\aleph_0)$ is false, but for every $\alpha > 0$, $F(\aleph_\alpha)$ is true. (This is the basic fact used in proving T 1127). In a Sacks model, $F(\aleph_0)$ is true and $F(\aleph_\alpha)$ also holds for every $\alpha \geq 2$.

* P 1128 (Friedman) Is $F(\aleph_1)$ true in a Sacks model?

If the answer is yes, then some special properties of Sacks' model must be used in the proof in view of

T 1129 Con(ZF) implies $Con(ZF + Va(a \subseteq \omega \wedge V = L[a]) + \neg F(\aleph_1))$,

which is a consequence of the following general method of Solovay, hereinafter called "Solovay's trick":

* T 1130 Let M be a c.s.m. of $ZF + AC + \omega_1^L = \omega_1$, and let $A \in M$, $A \subseteq \omega_1$. Then there is an $t \subseteq \omega$ such that in M[t], "$A \in L[t]$" is true: in other words A is constructible from t.

Proof of T 1130: Let $\langle B_\nu | \nu < \omega_1 \rangle$ be a constructible family of reals, in M, s.t. $\nu < \mu < \omega_1 \rightarrow B_\nu \cap B_\mu < \omega$, and each B_ν is infinite. We aspire to find a real t for which

$$v(\nu < \omega_1 \rightarrow [\nu \in A \longleftrightarrow B_\nu \cap t < \omega]) .$$

A condition will be a pair $\langle X,Y \rangle$ where X is a finite subset of ω and Y is a finite subset of A, the intended interpretation being

$$t \cap \bigcup_{\nu \in Y} B_\nu = X \cap \bigcup_{\nu \in Y} B_\nu .$$

We therefore define

+ D 1117 The function $f: S(\omega) \to S(\omega) \times S(\omega)$ defined by $f(a) = \langle f_1(a), f_2(a)\rangle$, where

$$n \in f_1(a) \text{ iff } 2n \in a$$

$$n \in f_2(a) \text{ iff } 2n+1 \in a ,$$

is said to <u>split</u> the real a.

Now if we split a random (or generic) real, we get two random (or generic) reals of incomparable L-degrees, but if a is Silver or Sacks, $f_1(a)$ and $f_2(a)$ have comparable L-degrees. Further, as Solovay has shown, there are no minimal L-degrees in M^{B_1}/F_1 or M^{B_2}/F_2, and if N is a c.s.m. of ZF + GCH + "there are precisely two L-degrees," then N contains no reals random or generic over L^N. Thus

T 1118 Let M be a c.s.m. of ZF + V = L. Then the properties of a real number being generic, random, Silver or Sacks over M are mutually exclusive; moreover if a is a real having any one of these properties, M[a] contains no real having any of the other three.

Silver and Sacks' forcing have another interesting property; which we state in the language of Boolean algebras. A useful theorem in making Boolean valued extensions of L is that if $\mathbb{B}$ satisfies the countable chain condition, then cardinals are preserved in the extension $L^{(\mathbb{B})}$. Now both Silver and Sacks' forcing preserve cardinals, but neither satisfies the countable chain condition.

Let K denote the universe of hereditarily ordinally definable sets. (The paper of Myhill and Scott [*] gives a detailed account of this notion.)

T 1119 $ZF \vdash V = K \to AC$.

T 1120 $ZF \vdash L \subseteq K \subseteq V$.

T 1121 (Lévy) Con(ZF) implies Con(ZF + V $\neq$ K + K = L + GCH).

In fact there is the general principle:

T 1122 (Kripke) Let $\mathbb{B}$ be homogeneous. Then

$$V = L \to [\![L \equiv K]\!]^{\mathbb{B}} = 1 .$$

More generally, the conclusion of T 1122 holds if $\mathbb{B}$ is the direct sum of homogeneous algebras. Thus (see Kripke [*]) a counterexample to the conjecture of Sikorski that every complete Boolean algebra is the direct sum of homogeneous algebras is given by an algebra $\mathbb{B}$ (which may readily be constructed using the methods of Easton (T 1202)) such that $L^{\mathbb{B}}$

$$[\![(n | 2^{\aleph_n} = \aleph_{n+1}) \notin L]\!]^{\mathbb{B}} = 1 .$$

T 1122 and its corollary have been strengthened:

T 1123 (Vopěnka) In a Boolean-valued model $V^{(\mathbb{B})}$, the class K is precisely $V^{(\mathbb{B}^*)}$, where

$$\mathbb{B}^* = \{b \in \mathbb{B} | b \text{ is fixed under all automorphisms of } \mathbb{B}\} .$$

Solovay remarked, after hearing of Kripke's work, that if $\mathbb{B}$ is the algebra used in proving

T 1124 (McAloon) Con(ZF) implies Con(ZF + GCH + V = K + K $\neq$ L).

D 1131 $\langle X,Y\rangle \Vdash \check{n} \in \dot{t}$ iff $n \in X \cap \bigcup_{v \in Y} B_v$;

$\langle X,Y\rangle \Vdash \check{n} \notin \dot{t}$ iff $n \in (\bigcup_{v \in Y} B_v) \setminus X$.

The partial ordering of conditions is given by

D 1132 $\langle X,Y\rangle \leq \langle X',Y'\rangle$ iff $X' \subseteq X \wedge Y' \subseteq Y \wedge X \setminus X' \subseteq \omega \setminus \bigcup \{B_v \mid v \in Y\}$,

(where "$\leq$" is read "is a stronger condition than," in sympathy with the Boolean algebraic approach to forcing.) For any Y, Y', $\langle X,Y\rangle$ and $\langle X,Y'\rangle$ are compatible, so the countable chain condition is easily shown to hold; the remainder of the proof is elementary.

To prove T 1129, let N be a c.s.m. of $ZF + V = L$; A a subset of ω_1^N generic with respect to the regular open algebra of 2^{ω_1} with the ω topology; and let t be as given by T 1130, with $M = N[A]$. Then $F(\aleph_1)$ is false in $M[t]$.

§ 2. <u>Doing without the Continuum Hypothesis</u>.

We now consider ways and means of violating the GCH while preserving AC.

T 1200 (Cohen) Con(ZF) implies $\mathrm{Con}(ZF + AC + 2^{\aleph_0} = \aleph_2)$.

It is known that $2^{\aleph_0}$ cannot be $\aleph_\omega$; more generally,

T 1201 (König) $ZF + AC \vdash \bigwedge\alpha[\mathrm{cf}(\aleph_\alpha) < \mathrm{cf}(2^{\aleph_\alpha})]$.

The following theorem shows that for regular $\aleph_\alpha$, T 1201 is the only restriction.

T 1202 (Easton) Let M be a c.s.m. of NBG + GCH, and let F be any function (in M) with domain On^M and range $\subseteq On^M$ such that

$$\bigwedge\alpha(\alpha < F'\alpha);\bigwedge\alpha\beta(\alpha < \beta \rightarrow F'\alpha \leq F'\beta);$$

and

$$\bigwedge\alpha(\mathrm{cf}(\aleph_\alpha) < \mathrm{cf}(\aleph_{F'\alpha})).$$

Then there is an extension N of M in which $On^M = On^N$; $\mathrm{card}^M = \mathrm{Card}^N$; $\mathrm{cf}^M = \mathrm{cf}^N$; AC is true and for every regular $\aleph_\alpha$, $2^{\aleph_\alpha} = \aleph_{F'\alpha}$.

Little is known about singular cardinals: for instance,

P 1203 (Solovay) Does Con(ZF) imply $\mathrm{Con}(ZF + AC + \bigwedge n < \omega(2^{\aleph_n} = \aleph_{n+1}) + 2^{\aleph_\omega} = \aleph_{\omega+2})$?

P 1204 (Solovay) Con(ZF + AC + $\bigwedge x \bigwedge n < \omega(x \subseteq \aleph_n \to x \in L)$ + $\bigvee y(y \subseteq \aleph_\omega \wedge y \notin L)$)? ; or rephrased model-theoretically, can one add a subset of $\aleph_\omega$ in an extension of a model of ZF + V = L without adding subsets of $\aleph_n$, any n? .

These are the simplest cases of the "singular cardinals problem." There is some restriction on the behaviour of $2 \exp \aleph_\omega$, for as pointed out by Hechler and Bukovský independently, if $\bigwedge n \geq n_0$, $2 \exp \aleph_n = \aleph_{\omega+1}$, then $2 \exp \aleph_\omega = 2 \exp(\Sigma \aleph_n) = \Pi(2 \exp \aleph_n) \leq (2 \exp \aleph_{n_0}) \exp \aleph_0 = 2 \exp \aleph_{n_0} = \aleph_{\omega+1} \leq 2 \exp \aleph_\omega$, so $2 \exp \aleph_\omega = \aleph_{\omega+1}$.

This remark is a special case of

* T 1205 (Bukovský [H25]) ZF + AC $\vdash$ Let $cf(\aleph_\alpha) \leq \aleph_\beta < \aleph_\alpha$. Then

(i) $2 \exp \aleph_\alpha = \sum_{\gamma \in \alpha} (2 \exp \aleph_\gamma)$ if there is a $\gamma_0 < \alpha$ such that $\bigwedge \gamma(\gamma_0 < \gamma < \alpha \to 2 \exp \aleph_\gamma = 2 \exp \aleph_{\gamma_0})$;

$2 \exp \aleph_\alpha = (\sum_{\gamma<\alpha} (2 \exp \aleph_\gamma)) \exp(cf(\sum_{\gamma<\alpha} (2 \exp \aleph_\gamma)))$

otherwise.

(ii) $\aleph_\alpha \exp \aleph_\beta = \sum_{\gamma<\alpha} (\aleph_\gamma \exp \aleph_\beta)$, if there is a $\gamma_0 < \alpha$ such that for all γ with $\gamma_0 < \gamma < \alpha$, $\aleph_\gamma \exp \aleph_\beta = \aleph_{\gamma_0} \exp \aleph_\beta$;

$\aleph_\alpha \exp \aleph_\beta = (\sum_{\gamma<\alpha} (\aleph_\gamma \exp \aleph_\beta)) \exp(cf(\sum_{\gamma<\alpha} (\aleph_\gamma \exp \aleph_\beta)))$

otherwise.

Thus in a precise sense the function $\lambda\alpha\beta\ \aleph_\alpha \exp \aleph_\beta$ can be calculated from $\lambda\alpha\ \aleph_\alpha \exp cf(\aleph_\alpha)$. Hence the general problem:

P 1206 Investigate the behaviour of $\lambda\alpha\ \aleph_\alpha \exp cf(\aleph_\alpha)$ in various models.

* P 1207 Do there exist two models $M \subseteq N$ of ZF + AC with the same cardinals but different cofinalities?

Příkrý has shown (T 2118) that if there is a two-valued measurable cardinal κ in M, a c.s.m. of ZF + AC, then there is an $N \supseteq M$ with the same cardinals as M in which $cf^N(\kappa) = \omega$.

P 1208 (Bukovský) Show

Con(ZF + AC + $2 \exp \aleph_0 = \aleph_1 + \bigwedge n \geq 1(2 \exp \aleph_n = \aleph_{\omega+2} = 2 \exp \aleph_{\omega+1})$ $+ \aleph_\omega \exp \aleph_0 = \aleph_{\omega+2} + \bigwedge \alpha > \omega(2 \exp \aleph_\alpha = \aleph_{\alpha+1})$) .

P 1209 (Takeuti) Does Con(ZF) imply

Con(ZF + AC + $\bigwedge n < \omega \bigwedge m < \omega(2^{\aleph_n} = 2^{\aleph_m} \wedge L(S(\aleph_n)) = L(S(\aleph_m)))$)?

P 1210 . Find a real t and a c.s.m. M of ZF + GCH such that $M[t] \models$ ZF + AC + $2^{\aleph_0} = \aleph_2$ and the cardinals of M[t] are those of M.

T 1211 (Silver; Solovay) Let M be a c.s.m. of ZF + V = L. Then there is a real t such that M[t] is a model of ZF + AC, and

$$\aleph_1^{M[t]} = \aleph_1^M ;\ \aleph_2^{M[t]} = \aleph_2^M \approx \aleph_3^M \text{ in } M[t] .$$

T 1211 is proved by using Solovay's trick, T 1130, as is

* T 1212 (Jensen). Let M be a c.s.m. of $ZF + V = L$. Let θ be any ordinal strongly inaccessible in M. Then there is a $t \subseteq \omega$ such that $M[t] \models ZF + GCH$ and in $M[t]$, $\theta = \aleph_1$.

(Solovay had obtained T 1212 for the special case when θ is the first inaccessible in M).

T 1213 (Kunen) $ZF \vdash$ If there is a c.s.m. of $ZF + V = L$, then there is a c.s.m. N of $ZF + AC$ such that no submodels of N with the same cardinals satisfies GCH.

We have seen McAloon's methods for making sets of ordinals ordinal-definable, and have noted that we need only use the higher reaches of the ordinals for coding; for example given $t \subseteq \omega$ we can select an α and make

$$t = \{n \mid 2 \exp \aleph_{\alpha+n} = \aleph_{\alpha+n+1}\} ;$$

by intelligent iteration we may prove

T 1214 (McAloon) Con(ZF) implies

$$\mathrm{Con}(ZF + AC + 2^{\aleph_0} = \aleph_{21} + 2^{\aleph_1} = \aleph_{\omega+5} + V = K + K \neq L)$$

T 1215 (McAloon) Con(ZF) implies $\mathrm{Con}(ZF + AC + 2^{\aleph_0} = \aleph_2 + V \neq K + K \neq L)$.

Throughout 1216-1220, N, M are c.s.m's of $ZF + AC$ with the same ordinals, and $N \subseteq M$.

* D 1216 (Vopěnka) A set $Z \in N$, $Z \subseteq M$ is a <u>support</u> for N over M iff

$$\bigwedge X \in N(X \subseteq M \rightarrow \bigvee R \in M(R''Z = X)) .$$

T 1217 (Vopěnka) A support for N over M exists iff there exists a complete Boolean algebra $\mathbb{B} \in M$ and an M-complete ultrahomomorphism F such that $N = M^{\mathbb{B}}/F$.

T 1218 (Vopěnka) Let K be the class of hereditarily ordinal definable sets of M. Let $A \in M$. If $L[A] \supseteq K$, then there is a $\mathbb{B} \in K$ and an F such that $L[A] = K^{\mathbb{B}}/F$.

T 1219 (Vopěnka) Let $Z \subseteq A \in M$. Then Z is a support over M for some N iff

(1) $\bigvee R_1 \in M(R_1''Z = S^M(Z))$

and

(2) $\bigvee R_2 \in M(R_2''Z = S^M(A \setminus Z))$.

We may also treat the notion of a Boolean-valued model when $\mathbb{B}$ is a proper class of M. $\mathbb{B}$ is <u>L-complete</u> iff it contains suprema and infima of constructible subsets.

T 1220 (Vopěnka). Suppose that $M \models V = L$ and that Gödel's axiom E, the class form of the axiom of choice, holds in N. Then there is a constructible L-complete Boolean algebra $\mathbb{B}$ and an L-complete ultrahomomorphism F such that $N = M^{\mathbb{B}}/F$. Moreover, $S(L) = \{r''F \mid r \in L\}$; that is, F is a support for N over M, except that F is not necessarily a set of N.

Easton has shown that axiom E is independent of $NBG + AC$; see T 5002.

Souslin originally formulated his hypothesis as "Every complete total ordering with neither first nor last elements which satisfies the countable chain condition on open intervals in order-isomorphic to the open interval (0,1)": we shall call this form SH^1.

* D 1221 A Souslin tree is a partial ordering $\langle T, <\rangle$ such that for all $x \in T$, $\langle\{y|y < x\}, <\rangle$ is a well-ordering, every chain and anti-chain is countable, but $\overline{\overline{T}} = \aleph_1$.

SH^1 was shown by E.C. Miller to be equivalent (in ZF + AC) to the **assertion** that there are no Souslin trees, which form we shall call SH^2.

D 1222 A normal Souslin tree $\langle T,<_T\rangle$ is one which has precisely one minimal element, in which there are infinitely many branches at each point, whose field T is $\aleph_1$, the set of countable ordinals, whose ordering is such that $\alpha <_T \beta \rightarrow \alpha \in \beta$, and in which

$$\bigwedge\alpha\, \overline{\overline{\{\beta|\alpha <_T \beta\}}} = \aleph_1$$

D 1223 We write SH for "There are no normal Souslin trees." SH is equivalent in ZF + AC to SH^1 and SH^2.

T 1224 (Tennenbaum; Jech) Con(ZF) implies Con(ZF + GCH + ¬ SH).

T 1225 (Tennenbaum; Jech) Con(ZF) implies Con(ZF + AC + ¬ SH + $2^{\aleph_0} = \aleph_2$).

T 1226 (Solovay, Tennenbaum) Con(ZF) implies Con(ZF + AC + SH).

We shall first make some remarks about the original proofs of these theorems, and discuss certain generalizations of SH to higher cardinals; and then we shall consider an axiom that implies SH and has other virtues.

T 1224 may be proved by adding a Souslin tree to a c.s.m. M of ZF + V = L: Tennenbaum worked with finite conditions, and it was remarked by Solovay that the algebra he used was isomorphic to the one used to add $\aleph_1$ generic reals, viz. the regular open algebra of $2^{\aleph_1}$ with the $\aleph_0$ topology. Another method, developed in Czechoslovakia by Jech [H 50], and modified by Přikrý, is to take as conditions trees which satisfy the requirements of D 1221 and D 1223 save that they are countable. One tree is stronger than another (quâ conditions) if it is an end extension of it. Jech mentions in a letter to the author that his algebra is the regular open algebra of $2^{\aleph_1}$ with the $\aleph_1$ topology: thus no new reals are added with his method.

The proof of T 1226 is much harder: suppose there is a normal Souslin tree T in M. We may destroy it by adding a generic path through T, a condition being an initial segment of a path through T, ending at some point of T, and the partial ordering of conditions being that given by inclusion (the countable chain condition follows from the fact that in T, all antichains are countable); but new trees may be produced in the act. Now in M there are at most $\aleph_2$ normal Souslin trees. We destroy one; then in the extension there are still $\leq \aleph_2$ trees, and as cardinals are preserved, it is the same $\aleph_2$. We now iterate the process, zigzagging back and forth so that every normal Souslin tree in the sequence of models, which will be of length $\aleph_2$ is destroyed precisely once. Then provided that no cardinals have collapsed in the meantime, SH will be true in the final model. The original demonstration has been considerably simplified by the following general principle of iteration:

T 1227 (Solovay) ZF + AC $\vdash$ Let κ be an infinite ordinal; let $\langle \mathbb{B}_\nu \mid \nu \le \kappa \rangle$ be a sequence of complete Boolean algebras such that $\mathbb{B}_0 = \mathbf{2}$; each $\mathbb{B}_\nu$ is a regular subalgebra of $\mathbb{B}_{\nu+1}$; for λ a limit ordinal $\le \kappa$, $\mathbb{B}_\lambda$ = the minimal completion of $\bigcup_{\nu<\lambda} \mathbb{B}_\nu$; then the first ν such that $\mathbb{B}_\nu$ does not satisfy the countable chain condition is not a limit ordinal.

But after $\aleph_2$ iterations we find that possibly $2^{\aleph_0} > \aleph_1$; thus the main problem now is this:

P 1228 Does Con(ZF) imply Con(ZF + GCH + SH)?

P 1229 Is SH true in L?

P 1230 (Solovay) If one extends a c.s.m. of ZF + V = L by adding $\aleph_1$ random reals, is SH necessarily false in the extension?

In T 1226 using Easton's method, one can get $2^{\aleph_0} = 2^{\aleph_1}$ = anything permitted by T 1201, but it is not known how to get $2^{\aleph_0} < 2^{\aleph_1}$. The function $\langle 2^{\aleph_\alpha} \mid \alpha > 1 \rangle$ in T 1226 and the function $\langle 2^{\aleph_\alpha} \mid \alpha > 0 \rangle$ in T 1225 can be made to do anything sensible.

T 1231 (Rudin [8]) ZF + AC $\vdash$ If every normal Hausdorff space is countably paracompact, then SH.

P 1232 (Tennenbaum) Prove that Con(ZF) implies Con(ZF + GCH + every normal Hausdorff space is countably paracompact).

D 1233 Let κ be an infinite cardinal. A <u>Souslin κ-tree</u> is a tree of power κ such that every chain and antichain is of power $< \kappa$.

T 1234 ZF + AC $\vdash$ If κ is weakly compact, then there are no Souslin κ-trees.

T 1235 (Hrbáček [H 51]) Let κ be a regular infinite cardinal in a c.s.m. M of ZF + V = L. In M, let $\mathbb{B}$ be the regular open algebra of 2^{κ^+} with the κ^+ topology. Then

$$[\![\text{There is a Souslin } \kappa^+\text{-tree}]\!]^{\mathbb{B}} = \mathbf{1}.$$

Jech's method is the case $\kappa = \omega$. Příkrý has shown that the method works also for adding a κ-tree where κ is inaccessible or Mahlo, (and preserving these properties in the extension); Silver has discovered how to add a κ^+-tree if κ is a singular cardinal.

Several relative consistency results have been obtained by iteration techniques similar to the one just described: in each case, a series of extensions of L is made, each adding an object generic with respect to some collection of subsets of a complete Boolean algebra. A number of people (Kunen; Martin; Rowbottom; Silver) independently realized that much of the pain of iteration arguments could be removed by working with an axiom asserting that the required generic objects exist in the real (and not a Boolean-valued) universe. We select one possible formulation of such a principle: Solovay and Martin intend to write a paper about Martin's version to be entitled "Internal Cohen Extensions."

D 1236 We write (*) for the following assertion: for every complete Boolean algebra $\mathbb{B}$ satisfying the countable chain condition and every family of cardinality at most $\aleph_1$ of subsets of $\mathbb{B}$, there is a homomorphism of $\mathbb{B}$ onto $\mathbf{2}$ complete with respect to every set in the family.

The following result can be proved by iterated forcing:

T 1237 Con(ZF) implies Con(ZF + AC + (*)).

T 1238 ZF + AC + (*) ⊢ $2^{\aleph_0} > \aleph_1$. Both $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_0} > \aleph_2$ are consistent relative to ZF + AC + (*).

T 1226 may now be obtained as a corollary of T 1237 and

T 1239 ZF + AC + (*) ⊢ SH.

T 1240 (Solovay) ZF + AC + (*) ⊢ Every subset of $\aleph_1$ is constructible from a real.

T 1239 is readily seen: if T were a normal Souslin tree, the Tennenbaum destruction algebra described in the discussion of the proof of T 1226 would be a counterexample to (*). T 1240 uses T 1130.

Further consequences of (*) will be given in §3: we note here some topological applications.

T 1241 (Silver) ZF + AC + (*) ⊢ There is a non-metrizable separable normal Moore space.

(For Moore spaces, see Bing [1]).

T 1242 (Booth) ZF + AC + (*) ⊢ 2^{ω_1} is sequentially compact, where 2 has the discrete topology and 2^{ω_1} the product topology.

In contrast to T 1242, 2^c cannot be sequentially compact, where $c = 2^{\aleph_0}$.

T 1243 (Booth) ZF + AC + (*) ⊢ No compact separable metric space of positive dimension is the union of $\aleph_1$ closed sets of dimension 0.

T 1244 (Bukovský) [H55] Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} = 2^{\aleph_1}$ + every uncountable separable metric space contains a non-Borel set).

¶ 3. Life Without Choice

We use Lévy's notation [5] for various weak forms of the axiom of choice, though we abbreviate "$\bigwedge \alpha C_n^{\alpha}$" by "$C_n^{w\alpha}$". By DC^{α} (where α is finite or an aleph) we mean the following:

given a relation R such that for every subset Y of a set X with $\bar{\bar{Y}} < \alpha$ there is an $x \in X$ with $\langle Y, x\rangle \in R$, there is a function $f: \alpha \to X$ such that $\bigwedge \beta < \alpha \, \langle f''\beta, f'\beta\rangle \in R$.

Thus DC^{ω} is the axiom of dependent choices.

[5] gives a thorough treatment of the Fraenkel-Mostowski family of results, some remarks on which will be found in a later section of this saga.

T 1300 (Cohen) Con(ZF) implies Con(ZF + ¬AC).

T 1301 (Cohen) Con(ZF) implies Con(ZF + $\neg C_2^{\omega}$).

T 1302 (Jensen) Con(ZF) implies Con(ZF + C^{ω} + $\neg DC^{\omega}$);

1302 can be strengthened to

T 1303 (Jensen) Let α be an infinite cardinal in a c.s.m. M of ZF + V = L; then M can be extended to an N in which $On^M = On^N$; $Card^M = Card^N$; C^{α} is true but DC^{ω} is false.

On the other hand,

T 1304 (Jensen) ZF ⊢ $\bigwedge \alpha \, DC^{\alpha} \to DC^{\omega}$

T 1305 ZF ⊢ $\bigwedge \alpha \, DC^{\alpha} \to AC$

T 1306 (Solovay, based on work of Feferman and Jensen) Con(ZF) implies Con(ZF + $\bigwedge \alpha C^{\alpha}$ + $\neg DC^{\omega_1}$).

T 1307 (Jech [H 44]) $ZF \vdash$ Let M be a c.s.m. for $ZF + AC$, and α a regular infinite cardinal in M. Then there is an $N \supseteq M$ with the same ordinals such that in N, $\neg DC^{\alpha}$, $\neg C_2^{\alpha}$ and there is a cardinal incomparable with α, but for every $\beta < \alpha$, DC^{β}, C^{β} and β is comparable with every cardinal.

T 1308 (Derrick, Drake; Hájek, Vopěnka [H 31]). Let $\lambda \in On^M$, where M is a c.s.m. of $ZF + V = L$, be an infinite ordinal not the successor of a limit ordinal; then there is an $N \supseteq M$ with the same ordinals and in which V_{λ} is well ordered but $V_{\lambda+1}$ is not.

For a detailed discussion of the finite axioms of choice C_n, the reader is referred to the thesis of Zuckerman; see [*]. The present test problem is

* P 1309 (Mostowski) Does $ZF \vdash C_3 \wedge C_5 \wedge C_{13} \rightarrow C_{15}$?

Gauntt has shown, though, that

T 1310 $ZF \vdash \bigwedge m, n < \omega[C_{m+n} \rightarrow C_m$ <u>or</u> $C_n^{wo}]$, and that consequently

T 1311 (Gauntt) $ZF \vdash C_3 \wedge C_5 \wedge C_{13} \rightarrow C_{15}^{wo}$.

P 1309 would be true if the following were:

P 1312 $ZF \vdash C_{m+n} \rightarrow C_m$ <u>or</u> C_n.

P 1313 Find a finitary procedure for deciding of any $n, n_1, \ldots, n_k$, whether $ZF \vdash C_{n_1} \wedge \ldots \wedge C_{n_k} \rightarrow C_n$.

Gauntt has a partial answer to P 1313 for the system ZFU, which is ZF weakened to admit Urelemente: it is conjectured that his work will transfer to ZF. His results are revealed in §5. We now discuss the relationships of sundry consequences of AC. Some definitions: we write

D 1314 O for "Every set can be totally ordered"

OE for "Every partial ordering can be extended to a total ordering"

BPI for "Every Boolean algebra has a prime ideal"

S for "Every homomorphism of a subalgebra of a Boolean algebra $\mathbb{A}$ into a complete Boolean algebra $\mathbb{B}$ can be extended to an homomorphism of $\mathbb{A}$ into $\mathbb{B}$"

T 1315 (Sikorski) $ZF \vdash AC \rightarrow S$.

T 1316 (Luxemburg) $ZF \vdash S \rightarrow BPI$.

T 1317 $ZF \vdash BPI \rightarrow OE$ (via the compactness theorem) $. \wedge . OE \rightarrow O$.

T 1318 (Cohen) Con(ZF) implies Con(ZF + $\neg$O).

T 1319 (Mathias) Con(ZF) implies Con(ZF + O + $\neg$OE).

T 1320 (Halpern, Läuchli, Lévy) Con(ZF) implies Con(ZF + BPI + $\neg$AC).

* P 1321 Does Con(ZF) imply Con(ZF + OE + $\neg$BPI)?

P 1322 or Con(ZF + BPI + $\neg$S)?

P 1323 or Con(ZF + S + $\neg$AC)?

P 1324 or Con(ZF + BPI + DC^{ω} + $\neg$AC)?

Felgner has proved in ZF without the axiom of foundations that if every total ordering has a cofinal sub-well-ordering, then the power set of a well-ordered set is well-orderable; as a corollary,

T 1325 (Felgner) ZF $\vdash$ If every total ordering has a cofinal sub-well-ordering, then AC.

T 1326 ZF $\vdash$ BPI $\rightarrow$ the Hahn Banach theorem.

P 1327 Is BPI provable in ZF + the Hahn Banach theorem?

T 1328 (Solovay) Con(ZF) implies Con(ZF + DC^{ω} + every endomorphism of the additive subgroup of the real line is continuous.

T 1329 (Lévy) Con(ZF) implies Con(ZF + $cf(\omega_1) = \omega$ + the real line is the countable union of countable sets + $\neg\Sigma^1_3$ - AC)

P 1330 Find a model M such that no extension with the same ordinals satisfies AC. Even better,

P 1331 Find a model in which $\bigwedge \alpha \in On\ (cf(\aleph_\alpha) = \omega)$.

We shall now use German letters m, n, p to denote cardinals, (not necessarily alephs).

P 1332 Does $\bigwedge \mathfrak{m}(\mathfrak{m} \geq \omega \rightarrow \mathfrak{m} + \mathfrak{m} = \mathfrak{m})$ imply AC in ZF?

For other problems of this sort, see Lévy [illegible].

D 1333 $\mathfrak{m}$ adj $\mathfrak{n}$ iff $\mathfrak{m} < \mathfrak{n} \wedge \neg \bigvee \mathfrak{p}[\mathfrak{m} < \mathfrak{p} < \mathfrak{n}]$.

T 1334 (Specker) ZF $\vdash \neg \bigvee \mathfrak{m}[2^{\mathfrak{m}} \leq \mathfrak{m}^2]$; hence

T 1335 (Specker; Lindenbaum, Tarski) ZF $\vdash \bigwedge \mathfrak{m}[\mathfrak{m}$ adj $2^{\mathfrak{m}}$ adj $2^{2^{\mathfrak{m}}} \rightarrow 2^{\mathfrak{m}}$ is an aleph].

P 1336 Does Con(ZF) imply Con(ZF + $\bigvee \mathfrak{m}[\mathfrak{m}^2 \not\leq 2^{\mathfrak{m}}]$)?

P 1337 Does ZF $\vdash \mathfrak{m}$ adj $2^{\mathfrak{m}} \rightarrow \mathfrak{m}$ is an aleph?

T 1338 (Solovay) Con(ZF + V = L + there is an inaccessible cardinal) implies Con(ZF + $\bigvee \mathfrak{m}(\mathfrak{m}$ adj $2^{\mathfrak{m}}$ and $2^{\mathfrak{m}}$ not an aleph)).

P 1339 Is the conclusion of 1338 deducible from Con(ZF)?

P 1340 (Conway) Does Con(ZF) imply Con(ZF + $\neg$ AC + $\bigwedge \mathfrak{m} \bigwedge \mathfrak{n}(\mathfrak{m}\mathfrak{n} = \mathfrak{m} + \mathfrak{n} \rightarrow \mathfrak{m} \leq \mathfrak{n}$ or $\mathfrak{n} \leq \mathfrak{m}$))?

D 1341 (Hartogs) $\aleph(\mathfrak{m})$ = the least ordinal α s.t. $\bar{\bar{\alpha}} \not\leq \mathfrak{m}$.

T 1342 ZF + AC $\vdash \aleph(2^{\aleph_0}) \neq \aleph_\omega$.

T 1343 (Derrick, Drake) Con(ZF) implies Con(ZF + $\aleph(2^{\aleph_0}) = \aleph_\omega$). In fact, given a c.s.m. M of ZF + GCH, and an $\alpha > 0$, $\alpha \in On^M$, there is an $N \supseteq M$ with $Card^N = Card^M$ and $N \models ZF + \aleph(2^{\aleph_0}) = \aleph_\alpha$.

T 1344 ZF $\vdash \mathfrak{m}$ adj $2^{\mathfrak{m}} \wedge 2^{\mathfrak{m}}$ not an aleph $\rightarrow \aleph(\mathfrak{m})$ is a limit cardinal in L and $\aleph(\mathfrak{m}) = \aleph(2^{\mathfrak{m}}) \leq^* 2^{\mathfrak{m}}$.

In the model constructed by Lévy to prove T 1329, $\aleph(2^{\aleph_0}) = \aleph_1$.

P 1345 (Hájek) In Lévy's model, $\neg(\aleph_0$ adj $2^{\aleph_0})$.

By T 1344, $2^{\aleph_0}$ is not adjacent to $2 \exp 2^{\aleph_0}$ in Lévy's model.

D 1346 $DF(\mathfrak{m})$ for $\aleph(\mathfrak{m}) = \aleph_0$ ("Dedekind finite").

P 1347 Does Con(ZF) imply

$$Con(ZF + \bigvee \mathfrak{m}(DF(\mathfrak{m})) + \bigwedge \mathfrak{m} \bigwedge \mathfrak{n}(DF(\mathfrak{m}) \wedge DF(\mathfrak{n}) \rightarrow . \mathfrak{m} < \mathfrak{n} \text{ or } \mathfrak{n} < \mathfrak{m}))?$$

The paper of Ellentuck [2] describes researches into the theory of Dedekind finite cardinals.

Other results for theories negating the axiom of choice are described in §3.

T 1348 (Derrick, Drake) Con(ZF) implies

$$Con(ZF + \neg AC + \bigwedge \alpha(\aleph(2^{\aleph_\alpha}) = \aleph_{\alpha+2})).$$

P 1349 Does $ZF \vdash \neg \bigwedge \alpha(\aleph(2^{\aleph_\alpha}) = \aleph_{\alpha+1})$?

We end with a theorem about the possible patterns of cardinals, and an answer to a question of Tarski.

T 1350 (Jech [H 36]) $ZF \vdash$ Let M be a c.s.m. for $ZF + AC$; $\langle A, \preceq \rangle$ a partially ordered set in M. Then there is an $N \supseteq M$ with the same ordinals and a set $B \in N$ of cardinals such that

$$\langle A, \preceq \rangle \cong \langle B, \leq \rangle .$$

P 1351 Under what conditions on $\langle A, \preceq \rangle$ is there a $B \in N$, such that

$$\langle A, \preceq \rangle \cong \langle C, \leq \rangle$$

where C is the set of cardinals of subsets of B?

D 1352 (Tarski) $\mathfrak{m}$ is a strong successor of $\mathfrak{n}$ iff

$$\mathfrak{m} > \mathfrak{n} \wedge \bigwedge \mathfrak{p}(\mathfrak{p} < \mathfrak{m} \rightarrow \mathfrak{p} \leq \mathfrak{n}).$$

T 1353 (Jech [H 40]) Con(ZF) implies $Con(ZF + \bigvee \mathfrak{m}(\mathfrak{m}$ has no strong successor)).

P 1354 Does ZF + "Every cardinal has a strong successor" $\vdash$ AC?

¶4. Proving two-quantifier statements.

Lévy has classified the sentences AC, GCH, $V = L$ in his hierarchy [6].

T 1400 (Lévy) AC, GCH, $V = L$ are all Π_2 sentences. Let $\Phi \in \Sigma_2$.

T 1401 (Lévy) If $ZF + GCH \vdash \Phi \to V = L$ then $ZF + GCH \vdash \neg\Phi$.

T 1402 (Lévy) If $ZF + AC \vdash \Phi \to GCH$ then $ZF + AC \vdash \neg\Phi$.

T 1403 (Lévy) If $ZF \vdash \Phi \to AC$ then $ZF \vdash \neg\Phi$.

T 1404 (Lévy) If $ZF \vdash \Phi \to C_2^{\omega}$, then $ZF + AC \vdash \neg\Phi$.

The proofs are described in Lévy's lecture [*].

The predicate "$x = 2^{\aleph_0}$" is Π_2: we may consider characterizing cardinals κ by properties expressible in higher order logic. Let Σ^1_n ... refer to this hierarchy. Then the cardinal $2^{\aleph_0}$ can be characterized by the Σ^1_2 property that it admits relations giving it the structure of a complete ordered field.

T 1405 (Kunen) Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} < \aleph_{\omega_1}$ + the cardinal $2^{\aleph_0}$ is not Π^1_2-characterizable).

The lecture of Kunen [*] gives a full discussion.

§2. LARGE CARDINALS

0. Measurability and Compactness

D 2000 κ is 2VM iff κ is a 2-valued measurable cardinal.

κ is RVM iff κ is real-valued - but not 2-valued measurable.

κ is A-Mahlo (where $A \subseteq \kappa$) iff every nonempty closed unbounded subset of κ contains a member of A and κ is weakly inaccessible.

If $A = \{\alpha < \kappa | \alpha$ is regular$\}$, $\{\alpha < \kappa | \alpha$ is weakly inaccessible$\}$, $\{\alpha < \kappa | \alpha$ is strongly inaccessible$\}$, then κ is Mahlo, weakly Mahlo, strongly Mahlo, respectively.

$i(\kappa,\lambda)$ iff κ, λ are cardinals, and κ carries a λ-saturated κ-complete nontrivial ideal.

An ideal I on κ is normal iff for every function $f: \kappa \to \kappa$ such that $\{\alpha | f(\alpha) < \alpha\} \notin I$, there is a $\beta < \kappa$ with $\{\alpha | f(\alpha) = \beta\} \notin I$.

A filter on κ is normal iff its dual ideal is.

A measure on κ is normal iff the ideal of sets of measure 0 is.

The following subsumes all previously popular partition properties.

D 2001 (Moschovakis) Let κ, λ, ν be cardinals, μ an ordinal.

$\kappa \underset{\lambda}{\to} (\mu)^{<\omega}_{\nu}$ iff for every f with domain the set of sequences of length λ of finite subsets of κ and range a subset of ν (in symbols, $f:([\kappa]^{<\omega})^{\lambda} \to \nu$), there is a sequence $\langle x_\eta | \eta < \lambda\rangle$ of subsets of κ, each of order type μ, (i.e. $\langle x_\eta, \in \restriction x_\eta\rangle \cong \mu$), with the property that for every s and t in the domain of f, if for all $\eta < \lambda$, $s_\eta \subseteq x_\eta$, $t_\eta \subseteq x_\eta$, and $\bar{\bar{s}}_\eta = \bar{\bar{t}}_\eta$, then $f(s) = f(t)$.

The case $\lambda = 1$ is the Erdös-Rado property $\kappa \to (\mu)^{<\omega}_{\nu}$. The case $\nu = 2$ will be written $\kappa \underset{\lambda}{\to} (\mu)^{<\omega}$.

κ is a <u>Ramsey</u> cardinal iff $\kappa \to (\kappa)^{<\omega}$. κ is a <u>Moschovakis</u> cardinal iff $\kappa \underset{\omega_1}{\longrightarrow} (\omega_1)^{<\omega}$.

D 2002 We write MC for $\forall\kappa(\kappa$ is 2VM)

RMC for $\forall\kappa(\kappa$ is RVM)

SCC for $\forall\kappa(\kappa$ is strongly compact)

SuCC for $\forall\kappa(\kappa$ is supercompact).

The next theorems list the basic relations between these properties.

T 2003 ZF + AC $\vdash$ κ supercompact $\to \kappa$ strongly compact; κ strongly compact $\to \kappa$ 2VM; κ 2VM $\to \kappa$ Ramsey; $\beta < \alpha \leq \kappa \wedge \kappa \to (\alpha)^{<\omega} . \to \kappa \to (\beta)^{<\omega}$; $i(\kappa,\lambda) \wedge \lambda' > \lambda . \to i(\kappa,\lambda')$; κ RVM $\to i(\kappa,\omega_1)$; $i(\kappa,\lambda^+) \to \kappa$ is 2VM or $\kappa \leq 2^{\lambda}$; κ 2VM iff $i(\kappa,2)$ κ is weakly compact and strongly inaccessible iff κ is Π^1_1 indescribable; κ Ramsey $\to \kappa$ Π^1_1 indescribable; κ Π^1_1 indescribable $\to$ Mahlo; κ 2VM $\to \kappa$ Π^2_1 indescribable.

D 2004 $\kappa(\alpha)$ = the least λ s.t. $\lambda \to (\alpha)^{<\omega}$, if one exists.

T 2005 (Rowbottom) ZF + AC $\vdash$ $\lim(\beta) \wedge \forall\lambda(\lambda \to (\beta)^{<\omega}) \wedge \alpha < \beta . \to \kappa(\alpha) < \kappa(\beta)$.

T 2006 (Silver) ZF + AC $\vdash$ If $\kappa(\omega)$ exists then there is a $\lambda < \kappa(\omega)$ which is Π^m_n indescribable for all m and n; $\kappa(\omega)$ itself is Π^1_1 describable.

T 2007 (Erdös) ZF + AC $\vdash$ If α is a limit ordinal and $\forall\lambda(\lambda \to (\alpha)^{<\omega})$ then $\kappa(\alpha)$ is strongly inaccessible.

T 2008 (Silver) ZF + AC $\vdash$ If $\lim(\alpha)$ and $\forall\lambda(\lambda \to (\alpha)^{<\omega})$, then for all cardinals $\nu < \kappa(\alpha)$, $\kappa(\alpha) \to (\alpha)^{<\omega}_{\nu}$. In particular, if α is Ramsey, (i.e. if $\alpha = \kappa(\alpha)$) then $\alpha \to (\alpha)^{<\omega}_{\nu}$ for all $\nu < \alpha$.

T 2009 ZF + AC $\vdash$ If κ is 2VM, there is a $\lambda < \kappa$ which is Ramsey.

T 2010 ZF + AC $\vdash$ If κ is weakly compact, there is a $\lambda < \kappa$ which is Mahlo.

Little is known about the size of Moschovakis cardinals relative to other large numbers; Silver has made the following simple observation:

T 2011 ZF + AC $\vdash$ If there are both Moschovakis and strongly compact cardinals, then the first Moschovakis cardinal is smaller than the first strongly compact.

We now ponder particulars.

Kunen has very recently solved the problem of the number of normal measures admitted by a 2VM cardinal.

T 2012 (Scott) ZF + AC $\vdash$ if κ is 2VM, then κ has at least one normal measure.

T 2013 (Kunen) ZF + GCH $\vdash$ Let κ be 2VM, μ a normal measure, R a set of regular infinite cardinals $< \kappa$ with $\mu(R) = 0$, F an increasing map from R into κ such that $\forall\alpha \in R(F(\alpha)$ is a cardinal $\wedge$ $cf(F(\alpha)) > \alpha)$, and λ a cardinal $\geq \kappa^{++}$ with $cf(\lambda) > \kappa^+$. Then there is a Boolean extension of the universe preserving cardinals in which

(i) $2^{\kappa} = \kappa^{+}; 2^{\kappa^{+}} = \lambda$

(ii) $\forall \alpha \in R(2^{\alpha} = F(\alpha))$

(iii) κ has λ distinct normal ultrafilters.

P 2014 Let κ be 2VM; can a measure on κ be defined in $V_{\kappa+n}$, for some $n < \omega$?

T 2015 (Solovay) $ZF + AC \vdash$ If κ is 2VM, there is always at least one normal μ such that $\mu(\{\alpha < \kappa | \alpha$ is 2VM$\}) = 0$.

Let κ be 2VM, μ, ν' normal measures. Denote by $L^{\mu'}$ the universe constructible from μ', in the sense of Levý: informally $L^{\mu'}$ is the smallest universe W containing all ordinals and $\mu' \cap W$. Then $\mu = \mu' \cap L^{\mu'}$ is a normal measure on κ in $L^{\mu'} = L^{\mu}$: further, setting $\nu = \nu' \cap L^{\nu'}$,

T 2016 (Kunen) $\mu = \nu$;

T 2017 (Silver [*]) GCH is true in L^{μ}, and

T 2018 (Kunen) μ is the only normal measure on κ in L^{μ}.

T 2019 It is also known that κ is the only 2VM cardinal in L^{μ}, and so (by T 2018, which was proved before but follows from T 2016) κ and μ are definable in L^{μ}.

Formally, T2017 states that GCH is a theorem of the theory $ZF + AC + \bigvee \kappa, \mu(\mu$ is a normal two-valued κ-additive measure on κ and $V = L^{\mu})$; in future this theory will be indicated by $ZF + AC + V = L^{\mu}$.

T 2020 (Solvay) $ZF + AC \vdash$ If κ is supercompact, then there is a normal measure μ on κ with $\mu(\{\alpha < \kappa | \alpha$ is 2VM$\}) = 1$.

Using this and T 2016, Solovay has proved

T 2021 $ZF + AC \vdash$ If κ is supercompact, then κ has more than 2^{κ} normal measures.

P 2022 Show that the first strongly compact cardinal κ has a normal measure μ for which $\mu(\{\alpha < \kappa | \alpha$ is 2VM$\}) = 1$.

The first indication that the presence of a 2VM cardinal affected the relationship of L to the universe was

T 2023 (Scott) $ZF + AC + MC \vdash V \neq L$.

(Much more startling things are known now - see T 2033.) A related result has been obtained assuming a strongly compact cardinal. First, a definition:

D 2024 An ultrafilter U on λ (U not necessarily λ additive) is <u>uniform</u> iff

$$\bigwedge x \subseteq \lambda(\bar{\bar{x}} < \lambda \rightarrow x \notin U).$$

T 2025 (Vopěnka, Hrbáček, [H41]) $ZF + AC \vdash$ If there is a uniform ultrafilter on λ^{+} then $V \neq L[a]$ for any $a \subseteq \lambda$.

This result has two corollaries:

T 2026 (Vopěnka, Hrbáček) $ZF + AC + SCC \vdash \bigwedge y \bigvee x[x \notin L^{y}]$.

T 2027 (Vopěnka, Horbáček) Con(ZF + AC + MC) implies Con(ZF + AC + $\bigvee \kappa(\kappa$ is 2VM and there is no uniform ultrafilter on $(2^{\kappa})^{+}$)).

P 2028 (Vopěnka, Hrbáček) Show that ZF + AC $\vdash$ If κ is 2VM, then there is a κ-additive uniform ultrafilter on 2^{κ}.

T 2026 says that if there is a strongly compact cardinal, the universe is not constructible from a set. Let now κ be 2VM, μ a normal measure on κ. By Silver's work (T 2017), κ is 2VM in L^{μ}, and by T 2026, $L^{\mu} \models \neg$ SCC, so that "strongly compact" and "2VM" are not provably the same. That SCC is stronger than MC was shown recently by Kunen, who obtained

T 2029 ZF + AC + SCC $\vdash$ Con(ZF + AC + MC)

as a corollary to a much stronger result, T 2044; he has very recently improved T 2029 to

T 2030 ZF + AC + SCC $\vdash \bigwedge \bar{\alpha} \bigvee M$(M is a transitive model for ZF + AC $\wedge\ \alpha \in M \wedge M \models$ "There are α 2VM cardinals").

The following problem, much milder than P 2022, is still open.

P 2031 Show ZF + AC + SCC + "the first 2VM is not strongly compact" consistent, relative to something else.

One of the more interesting features of large cardinals is their stunning effect on the constructible universe. Gaifman and Rowbottom independently strengthened Scott's theorem T 2023, showing inter alia that

T 2032 ZF + AC + MC $\vdash$ there are only countably many constructible reals.

We shall now state the Hauptsatz of Silver's doctoral dissertation, after which we shall note some of the striking consequences and indicate the advances made by Rowbottom and Gaifman.

T 2033 (Silver) ZF + AC + $\bigvee \kappa(\kappa \to (\omega_1)^{<\omega})$ $\vdash$ For every uncountable cardinal α there is an $X_{\alpha} \subseteq \alpha$ with the properties

(i) X_{α} is closed and unbounded in α;

(ii) every $x \in L_{\alpha}$ is definable in L_{α}, allowing parameters from X_{α}; and

(iii) X_{α} is a set of indiscernibles for L_{α}: viz., if $n \in \omega$; $\kappa_0 < \cdots < \kappa_{n-1}$, $\kappa'_0 < \cdots < \kappa'_{n-1}$ are two sequences of elements of X_{α}; and $\mathfrak{A}(\xi_0, \ldots, \xi_{n-1})$ is a wff of the language of set theory with precisely the n free variables shown, then

$$L_{\alpha} \models \mathfrak{A}[\kappa_0, \ldots, \kappa_{n-1}] \longleftrightarrow L_{\alpha} \models \mathfrak{A}[\kappa'_0, \ldots, \kappa'_{n-1}] .$$

Further, set $X = \bigcup_{\alpha = \bar{\alpha} > \omega} X_{\alpha}$; then

(iv) X is closed and unbounded and contains every uncountable cardinal;

(v) $\beta \in X \wedge \beta' \in X \wedge \beta < \beta' \to L_{\beta} \prec L_{\beta'}$;

(vi) $\beta \in X \to \beta \cap X$ is a set of indiscernibles for L_{β};

(vii) $\beta \in X \to L_{\beta} < L$; and

(viii) X is a class of indiscernibles for L.

Two striking corollaries;

T 2034 (Rowbottom) ZF + AC + $\bigvee \kappa(\kappa \to (\omega_1)^{<\omega}) \vdash$ If α is a countable ordinal, then V_{α} contains only countably many constructible sets.

T 2035 ZF + AC + $\bigvee \kappa(\kappa \to (\omega_1)^{<\omega}) \vdash$ Let X be as in T 2033. Then the first $\alpha \in X$ is countable; and every $\alpha \in X$ is weakly compact and $\to (\omega)^{<\omega}$ in L.

Gaifman working in the theory ZF + AC + MC had noticed that if α, β are uncountable cardinals and $\alpha < \beta$, then $L_\alpha \prec L_\beta \prec L$. Rowbottom had realized the significance of the Ramsey properties, had strengthened Erdös and Hajnal's theorem that κ 2VM $\rightarrow \kappa$ Ramsey to

T 2036 $ZF + AC \vdash$ Let κ be 2VM, μ a normal measure. Let $f:[\kappa]^{<\omega} \rightarrow \lambda$, where $\lambda < \kappa$, be a partition. Then there is a $y \subseteq \kappa$ with $\mu(y) = 1$ which is a set of indiscernibles for f.

and had noted that if $\bigvee \kappa(\kappa \rightarrow (\omega_1)^{<\omega})$, then there is a non-constructible real. Moreover, he formulated the notion of a remarkable set of sentences, one of the key concepts in the proof of T 2033 (cf. Silver's thesis). A model-theoretic property of measurable cardinals established by Rowbottom is discussed in §4.

D 2037 (Solovay [12]) Assume $\bigvee \kappa(\kappa \rightarrow (\omega_1)^{<\omega})$. Define

$$0^\# = \text{the theory of } \langle L_{\aleph_\omega}, \epsilon, \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots \rangle_{n<\omega},$$

(where $\aleph_i$ = the real $\aleph_i$).

By T 2033, $L_{\aleph_\omega} \prec L$, and the $\aleph_n$ $(1 \leq n < \omega)$ are a set of indiscernibles for L: so that $0^\#$, which by suitably Gödel-numbering the language may be regarded as a set of integers, is "the theory of L with ω indiscernibles."

T 2038 (Solovay) There is a Π^1_2 predicate A of reals such that $ZF \vdash \bigvee x A(x) \rightarrow \bigvee ! x A(x)$, and such that $ZF + AC + \bigvee \kappa(\kappa \rightarrow (\omega_1)^{<\omega}) \vdash A(0^\#)$. Furthermore, the assertion that $\bigvee_x A(x)$ is equivalent in ZF to the conclusion of T 2033.

Instead of $\bigvee x A(x)$, we shall say "$0^\#$ exists." Thus if $0^\#$ exists, there is a class X of ordinals satisfying (i) - (viii) of T 2033, where $X_\alpha = X \cap \alpha$ for an uncountable cardinal α. It follows from T 2038 that $0^\#$, if it exists, is a non-constructible real which is Δ^1_3, and in which every constructible real is recursive. As the theory of L may be read off from $0^\#$, it may be shown that every ordinal definable in L is either finite or isomorphic to a Δ^1_3 well ordering of ω, and so countable. In the theory ZF + "$0^\#$ exists", one may prove Rowbottom's theorem T 2034 and that every $\alpha \in X$ is strongly Mahlo in L.

The moral of the next theorem is that two measurable cardinals are much better than one.

T 2039 (Solovay) $ZF + AC \vdash$ Suppose κ and λ are two measurable cardinals with $\kappa < \lambda$, and let μ be a normal measure on κ. Then there is an $X \subseteq \kappa$ and a $Y \subseteq On$ such that

(i) X is closed and unbounded in κ

(ii) Y is closed and unbounded, and $\kappa < \min(Y)$

(iii) $X \cup Y$ contains every uncountable cardinal except κ

(iv) if $\mathfrak{A}(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is any formula of the language of set theory (with predicates $\mathcal{E}$, $\equiv$) enriched by a name for κ and a name for μ, with precisely the free variables shown, and if $\alpha_1, \ldots, \alpha_n$, $\alpha'_1, \ldots, \alpha'_n$ are in X, with $\alpha_1 < \cdots < \alpha_n$, $\alpha'_1 < \cdots < \alpha'_n$, and $\beta_1, \ldots, \beta_m$, $\beta'_1, \ldots, \beta'_m$ are in Y, with $\beta_1 < \cdots < \beta_n$, $\beta'_1 < \cdots < \beta'_m$, then

$$L^\mu \models \sigma[\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m]$$

$$\longleftrightarrow L^\mu \models \sigma[\alpha_1', \ldots, \alpha_n', \beta_1', \ldots, \beta_m'] .$$

(v) every element of L^μ is definable in L^μ, allowing parameters from $X \cup Y$.

Properties akin to the more precise clauses of T 2033 may be established for L^μ_α, where α is an uncountable cardinal. In particular, we may use the analogue for (vii) to formalize the following in set theory:

D 2040 $0^\dagger$ = the theory of $\langle L^\mu, \epsilon, \kappa, \mu, \aleph_1, \aleph_2, \ldots, \kappa^+, \kappa^{++}, \ldots\rangle$. $0^\dagger$ is the theory of L^μ with ω indiscernibles from X and ω indiscernibles from Y.

T 2041 (Solovay) There is a Π^1_2 predicate of reals B such that $ZF \vdash \bigvee x B(x) \rightarrow \bigvee ! x B(x)$ and such that $ZF + AC \vdash$ If κ is 2VM and $\kappa < \lambda$ and λ is 2VM, then $B(0^\dagger)$ where $0^\dagger$ is defined by D 2040. We shall write $\bigvee x B(x)$ as "$0^\dagger$ exists."

D 2042 $\mathcal{F}_\alpha$ = the filter generated by the closed unbounded subsets of α. $U_\alpha = \mathcal{F}_\alpha \cap L^{\mathcal{F}\alpha}$

T 2043 (Solovay) $ZF \vdash$ Suppose $0^\dagger$ exists. Then for every uncountable aleph α, U_α is a normal ultrafilter on α in L^{α}: so that α is 2VM in $L^{\mathcal{F}\alpha} = L^{U_\alpha}$. Further α^+ is strongly inaccessible in $L^{\mathcal{F}\alpha}$, so that the existence of a standard model of ZF + AC + MC may be proved.

T 2044 (Kunen) $ZF + AC + SCC \vdash 0^\dagger$ exists.

T 2029 is an immediate consequence of T 2043 and T 2044.

Chang has investigated large cardinal axioms using infinitary languages. L may be defined hierarchically using the ordinary language of set theory, $L_{\omega,\omega}$. Let $M^\alpha_{\kappa,\kappa}$ denote the α^{th} level of the corresponding hierarchy for $L_{\kappa,\kappa}$. The L_α mentioned above is in this new notation $M^\alpha_{\omega,\omega}$. Write $M^{On}_{\kappa,\kappa} = \bigcup_{\alpha \in On} M^\alpha_{\kappa,\kappa}$. Then

T 2045 (Solovay; Lévy; Silver) $ZF + AC \vdash \kappa$ 2VM $\rightarrow M^{On}_{\kappa^+,\kappa^+} \neq V$.

T 2046 (Chang) $ZF + AC + \bigvee \kappa(\kappa \rightarrow (\omega_1)^{<\omega}) \vdash$ Let λ, μ, ν be infinite regular cardinals with $\lambda < \mu < \nu$. Then $M_{\omega,\omega} \prec_{L_{\mu,\lambda}} M^\nu_{\omega,\omega}$.

✱ P 2047 (Chang) Show $ZF + AC + \bigvee \kappa(\kappa \rightarrow (\kappa)^{<\omega}) \vdash V \neq M^{On}_{\omega_1,\omega_1}$.

T 2047 (Silver) Con(ZF+AC+MC) implies

$$\mathrm{Con}(ZF{+}AC{+}MC{+}\neg M^{\omega_2}_{\omega_1,\omega_1} \prec M^{O_n}_{\omega_1,\omega_1}) .$$

Further results and problems are given in Chang's lecture notes [*]. We now discuss real-valued measurable cardinals. Inspired by the work of Hanf [3] on weakly compact cardinals, Keisler and Tarski [4] proved

T 2048 $ZF + AC \vdash$ If κ is 2VM, then κ is strongly Mahlo.

Three people independently adapted the methods of Keisler and Tarski to show that RVM's enjoy a similar remoteness:

T 2049 (Solovay; Jensen; Fremlin) ZF + AC $\vdash$ Let κ be RVM. Then κ has a normal measure, and is weakly Mahlo.

Solovay, indeed, obtained these results in a more general setting.

* T 2050 (Solovay) ZF + AC $\vdash$ Suppose $i(\kappa,\kappa)$. Then κ has a normal κ-complete κ-saturated ideal I, and for any $B \subseteq \kappa$, $B \notin I$,

$$\{\alpha < \kappa \mid \alpha \text{ is not } B\text{-Mahlo}\} \in I .$$

Further, $V \neq L$. Moreover, if $i(\kappa,\lambda)$ for some $\lambda < \kappa$, then κ has a normal κ-complete λ-saturated ideal, I, say. L^I satisfies GCH above λ, and $I \cap L^I$ is a λ-saturated ideal in L^I: it follows that κ is 2VM in L^I.

Indeed, Kunen's result T 2016 shows that $I \cap L^I$ is a normal prime ideal, and so the full GCH is true in L^I.

P 2051 What are the possible cardinalities of minimal generating sets for the measure algebra of a real-valued measurable cardinal, or for an ω_1-saturated Boolean-valued measurable cardinal?

If $\kappa = 2^{\aleph_0}$ is RVM, then every minimal generating set for its measure algebra is of cardinality $> \kappa$.

* D 2052 $k(\kappa)$ iff $i(\kappa,\kappa)$.^. $\bigwedge \lambda < \kappa(\neg i(\kappa,\lambda))$.

T 2053 (Lévy) ZF + AC $\vdash$ κ strongly inaccessible and $k(\kappa)$.→ κ not weakly compact.

It is readily seen that $k(\kappa) \to \kappa > \omega$. For the next two questions, consistency results and proofs from large cardinals are both interesting.

P 2054 Is there a κ such that $k(\kappa)$?

P 2055 Is there a strongly inaccessible κ such that $k(\kappa)$?

P 2056 Does ZF + GCH $\vdash \neg \bigvee \kappa\, k(\kappa)$

* P 2057 Prove Con(ZF+AC+ω_1 has an ω_1-complete ideal I (even better: ω_2-saturated) with $S(\omega_1)/I$ complete), assuming Con(ZF+AC+ some large cardinal).

Part of T 2050 was proved in Solovay's second lecture [*]; in his last, he applied properties of saturated ideals to solve a problem of Födor and Hajnal:

D 2058 Let κ be a cardinal. $B \subseteq \kappa$ is <u>stationary</u> iff it intersects every closed and unbounded subset of κ.

* T 2059 (Solovay [*] ZF + AC $\vdash$ Let κ be an uncountable cardinal. Then every stationary subset of κ is the union of κ disjoint stationary subsets of κ.

P 2060 (Erdös) Is there a stationary subset of $\aleph_1$ which is the union of $2^{\aleph_1}$ stationary subsets the intersection of any two of which is not stationary?

Lastly, a problem about a simplified form of the Moschovakis property.

P 2061 Does $\omega_2 \xrightarrow[\omega]{} (2)^1_2$?

¶1. Relative Consistency Results.

T 2100 (Silver) $\mathrm{Con}(ZF + AC + \bigvee\kappa(\kappa \rightarrow (\omega)^{<\omega})$ implies

$\mathrm{Con}(ZF + V = L + \bigvee\kappa(\kappa \rightarrow (\omega)^{<\omega}))$.

This undoes one weakening of the hypothesis of T 2033 and is a corollary of

T 2101 (Silver) $ZF + AC \vdash$ If $\kappa \rightarrow (\omega)^{<\omega}$ then it is true in L that $\kappa \rightarrow (\omega)^{<\omega}$.

T 2102 $ZF + AC \vdash$ If κ is weakly compact, then in L it is true that κ is weakly compact.

T 2103 Con(ZF + AC + there is a weakly compact cardinal) implies

Con(ZF + V = L + there is a weakly compact cardinal).

The problem of the relative consistency of CH to MC has been settled by Lévy and Solovay [7]: the results

T 2104 Con(ZF + AC + MC) implies Con(ZF + AC + MC + CH)

and

T 2105 Con(ZF + AC + MC) implies Con(ZF + AC + MC + $\neg$CH).

follow from their basic lemma that measurable cardinals remain measurable under small extensions by forcing:

T 2106 (Lévy; Solovay) $ZF + AC \vdash$ Let κ be 2VM and $\mathbb{B}$ a complete Boolean algebra of cardinality $< \kappa$. Then

$$[\![\check{\kappa} \text{ is 2VM}]\!]^{\mathbb{B}} = \mathbb{1}.$$

Thus for example both SH and its negation are consistent relative to ZF + AC + MC. The following theorems give similar results for the appropriate theories:

T 2107 (McAloon) $ZF + AC \vdash$ Let κ be strongly compact, and $\mathbb{B}$ a complete Boolean algebra of cardinality $< \kappa$; then

$$[\![\check{\kappa} \text{ is strongly compact}]\!]^{\mathbb{B}} = \mathbb{1}.$$

T 2108 $ZF + AC \vdash$ Let α be a limit ordinal for which $\bigvee\lambda(\lambda \rightarrow (\alpha)^{<\omega})$, and $\mathbb{B}$ a complete Boolean algebra of cardinality $< \kappa = \kappa(\alpha)$. Then

$$[\![\kappa \rightarrow (\check{\alpha})^{<\omega}]\!]^{\mathbb{B}} = \mathbb{1}.$$

In particular, if α is a Ramsey cardinal, then $\kappa = \alpha$ and

$$[\![\check{\alpha} \text{ is a Ramsey cardinal}]\!]^{\mathbb{B}} = \mathbb{1}.$$

T 2108 is proved by applying T 2008 with $v = \bar{\mathbb{B}}$.

The theorem

T 2109 (Jensen; Silver) Con(ZF + AC + MC) implies Con(ZF + GCH + MC)

is a consequence of both T 2017 and T 2110. Silver's method is to contract the universe: Jensen expands it.

T 2110 (Jensen) ZF $\vdash$ Every c.s.m. M of ZF + AC can be extended to a c.s.m. N of ZF + GCH which has the further properties

(i) $On^N = On^M$;

(ii) if κ is 2VM in M, then κ is 2VM in N;

(iii) if λ is a limit ordinal, κ an ordinal and $\kappa \to (\lambda)^{<\omega}$ in M, then $\kappa \to (\lambda)^{<\omega}$ in N;

in particular,

(iv) κ Ramsey in M implies κ Ramsey in N;

and

(v) if $\kappa \to (\omega_1)^{<\omega}$ in M, then $\kappa \to (\omega_1)^{<\omega}$ in N.

Clause (v) holds because ω_1 is preserved in the extension. Standard techniques translate the proof of T 2110 into proofs of

T 2111 (Jensen) Con(ZF + AC + there is a Ramsey cardinal) implies Con(ZF + GCH + there is a Ramsey cardinal)

T 2112 (Jensen) Con(ZF + AC + $\bigvee\kappa(\kappa \to (\omega_1)^{<\omega})$) implies Con(ZF + GCH + $\bigvee\kappa(\kappa \to (\omega_1)^{<\omega})$).

P 2113 Does Con(ZF + AC + SCC) imply Con(ZF + GCH + SCC)?

P 2114 Does Con(ZF + AC + SuCC) imply Con(ZF + GCH + Succ)?

T 2115 (Scott) ZF + AC $\vdash \kappa$ 2VM $\wedge \bigwedge\lambda < \kappa\ 2^\lambda = \lambda^+ . \to 2^\kappa = \kappa^+$.

More generally, if κ is 2VM, μ a normal measure and $\mu\{\lambda < \kappa | \lambda = \bar{\lambda} \wedge 2^\lambda = \lambda^+\} = 1$ then $2^\kappa = \kappa^+$.

A number of results of the following type may also be proved.

T 2116 (Vopěnka [H27]) ZF + AC $\vdash$ Let κ be 2VM. Then

(i) if $\gamma < \kappa$ and $\bigwedge\alpha < \kappa\ 2^{\aleph_\alpha} \leq \aleph_{\alpha+\gamma}$, then $2^\kappa \leq \aleph_{\kappa+\gamma}$;

(ii) if $\bigwedge\alpha < \kappa\ 2^{\aleph_\alpha} \leq \aleph_{\alpha+\alpha}$, then $2^\kappa \leq \aleph_{\kappa+\kappa}$.

T 2115 and T 2116 relate to the following important questions.

P 2117 Does ZF + AC $\vdash \kappa$ 2VM $\to 2^\kappa = \kappa^+$?

T 2118 (Přikrý) ZF $\vdash$ Let M be a c.s.m. of ZF + AC + MC, and let κ be 2VM in M; then there is a c.s.m. N of ZF + AC extending M, such that

(i) $On^N = On^M$;

(ii) $Card^N = Card^M$;

(iii) for all cardinals $\lambda < \kappa$, $S^N(\lambda) = S^M(\lambda)$; and

(iv) $cf^N(\kappa) = \omega$.

To prove this we use forcing to add an ascending ω-sequence of ordinals, f, with $f''\omega$ cofinal in κ. A condition is a pair, the first element of which is a finite ascending sequence of ordinals $\alpha_0 < \dots < \alpha_n < \kappa$; the second is a set $A \subseteq \kappa$, with $\mu(A) = 1$, where μ is some fixed normal measure on κ, and $\lambda \in A \to \alpha_n < \lambda$. The intended interpretation is "$f(0) = \alpha_0 \wedge \dots \wedge f(n) = \alpha_n \wedge \bigwedge m > n\ f(m) \in A$", and the set of conditions is partially ordered accordingly. Evidently in the extension f is an ω-sequence cofinal in κ; that no new subsets of earlier ordinals are introduced follows from the lemma:

T 2119 (Přikrý) Let $\mathfrak{A}$ be a sentence of the forcing language, and $\langle\langle\alpha_0, \ldots, \alpha_n\rangle, A\rangle$ a condition: then there is an $A' \subseteq A$ of μ-measure 1 such that $\langle\langle\alpha_0,\ldots,\alpha_n\rangle, A'\rangle \Vdash \mathfrak{A}$ or $\langle\langle\alpha_0,\ldots,\alpha_n\rangle, A'\rangle \Vdash \neg\mathfrak{A}$,

which is proved using the Ramsey properties of κ.

T 2118 gives a partial solution to P 1207 and, if the answer to P 2117 is negative, may lead to a solution of the singular cardinals problem. Přikrý has proved the following by related methods:

T 2120 $ZF \vdash$ Let M be a c.s.m. of ZF + AC + SCC. Let κ be strongly compact in M, and let $\lambda \in M$ be a regular cardinal greater than κ. Then there is an extension N of M with the same ordinals such that

(i) N is a model of ZF + AC;

(ii) $\bigwedge \alpha < \kappa(S^N(\alpha) = S^M(\alpha))$;

(iii) $\bar{\lambda}^N = \kappa$;

(iv) $cf^N(\kappa) = \omega$; and

(v) $\bigwedge \beta > \lambda$ (β a cardinal in $M \to \beta$ a cardinal in N).

Thus a negative answer to the following would be useful:

P 2121 Does $ZF + AC \vdash \kappa$ strongly compact $\to \bigwedge \lambda > \kappa(2^\lambda = \lambda^+)$?

P 2122 Does $ZF + AC \vdash \kappa$ weakly compact $\to 2^\kappa = \kappa^+$?

T 2123 (Kunen) Let κ be 2VM; then $\{\alpha < \kappa \mid 2^{\aleph_\alpha} > \aleph_{\alpha+1}\}$ may be of normal measure 1.

For by T 2013 (iii) κ may have two normal measures μ and ν; apply T 2013 (ii) with an R for which $\mu(R) = 0$, $\nu(R) = 1$.

By T 2049 and T 2003 we know that GCH implies that there are no RVM cardinals: but

T 2124 (Solovay) Con(ZF + AC + MC) implies and is implied by Con(ZF + AC + $2^{\aleph_0}$ is RVM).

We give a sketch of the argument. Silver's proof of T 2017 is the paradigm for one direction: if μ is a normal real-valued measure on κ, form L^μ. The theorem of Rowbottom used in Silver's proof, T 4009, can be adapted to show that GCH holds in L^μ; or we can use Kunen's remark following T 2050 that $\mu \cap L^\mu$ is a two-valued measure in L^μ. The method for the other direction parallels an earlier proof of Přikrý where κ generic reals were added to a model in which κ was 2VM: in the extension $2^{\aleph_0} = \kappa$, $i(\kappa, \omega_1)$, but κ was not RVM. Thus

T 2125 (Přikrý) Con(ZF + AC + MC) implies Con(ZF + AC + $\neg$RMC + $i(2^{\aleph_0}, \aleph_1)$).

T 2124 is proved by adding κ random reals: then in the extension $2^{\aleph_0} = \kappa$ and κ is RVM.

T 2126 (McAloon) Con(ZF + AC + MC) implies Con(ZF + GCH + MC + V = K).

T 2127 (McAloon) Con(ZF + AC + MC) implies Con(ZF + GCH + MC + V $\neq$ K). (K is the universe of hereditarily ordinal definable sets).

P 2128 Does Con(ZF + AC + SCC) imply Con(ZF + V = K + SCC)?

§3. THE STRUCTURE OF THE REAL LINE

If ZF is consistent, then it is impossible to prove that every real is constructible. It is natural to seek the "level of complexity" at which non-constructible reals may first occur. Similarly, one may ask "What is the simplest Lebesgue non-measurable set of reals?" Partial answers to these and related questions have been obtained, when "level of complexity" is taken to mean level of the analytical or projective hierarchy: some are deductions from large cardinal axioms, like "If there is a measurable cardinal, then every Σ^1_2 set of reals is Lebesgue measurable"; some are relative consistency results. These results are arranged in five paragraphs: in ¶0 the projective hierarchy is discussed, and recent applications of the axiom of determinacy reported; in ¶1, answers are given to the question "What reals are constructible?". ¶2 expounds the work of Mansfield and Solovay on perfect subsets of sets of reals. ¶3 deals with Lebesgue measure and the property of Baire, and in ¶4, some problems raised by Hausdorff are treated.

¶0. The Projective Hierarchy.

For notation see Addison [*].

An important problem is the pattern of the separation principles for the projective hierarchy. We write $Sep(\mathcal{C})$, ($Red(\mathcal{C})$) for "$\mathcal{C}$ has the first separation (the reduction) property"; $Unif(\mathcal{C})$ for "every set in $\mathcal{C}$ can be uniformized by a set in $\mathcal{C}$." It is easily seen that in ZF, for every $n < \omega$, $Unif(\Pi^1_n)$ implies $Unif(\Sigma^1_{n+1})$, that for any class $\mathcal{C}$, $Red(\mathcal{C})$ implies $Sep(\breve{\mathcal{C}})$, and that for all reasonable $\mathcal{C}$ (in particular the classes Σ^1_n, Π^1_n), $Unif(\mathcal{C})$ implies $Red(\mathcal{C})$.

T 3000 (Kondo-Addison) $ZF \vdash Unif(\Pi^1_1)$: hence

T 3001 $ZF \vdash Sep(\Sigma^1_1) \wedge Sep(\Pi^1_2)$.

T 3002 (Addison) $ZF + V = L \vdash \bigwedge n \geq 2\ Unif(\Sigma^1_n)$.

T 3003 (Gödel) $ZF + V = L \vdash$ There is a Δ^1_2 well ordering of the reals of order type ω_1.

The well ordering $\prec$ constructed by Gödel to prove T 3003 has the useful property that the quantifier $\bigwedge y \prec x$ (for any $x \subseteq \omega$) when applied to a Δ^1_2 formula yields a Δ^1_2 formula: this fact is used to prove T 3002, of which an immediate corollary is

T 3004 (Addison) $ZF + V = L \vdash \bigwedge n \geq 2\ Sep(\Pi^1_n)$.

By AD we mean the axiom of determinacy, of Mycielski and Steinhaus, discussed in Solovay's lecture [*]. Addison and Moschovakis and, independently, Martin have recently shown

T 3005 $ZF + AD \vdash \bigwedge n(Red(\Sigma^1_{2n}) \wedge Red(\Pi^1_{2n+1}))$, and so

T 3006 $ZF + AD \vdash \bigwedge n(Sep(\Pi^1_{2n}) \wedge Sep(\Sigma^1_{2n+1}))$.

Let us pause to note some other consequences of AD.

T 3007 (Mycielski, Swierczkowski) $ZF + AD \vdash$ All sets of reals are Lebesgue measurable.

D 3008 A set of reals has the <u>property of Baire</u> iff it is equal to an open set up to first category.

Now every Borel set has the property of Baire: hence

T 3009 A set of reals has the property of Baire iff it is equal to a Borel set up to first category.

T 3010 A set of reals is Lebesgue measurable iff it is equal to a Borel set up to a set of measure 0.

Henceforth we shall write "LM" for "Lebesgue measurable" and "is PB" for "has the property of Baire." The duality suggested by T 3009, 3010 will be brought out in ¶3.

T 3011 (Mycielski) ZF + AD ⊢ Every uncountable subset of the reals is PB and has a perfect subset; hence $\aleph_0$ adj $2^{\aleph_0}$.

T 3012 (Solovay [*]) ZF + AD ⊢ Every subset of $\aleph_1$ is constructible from a real, whence every subset of $\aleph_1$ contains or is disjoint from a closed unbounded subset, and so

T 3013 (Solovay) ZF + AD ⊢ $\aleph_1$ is 2VM.

A simpler proof by Martin of T 3013 is based on

T 3014 (Martin) ZF + AD ⊢ Let $\mathcal{E} \subseteq \mathcal{D}$, the set of all degrees of insolubility. There is a degree d_0 such that either $\bigwedge d \geq d_0 (d \in \mathcal{E})$ or $\bigwedge d \geq d_0 (d \notin \mathcal{E})$.

$\mathcal{E}$ defines a game: I wins if the play has degree in $\mathcal{E}$, II otherwise. Let d_0 be the degree of a winning strategy: if I wins, then $\bigwedge d \geq d_0 (d \in \mathcal{E})$; otherwise $\bigwedge d \geq d_0 (d \notin \mathcal{E})$. Let $\mu(\mathcal{E}) = 1$ in the first case, = 0 in the second. Then μ is a measure on $\mathcal{D}$: it is countably additive because no ω-sequence of degrees can be cofinal in $\mathcal{D}$. T 3014 also leads to a reasonably short proof of

T 3015 (Solovay) ZF + AD ⊢ $\aleph_2$ is 2VM; as a corollary,

T 3016 ZF + AD ⊢ Con(ZF + AC + MC).

Using T 3015 and the methods of T 2041, we can show that the class of cardinals $\kappa > \aleph_1$ forms a class of indiscernibles for $L^{\mu'}$, where μ' is the canonical measure on $\aleph_1$ obtained from the μ above.

D 3017 We write ADD for the assertion that every set of reals ordinal definable from a real is determinate.

The interest in ADD is that T 3005 and T 3006 may be proved in ZF + ADD, and that ADD is not known to violate AC. Indeed

T 3018 (Solovay) Con(ZF + AD + DC^{ω}) implies and is implied by Con(ZF + ADD + AC).

T 3019 Con(ZF + AD) implies and is implied by Con(ZF + ADD).

P 3020 Prove that Con(ZF + AD) implies Con(ZF + AD + DC^{ω}).

It is known that all open games, and more generally, all $F_{\sigma\delta}$ games are determined.

P 3021 Does ZF + DC^{ω} ⊢ Every Borel game is determined?

Martin has recently made important progress by showing that

T 3022 ZF + AC + MC ⊢ Every Σ^1_1 game is determined.

In fact MC can be weakened to $\bigvee \kappa (\kappa \to (\omega_1)^{<\omega})$; and

T 3023 ZF + AC ⊢ If $\bigvee \kappa \bigwedge \alpha < \omega_1 (\kappa \to (\alpha)^{<\omega})$, then every Borel game is determined.

It is worth remarking here that the statement "every Borel game is determined" relativizes to L by Shoenfield's theorem T 3100; and "$\forall \kappa (\kappa \to (\alpha)^{<\omega}$ for every countable α)" is true in L if true in V by Silver's result that if α is constructibly countable and $\kappa \to (\alpha)^{<\omega}$, then in L, $\kappa \to (\alpha)^{<\omega}$.

Friedman has shown that set theoretical arguments are necessary to prove that every Borel game is determinate. Let us call second order arithmetic with the full comprehension schema <u>analysis</u>. The assertion that every Borel game has a strategy may be formulated as a schema in analysis. Then Friedman's theorem states that

T 3024 If analysis is consistent, then so is analysis + "there is a Δ^0_5 indeterminate Borel set."

T 3025 (Moschovakis; Prikrý) ZF + AC $\vdash$ If there is a Moschovakis cardinal, then every Σ^1_2 game is determined.

It is easy to see, by considering L^μ, where μ is a normal measure on κ, that Con(ZF + AC + MC) implies Con(ZF + AC + MC + not every Σ^1_2 game is determined).

P 3026 (Addison) Show ZF + AD $\vdash \bigwedge n$ Unif(Π^1_{2n+1}).

T 3027 ZF + V = L^μ $\vdash$ Unif(Σ^1_3).

Red(Σ^1_3) and Sep(Σ^1_3) cannot hold together; so T 3027 shows that Sep(Σ^1_3) cannot be proved in ZF + AC + MC.

P 3028 Show ZF + AC + some large cardinal $\vdash$ Unif(Π^1_3).

P 3029 Show Con(ZF + AC + Sep(Σ^1_3)), assuming anything.

P 3030 Show Con(ZF + AC + $\neg$Sep(Σ^1_3) + $\neg$Sep(Π^1_3)), assuming anything.

§ 1. <u>Constructibility of Reals.</u>

T 3100 (Mostowski) ZF $\vdash$ If M is a transitive model of ZF, then any Σ^1_1 statement, allowing real parameters from M, is true in M iff it is true in the universe.

T 3101 (Shoenfield) ZF $\vdash$ If M is a transitive model containing all countable ordinals, then any Σ^1_2 statement, with real parameters from M, is true in M iff it holds in the universe.

T 3100, 3101 hold also when M is a class term defining an inner model containing all ordinals. In particular, taking $M = L$, in T3102, and $M = L[b]$ in T 3103, we have

T 3102 (Shoenfield) ZF $\vdash$ Every Σ^1_2 or Π^1_2 real is constructible;

T 3103 (Shoenfield) ZF $\vdash a \subseteq \omega \wedge b \subseteq \omega \wedge a\ \Sigma^1_2$ in $b \to a \leq_L b$.

Moschovakis has remarked that if $M \subseteq N$ are transitive models of ZF with the same ordinals, and if both M and N satisfy "Every Δ^1_2 game is determined" then every Σ^1_3 or Π^1_3 sentence with parameters from M is true in M iff it is true in N; and has raised the general problem of finding conditions on M and N that would ensure that Σ^1_n and Π^1_n sentences relativize from N to M.

In §2 we remarked Solovay's theorem that ZF + AC + MC $\vdash$ there is a Δ^1_3 non-constructible real; Solovay has recently shown by a relative consistency result that Shoenfield's theorem T 3102 is best possible. The sequence of events is this:

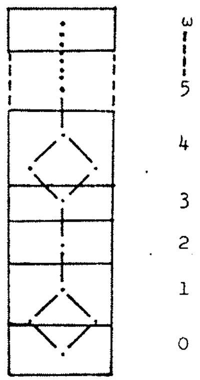
T 3104 (Kripke, Martin, Sacks) Con(ZF) implies Con(ZF + $\neg$ AC + there is a non-constructible Δ^1_4 real).

T 3105 (Solovay) Con(ZF) implies
Con(ZF + AC + $\bigvee x \subseteq \omega(x \notin L \wedge V = L[x] \wedge x \in \Delta^1_3 \wedge \omega^L_1 = \omega_1 \wedge$ x is the unique solution of a Π^1_2 predicate)).

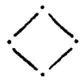
T 3106 (Jensen) Con(ZF) implies
Con(ZF + AC + $\bigvee x \subseteq \omega(x \notin L \wedge V = L[x] \wedge x \in \Delta^1_4 \wedge x$ is the unique solution of a Π^1_3 predicate $\wedge\ \omega^L_1 = \omega_1$)).

T 3107 (Jensen) Con(ZF) implies
Con(ZF + AC + $\bigvee x \subseteq \omega(x \notin L \wedge V = L[x] \wedge x \in \Delta^1_4 \wedge x$ is the unique solution of a Π^1_3 predicate) $\wedge \bigwedge y \subseteq \omega(y \in L \rightarrow y \in \Delta^1_4$)).

Jensen obtained T 3106 in ignorance of Solovay's result 3105, and about a week later. Jensen's method for 3106 is to construct a model in which the L-degrees form a pattern with $\omega + 1$ levels; each level has either a single point $\cdot$ or a triangle .⁄˙\., thus:



The model contains a real t, of the maximum L-degree, with t = {n: the n^{th} level of type .⁄˙\.}. This is done by iteration of Sacks forcing. (Martin showed that models with L-degree patterns

 and  could be constructed by appropriate iterations.)

3107 is proved similarly, but the pattern is drawn in the degrees of <u>collapsing functions</u> , viz, functions mapping ω onto ω^L_1. The notion used here, that of minimal collapsing function, is due to Přikrý.

3105 is proved by a different method. First Solovay shows that ω predicates P_i of reals can be defined such that for each $i < \omega$, $ZF + V = L \vdash \neg \bigvee x \subseteq \omega\, P_i(x)$. Now let M be a c.s.m. of ZF + V = L: then a sequence $\langle b_i | i < \omega \rangle$ of reals is added to M to form a model N of ZF + AC, and such that for each i, $P_i(b_i)$ holds in N; but if $N_i = M[\langle b_j | j \neq i \rangle]$, then $N_i \models \neg \bigvee x \subseteq \omega\, P_i(x)$. Thus the predicates P_i give ω "independent" notions of genericity over L. Now following McAloon's methods for obtaining models of V = K, a set $t \subseteq \omega$ is constructed such that $M[\langle b_i | i \in t \rangle] =_{df} M'$ is the desired model: t codes the construction of $M[\langle b_i | i \in t \rangle]$ from M, and in M', t will be a non-constructible Δ^1_3 real; M' = M[t].

P 3108 Find a model in which V = L[x], some $x \subseteq \omega$, $x \notin L$, x of minimal L-degree and $x \in \Delta^1_n$, some n.

T 3109 (Jensen) Con(ZF) implies Con(ZF + AC + every constructible real is Δ^1_3 + there is a non-constructible Δ^1_3 real).

P 3110 Find a model in which the constructible reals are precisely the Δ^1_3 reals, (or the Δ^1_n, some fixed n; or precisely the second-order definable reals).

P 3111 Find a model of ZF + AC + $\bigwedge x, y \subseteq \omega (x \leq_L y \rightarrow x\ \Delta^1_3$ in $y)$.

P 3112 Find a model of ZF + AC + the set of second-order definable reals is second-order definable.

T 3113 (Addison) ZF + V = L $\vdash$ The set of second-order definable reals is not second order definable.

T 3114 (Addison) The predicate "$x \leq_L y$" is Σ^1_2.

Thus the set of constructible reals is Σ^1_2; by T 3101, if it is also Π^1_2, then every real is constructible.

¶ 2. Perfect Sets.

There is a method of coding Σ^1_2 sets by reals (see Solovay [10, 11] for a description) which is useful in stating criteria for a set of reals to contain a perfect subset.

T 3200 ZF $\vdash$ A Σ^1_1 set with an element not Δ^1_1 in its code contains a perfect subset, and is therefore of power $2^{\aleph_0}$.

As particular cases of T 3200 we have the classical results that every Borel or analytic (Σ^1_1) set of reals is either countable or contains a perfect subset, and then is of cardinality $2^{\aleph_0}$. The same need not be true for Π^1_1 sets.

T 3201 (Gödel) ZF + V = L $\vdash$ There is an uncountable Π^1_1 set with no perfect subset.

More generally, by T 3000 and T 3114 (cf. [H 13])

T 3202 ZF $\vdash \omega^L_1 = \omega_1 \wedge 2^{\aleph_0} > \omega_1 \rightarrow$ there is a Π^1_1 set of cardinality $\aleph_1$ with therefore no perfect subset.

Every Π^1_1 set is the union of $\aleph_1$ Borel sets, and therefore has cardinality $\leq \aleph_1$ or is of cardinality $2^{\aleph_0}$.

T 3203 (Lévy) Let M be a c.s.m. of ZF + V = L; let $\kappa \in M$, $cf(\kappa) > \omega$. Then there is an extension N of M with $On^N = On^M$, and in N, ZF + AC are true, $2^{\aleph_0} = \kappa$, $\omega^L_1 = \omega_1$, and every set of reals which is ordinally definable from a real (in particular, every projective set of reals) has cardinality κ or $\leq \omega_1$.

T 3204 (Lévy) Let M be a c.s.m. of $ZF + V = L$ + there is an inaccessible cardinal. Let $\kappa \in M$ be a cardinal of cofinality at least θ, where θ is the first inaccessible in M. Then there is an extension N of M which is a model of $ZF + AC$, with the same ordinals as M in which $2^{\aleph_0} = \kappa$ and every set of reals ordinally definable from a real has cardinality κ or $\leq \omega$.

Note that the hypothesis of an inaccessible cannot be dropped, for

T 3205 $ZF + AC \vdash$ If every Π^1_1 set of reals has cardinality $2^{\aleph_0}$ or $\leq \omega$, and $\omega_1 < 2^{\aleph_0}$, then ω_1 is inaccessible in L.

Proof: ω_1 is regular in L, being regular. Suppose in L, $\omega_1 = \lambda^+$: there is a real a which codes a well-ordering of ω of type λ. In $L[a]$, λ is countable, and $\omega_1 = \omega_1^{L[a]}$. The set of reals in $L[a]$ is Σ^1_2 in a, and so Σ^1_2, but is of cardinality $\aleph_1^{L[a]} = \aleph_1$. Using T 3000, we obtain a Π^1_1 set of reals of cardinality $\aleph_1$.

The next theorem (most of which was proved independently by Solovay) parallels T 3200.

T 3206 (Mansfield) $ZF \vdash$ Let A be a Σ^1_2 set with code t: if A has a member not constructible from t, then A contains a perfect subset with code Δ^1_2 in t.

T 3207 (Mansfield; Solovay) $ZF \vdash$ The following are equivalent:

(i) every uncountable Σ^1_2 set has a perfect subset;

(ii) every uncountable Π^1_1 set has a perfect subset;

(iii) for each real t only countably many reals are constructible from t.

Condition (iii) is a consequence of $AC + \bigvee \kappa(\kappa \rightarrow (\omega_1)^{<\omega})$: thus

T 3208 (Solovay) $ZF + AC + \bigvee \kappa(\kappa \rightarrow (\omega_1)^{<\omega} \vdash$ Every uncountable Π^1_1 set has a perfect subset.

P 3209 (Solovay) Show that $ZF + AC + SuCC \vdash$ Every projective set of reals has a perfect subset.

Let us contrast Mansfield's result with the following theorem of Friedman:

T 3210 $ZF \vdash$ There is an infinite Π^1_1 set of reals, every member of which except one is Δ^1_2.

Solovay points out in his paper [11] on Σ^1_2 sets of reals that using T 2017, one can show

T 3211 Con(ZF + AC + MC) implies Con($ZF + MC + AC + 2^{\aleph_0} = \aleph_1$ + there is a Δ^1_2 set of reals with no perfect subset) and Con($ZF + MC + AC + 2^{\aleph_0} > \aleph_1$ + there is a Δ^1_2 set of reals of cardinality $\aleph_1$).

T 3212 (Lévy) Con(ZF) implies Con(ZF + GCH + every well ordering of reals which is ordinal definable without real parameters is denumerable or finite).

T 3213 (Lévy) Con(ZF + V = L + there is an inaccessible cardinal) implies Con(ZF + GCH + every projective well ordering (more generally, every well ordering ordinal definable from a real) of reals is finite or denumerable)).

P 3214 Find a model of ZF + AC in which $2^{\aleph_0} > \aleph_1$ and there is a projective well ordering of the continuum.

T 3215 (Mansfield) ZF ⊢ Let A be a Σ^1_2 well ordering. Then every real in the domain of A is constructible from a code for A.

I 3216 Write P(κ) for "Every subset of $\kappa \times \kappa$ is in the σ-field generated by all rectangles."

T 3217 (Kunen; Silver) ZF + AC ⊢ $P(\omega_1)$.

T 3218 (Silver) ZF + AC + (*) ⊢ $P(2^{\aleph_0})$.

T 3219 (Kunen) ZF + AC ⊢ $P(\kappa) \rightarrow \kappa \leq 2^{\aleph_0} \vee \kappa$ not RVM.

T 3220 (Kunen; Silver) Con(ZF + AC + MC) implies Con(ZF + AC + $\bigvee \kappa (2^{\aleph_0} = \kappa \wedge P(\kappa) \wedge i(\kappa,\kappa_1))$).

T 3221 (Kunen) In any Cohen generic or measure algebra extension of a model of ZF + AC + $2^{\aleph_0} = \aleph_1$, $P(\aleph_2)$ is false. Hence Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} = \aleph_2$ + $\neg P(2^{\aleph_0})$).

T 3222 (Mansfield) ZF + AC ⊢ The universal analytic subset of $N^N \times N^N$ does not lie in the σ-field generated by the analytic rectangles.

T 3222 answers a problem of Ulam.

P 3223 (Mansfield) Show that Con(ZF) implies Con(ZF + AC + the universal analytic set is not in the σ-field generated by all rectangles).

T 3224 (Martin, Solovay) ZF + AC + $\bigvee \kappa(\kappa \rightarrow (\kappa)^{<\omega})$ ⊢ Every Σ^1_3 set of reals has a Δ^1_4 element.

T 3225 (Solovay) ZF + AC + (*) ⊢ Every set of reals of power at most $\aleph_1$ is Π^1_1.

T 3225 is proved by using Solovay's trick.

Let κ be 2VM, μ a normal measure.

T 3226 (Silver) ZF + V = L^μ ⊢ There is a Δ^1_3 well ordering ≺ of the continuum, which has the property that the quantifier $\bigwedge y \prec x$ applied to a Δ^1_3 formula yields a Δ^1_3 formula.

T 3027 is an immediate corollary; more generally,

T 3227 ZF + V = L^μ ⊢ $\bigwedge n \geq 3$ Unif(Σ^1_n).

¶ 3. Measurability and the Property of Baire.

We begin by stating what is known about the set of constructible reals ('S' for short) in various models.

T 3300 (Addison) ZF ⊢ $S \in \Sigma^1_2$. If there are two L-degrees, then by T 3114, $S \in \Delta^1_2$, for then $S\ \omega\setminus S = \{x \mid a \leq_L x\} \in \Sigma^1_2$, where a is any non-constructible real.

T 3301 (Přikrý) In Sack's model D 1109, $S \in \Sigma^1_2 \cap \Pi^1_2$, and is neither LM nor PB.

The next two theorems were proved independently by Kunen; see also Bukovský [H 26] and Vopěnka, Hrbáček [H 47]. Recall D 1101, D 1102, and the remark after T 3010.

T 3302 (Solovay) In $V^{\mathbb{B}_1}$, S is of measure 0 and does not have the property of Baire.

T 3303 (Solovay) In $V^{\mathbb{B}_2}$, S is of inner measure 0, outer measure 1, and is of first category.

T 3304 ZF ⊢ Every Σ^1_1 or Π^1_1 set is LM and PB.

T 3305 (Gödel) ZF + V = L ⊢ There is a Δ^1_2 set of reals which is neither LM nor PB.

T 3306 (Solovay) Con(ZF + AC + there is a strongly inaccessible cardinal) implies Con(ZF + AC + GCH + every uncountable set of reals which is ordinally definable from a real is LM, PB and contains a perfect subset).

T 3307 (Solovay) Con(ZF + AC + there is a strongly inaccessible cardinal) implies Con(ZF + DC^{ω} + every uncountable set of reals is LM and PB and contains a perfect subset).

Solovay proved T 3306 by examining the model constructed by Lévy to prove T 3204. T 3307 follows by considering the submodel of all sets hereditarily ordinally-definable-from-a-real.

T 3308 (Hájek [H 39]) Con(ZF + AC + there is a strongly inaccessible cardinal) implies Con(ZF + there is no choice set for the Lebesgue decomposition of the real line + ω_1 is regular + there is an uncountable set of reals without a perfect subset).

P 3309 The hypothesis of T 3307 cannot be weakened to Con(ZF): but does Con(ZF) imply Con(ZF + DC^{ω} + every uncountable set of reals is LM)?

P 3310 Does ZF + DC^{ω} + every set of reals is LM ⊢ $\aleph_0$ adj $2^{\aleph_0}$?

T 3311 (Solovay) Con(ZF) implies Con(ZF + GCH + there is a translation invariant extension of Lebesgue measure defined (at least) on all sets of reals ordinal definable from a real).

T 3312 (Solovay) Con(ZF) implies Con(ZF + $\bigwedge_{\alpha} DC^{\alpha}$ + not every set of reals is Lebesgue measurable but there is a translation invariant extension of Lebesgue measure defined on all sets of reals).

T 3313 (Solovay) Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} > \aleph_1$ + the real line is the union of $\aleph_1$ sets of measure 0).

For the next three theorems, let M be a c.s.m. of ZF + V = L; and let κ be an infinite regular cardinal in M. (Theorems of this sort have also been proved by Kunen; for T 3316 see Bukovský [H 26] and Vopěnka, Hrbáček [H 47]).

T 3314 (Martin, Solovay) There is an extension N of M with $Card^N = Card^M$ in which $ZF + AC$ is true, $2^{\aleph_0} = \kappa$, the ideal of sets of reals of measure 0 is κ-complete and the ideal of sets of reals of 1st category is κ-complete.

T 3315 (Solovay) There is an extension of M with the same cardinals as M which is a model of $ZF + AC$ in which $2^{\aleph_0} = \kappa$, every set of reals of cardinality $< \kappa$ is of 1st category, but there is a non-measurable set of cardinality $\aleph_1$.

T 3316 (Solovay) There is an extension of M with the same cardinals as M which is a model of $ZF + AC$ in which $2^{\aleph_0} = \kappa$, every set of reals of cardinality $< \kappa$ is of measure 0, but there is a set of reals of cardinality $\aleph_1$ which is not PB.

D 3317 A set A of reals has <u>strongly measure</u> 0 iff for any sequence of positive reals $\{a_n\}_{n\in\omega}$ there is a sequence of open intervals $\{I_n\}$ such that $A \subseteq \bigcup I_n$ and $\mu(I_n) < a_n$.

P 3318 Find a c.s.m. of $ZF + AC$ + "every set of reals strongly of measure 0 is countable (Borel's conjecture)", given a model of $ZF + V = L$.

Gödel's theorem T 3305 should be contrasted with

T 3319 (Solovay) $ZF + AC \vdash$ Let $x \subseteq \omega$ and $\omega_1^{L[x]} < \omega_1$. Then every set of reals Σ^1_2 in x is LM and PB

which is proved using Shoenfield's theorem T 3101 and some very fortunate properties of random reals, and has the immediate corollary

T 3320 (Solovay) $ZF + AC + \bigvee\kappa(\kappa \to (\omega_1)^{<\omega}) \vdash$ Every Σ^1_2 set of reals is LM and PB.

T 3321 (cf T 3314) $ZF + AC + (*) \vdash$ The ideal of sets of measure 0 and the ideal of sets of first category are both $\aleph_2$ - additive; consequently every Σ^1_2 set of reals is LM and PB.

T 3322 (Tanaka [13]) $ZF \vdash$ The Lebesgue measure of a Σ^1_1 set of reals is a Σ^1_1 real.

The following answers a question of Přikrý:

T 3323 (Kunen; Solovay) Let M, κ be as in T 3314. Extend M by the regular open algebra of 2^κ (viz. add κ generic reals). Then in the extension $2^{\aleph_0} = \kappa$ and there is a Δ^1_2 nonmeasurable set.

P 3324 (Solovay) Show that $ZF + AC + SuCC \vdash$ Every projective set of reals is LM.

We end this paragraph with some results on RVM's.

T 3325 (Kunen) $ZF + AC \vdash$ Let κ be RVM. Then

(i) there is a set of reals of power $< \kappa$ which is not LM;

(ii) every set of reals of power $< \kappa$ is of first category;

(iii) the real line is the union of fewer than κ sets of first category; but

(iv) the real line is not the union of fewer than κ sets of Lebesgue measure 0.

§ 4. Some Problems of Hausdorff.

D 3400 Write $H(\kappa)$ for "There is a family F, $\bar{\bar{F}} = \kappa^+$, of functions $f\colon \kappa \to \kappa$ such that given any $g\colon \kappa \to \kappa$ there is an $f \in F$ such that $\bigwedge \nu < \kappa\ g(\nu) < f(\nu)$.

Hausdorff asked whether, even if the continuum hypothesis failed, it would still be possible to prove $H(\omega)$. Now

$$ZF \vdash H(\omega) \longleftrightarrow \bigvee F \subseteq N^N(\bar{\bar{F}} = \aleph_1 \text{ and } \bigwedge f \in N^N \bigvee g \in F$$
$$\bigvee m < \omega \bigwedge n > m(g(n) > f(m))):$$

that is, $H(\omega)$ holds iff there is a family of functions of cardinality $\aleph_1$ such that every function is eventually majorized by a function in the family.

T 3401 (Solovay) Let M be a c.s.m. of $ZF + V = L$. Let N_1 be obtained by adding $\aleph_2$ generic reals to M; then $N_1 \models AC \wedge 2^{\aleph_0} = \aleph_2 \wedge \neg H(\omega)$. Let N_2 be obtained by adding κ random reals, where in M, $\omega_1 < \kappa = \bar{\bar{\kappa}} \wedge cf(\kappa) > \omega$. Then $N_2 \models AC \wedge 2^{\aleph_0} = \kappa > \aleph_1 \wedge H(\omega)$. Indeed, the set of constructible functions $\in N^N$ will serve as the required $\mathcal{F}$.

D 3402 A set X of reals has Lusin's property (LP) iff $\bar{\bar{X}} = 2^{\aleph_0}$ and $\overline{\overline{X \cap Y}} \le \aleph_0$ for every Y of first category.

The existence of a set with LP implies (in ZF) that $H(\omega)$ is false. The first part of T 3401 has been strengthened to

T 3403 (Vopěnka, Hrbáček [H 47]) Let M be as in T 3401, and let N be obtained by adding κ generic reals, where $cf(\kappa) > \omega$. Then there is in N a set with LP.

Jensen noticed that models for $ZF + AC + 2^\kappa > \kappa^+ + H(\kappa)$ where κ is inaccessible in M (or $\kappa = \omega$), could be constructed using Sacks forcing. Silver then remarked that by collapsing cardinals underneath the inaccessible κ one could get (say) a model of $ZF + AC + 2^{\aleph_1} > \aleph_2 + H(\aleph_1)$. Hence

T 3404 (Jensen, Silver) $Con(ZF + V = L + \bigvee \kappa\ \kappa$ inaccessible) implies $Con(ZF + AC + 2^{\aleph_1} > \aleph_2 + H(\aleph_1))$.

P 3405 Does the conclusion of 3404 follow from $Con(ZF)$?

There is some relation between H and the property F discussed earlier (D 1126): by using Sacks' forcing one may show

T 3406 $Con(ZF)$ implies $Con(ZF + AC + \neg CH + F(\aleph_0))$.

For $f, g \in N^N$, define $f \prec g$ to hold iff $\bigvee m \bigwedge n > m(f(n) < g(n))$. $\prec$ has been investigated by Hechler [*] who shows

T 3407 (Hechler) Let M be a c.s.m. of $ZF + AC$ and let $\langle S, <_s \rangle$ be a partially ordered set in M with $S \le 2^{\aleph_0}$ in M; then there is an $N \supseteq M$ which is a model of $ZF + AC$, $Card^N = Card^M$, for all cardinals κ, $(2^\kappa)^N = (2^\kappa)^M$ and $\langle S, <_s \rangle$ is embeddable in $\langle N^N, \prec \rangle$.

T 3408 (Hausdorff) ZF + AC $\vdash$ The unit interval is the union of a properly increasing (nested) sequence of G_δ sets of length $\aleph_1$, but is not the union of a properly increasing sequence of sets in $G_\delta \cap F_\sigma$.

T 3409 (Solovay) Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} > \aleph_1$ + the unit interval is the union of a properly increasing nested sequence of F_σ sets of length $\aleph_1$)

(The last property is trivially true if $2^{\aleph_0} = \aleph_1$, as every countable set is an F_σ).

T 3410 (Solovay) Con(ZF) implies Con(ZF + AC + $2^{\aleph_0} > \aleph_1$ + the unit interval is not the union of a properly increasing sequence of F_σ sets of length $\aleph_1$).

* P 3411 (Keisler) Let κ be an uncountable cardinal. Does there exist an ultrafilter D on the set κ such that

(i) every $x \in D$ is of power κ;

(ii) for every set $E \subseteq D$ of power κ, some element of κ belongs to infinitely many elements of E?

The case $\kappa = \aleph_1$ is particularly interesting. Silver has shown that the answer is yes if κ is RVM.

P 3412 (Reyes) Let $R = \{\langle P_1, P_2, \ldots\rangle : P_i \subseteq \omega\} = (2^\omega)^\omega$. $S \subseteq R$ is <u>invariant</u> iff for every permutation π of ω, $\langle P_1, \ldots\rangle \in S$ iff $\langle \pi'' P_1, \ldots\rangle$ S. Is there an invariant $S \subseteq R$ which is neither meagre nor comeagre in the product topology?

The answer is yes if $2^{\aleph_0} = \aleph_1$ or (remarked by Solovay) if $\neg i(2^{\aleph_0}, \aleph_1)$.

§4. KUREPA'S CONJECTURE AND RESULTS IN MODEL THEORY.

¶0. <u>Conjectures of Kurepa and Chang</u>.

* D 4000 (Kurepa) $\aleph_\alpha$ has Kurepa's property (written $K(\aleph_\alpha)$) iff there is a family S with $\bar{\bar{S}} = \aleph_{\alpha+2}$ of subsets of $\aleph_{\alpha+1}$ such that for any $\beta < \aleph_{\alpha+1}$, $\overline{\overline{\{X \cap \beta | X \in S\}}} \leq \aleph_\alpha$.

D 4001 (ZF + AC) $\langle\kappa,\lambda\rangle \Rightarrow_0 \langle\kappa',\lambda'\rangle$ iff any theory with a model of type $\langle\kappa,\lambda\rangle$ has a model of type $\langle\kappa',\lambda'\rangle$, where $\kappa, \lambda, \kappa', \lambda'$ are infinite cardinals, $\kappa \geq \lambda$, $\kappa' \geq \lambda'$, and "type $\langle\kappa,\lambda\rangle$" is as defined in Vaught [ ].

D 4002 (ZF + AC) $\langle\kappa,\lambda\rangle \Rightarrow_1 \langle\kappa',\lambda'\rangle$ iff any relational structure $\langle A,U,R\rangle$ where $\bar{\bar{A}} = \kappa$, $\bar{\bar{U}} = \lambda$, $U \subseteq A$ and $R \subseteq A$ has an elementary substructure $\langle B,V,S\rangle$ where $\bar{\bar{B}} = \kappa'$, $\bar{\bar{V}} = \lambda'$, $V \subseteq B$, and $S \subseteq B$.

T 4003 (Lévy, Rowbottom; Bukovský [H45]) Con(ZF + V = L + there is an inaccessible cardinal) implies Con(ZF + GCH + $K(\aleph_0)$).

* T 4004 (Silver [*]) Con(ZF + V = L + there is an inaccessible cardinal) implies Con(ZF + GCH + $\neg K(\aleph_0)$ + $K(\aleph_1)$).

T 4005 (Vaught) ZF + AC $\vdash \bigwedge\alpha(\langle\aleph_{\alpha+1},\aleph_\alpha\rangle \Rightarrow_0 \langle\aleph_1,\aleph_0\rangle)$

* T 4006 (Chang) $\bigwedge\alpha\beta(\aleph_\beta$ regular $\rightarrow \langle\aleph_{\alpha+1},\aleph_\alpha\rangle \Rightarrow_0 \langle\aleph_{\beta+1},\aleph_\beta\rangle)$.

Now in [14], which discusses the property $\Rightarrow_0$ in greater detail, Vaught remarks that there is a sentence A in the first-order language with a distinguished unary predicate such that for all α, A has a model of type $\langle\aleph_{\alpha+2},\aleph_\alpha\rangle$ iff $K(\aleph_\alpha)$. For further theorems and problems see Silver [*].

D 4007 (ZF + AC) κ is a Rowbottom cardinal $(R(\kappa))$ iff

κ [illegible] $\bigwedge \lambda(\omega \leq \lambda < \kappa \rightarrow \langle \kappa, \lambda \rangle \Rightarrow_1 \langle \kappa, \omega \rangle)$.

Chang's conjecture (CC) is the assertion that $\langle \aleph_2, \aleph_1 \rangle \Rightarrow_1 \langle \aleph_1, \aleph_0 \rangle$.
Let us denote by Q the assertion that there is a family F of functions $f: \omega_1 \rightarrow \omega$ with

$$\bigwedge f, g \in F \bigvee \alpha < \omega_1 \bigwedge \beta (\alpha < \beta < \omega_1 \rightarrow f(\beta) \neq g(\beta)).$$

T 4008 (Silver) $ZF + AC \vdash CC \rightarrow \neg Q$.

T 4009 (Rowbottom) $ZF + AC \vdash \bigwedge \kappa (\kappa\ 2VM \rightarrow R(\kappa))$.

T 4010 (Příkrý) $ZF + AC \vdash \bigwedge n < \omega (\kappa_n 2VM \rightarrow R(\bigcup_{n<\omega} \kappa_n))$.

T 4011 (Rowbottom) $ZF + AC \vdash CC \rightarrow V \neq L$.

T 4012 (Keisler, Rowbottom) $ZF + AC + CC \vdash$ "There are several weakly compact cardinals" is true in L.

* T 4013 (Silver) $Con(ZF + AC + MC)$ implies $Con(ZF + GCH + CC)$.

T 4014 (Silver, using T 2118) $Con(ZF + AC + MC)$ implies $Con(ZF + GCH + \langle \aleph_1, \aleph_0 \rangle \Rightarrow_0 \langle \aleph_{\omega+1}, \aleph_\omega \rangle)$.

P 4015 Does $ZF + AC + Q \vdash K(\aleph_0)$?

P 4016 Is Q independent of something not very big?

1. End Extensions

D 4100 (ZF + AC) $\langle B, F \rangle$ is an end extension of $\langle V_\alpha, \in \rangle$ iff it is a proper elementary extension and $bFa \wedge a \in V_\alpha . \rightarrow b \in V_\alpha$.

P 4101 (Keisler) Is there an inaccessible α such that $\langle V_\alpha, \in \rangle$ has an end extension but no well-founded end extension?

P 4102 (Keisler) Let κ be the first Π^1_1 indescribable cardinal. Has $\langle V_\kappa, \in \rangle$ end extensions of arbitrarily large cardinality, or even of cardinality κ^{++}?

§5. OTHER SYSTEMS.

¶ 0. Variations of Zermelo-Fraenkel Set Theory.

By NBG we mean the axioms of blocks A to D of Gödel's monograph.

T 5000 Con(ZF) implies and is implied by Con(NBG).

T 5001 Any theorem of NBG in which only set variables occur is a theorem of ZF, and vice versa.

Let us note two facts about axiom E, the class form of AC:

T 5002 (Easton) Con(NBG) implies Con(NBG + GCH + $\neg$E).

T 5003 (Solovay) Any theorem of NBG + E in which only set variables occur is provable in ZF + AC.

Easton indeed showed more than T 5002, namely that if NBG is consistent, then so is NBG + GCH - "There is no class selecting one element from every unordered pair in the universe."

Let ZFU be the usual modification of ZF allowing individuals. The article of Lévy [5] gives all but the most recent results. A theorem of Gauntan ([9], Theorem 11.5.9) establishes the existence of a group of the sort needed to answer affirmatively the problem mentioned on the last page of [5].

D 5004 We write $C(S)$ ($C^{wo}(S)$) for the conjunction of the axioms C_n (C_n^{wo}) for $n \in S$, where S is a finite subset of ω.

T 5005 (Gauntt) Con(ZF) implies Con(ZFU + $\bigwedge n(1 < n \in \omega \to \neg C_n)$).

T 5006 (Gauntt) ZFU $\vdash \bigwedge n, m \in \omega(C_{n+m} \to C_n \vee C_m^{wo})$; $\bigwedge n, m \in \omega(C_{n+m}^{wo} \to C_n^{wo} \vee C_m^{wo})$.

P 5007 (Gauntt) Does ZFU $\vdash \bigwedge n, m \in \omega(C_{n+m} \to C_n \vee C_m)$?

Let $m \in \omega$ and S a finite subset of ω.

T 5008 (Gauntt) If ZFU $\vdash C^{wo}(S) \to C_m^{wo}$, then ZFU $\vdash C(S) \to C_m$.

T 5009 (Gauntt) ZFU $\vdash C^{wo}(S) \to C_m^{wo}$ iff for every subgroup G of Sym(m) without fixed points there exist finitely many proper subgroup $H_1 \cdots H_k$ of G such that $|G: H_1| + \cdots + |G: H_k| \in S$.

T 5009 gives a decision procedure for the relative consistency of ZFU + $C^{wo}(S) + \neg C_m^{wo}$; a procedure for ZFU + $C(S) + \neg C_m$ has yet to be found.

There is no general method for converting a proof of independence from ZFU to one from ZF, for consider the axiom of foundations itself, or less trivially the assertion X that every set which can be linearly ordered can also be well-ordered: ZF $\vdash$ X $\to$ AC, but if Con(ZFU) then Con(ZFU + X + $\neg$AC). However, some independence proofs are convertible:

T 5010 (Jech, Sochor [H37, H38]). Let a sentence A have one of the following forms:

$\bigvee a \bigwedge x \in aB(x,a)$

$\bigvee a \bigwedge x \in S(a)B(x,a)$

$\bigvee a \bigwedge x \in S(S(a))B(x,a)$, etc.,

where B is a restricted (Δ_0) formula. Then Con(ZFU + A) implies Con(ZF + A).

Let Z denote Zermelo set theory, D the axiom of foundations.

P 5011 Find a sentence A of set theory such that Z - D + A is consistent and has among its theorems all the axioms of ZF - D.

P 5012 (Lévy) Let T be the theory ZF - D + AC. For each formula A(x,y) of set theory (without parameters) add a new unary operation symbol f_A and the following axiom schemata:

S1. If A(x,y) is an equivalence relation, then for all x,y

$$f_A(x) = f_A(y) \longleftrightarrow A(x,y) .$$

S2. All instances of the axiom schema of replacement which contain the new symbols.

Call the extended theory S. Is there a sentence A of ZF (that is, without the new symbols) which is a theorem of S but not of T?

T 5013 (Hájek [H14]) NBG - D ⊢ Let R be a regular extensional relation. Then there is a model relation $S \supseteq R$ such that

(1) $S''\{x\} = R'\{x\}$ for all x in the field of R;

(2) $U \subseteq Fld(R) \rightarrow \bigvee Z(u = S''\{Z\})$.

¶ 1. Variations on Quine's Set Theory.

Let NFU be the system NF with extensionality replaced by

$$\bigvee z \in x \wedge \bigwedge z(z \in x \longleftrightarrow z \in y) \;.\!\rightarrow x = y \;.$$

Let T be ordinary type theory with ω levels. Let Inf be the axiom of infinity in the form "There is an ordering which is not a well-ordering," and AC the axiom of choice in the form "Every disjoint set of nonempty sets has a choice set."

T 5100 (Jensen) Con(T) implies Con(NFU)

T 5101 (Jensen) Con(T + Inf + AC) implies Con(NFU + Inf + AC)

T 5102 (Jensen) Con(T + Inf + ¬AC) implies Con(NFU + Inf + ¬AC).

These results and the consistency of T can be proved in elementary number theory. Hence

T 5104 (Jensen) ENT ⊢ Con(NFU).

Since NFU + Inf ⊢ Con(ENT), Inf is independent of NFU, in contradistinction to the following result:

T 5105 (Specker) NF ⊢ Inf ∧ ¬AC.

Let NFU^+ be NFU augmented by the following induction schema:

$$(+) \quad A(0) \wedge \bigwedge n(A(n) \rightarrow A(n+1)) \;.\!\rightarrow \bigwedge nA \;,$$

where the variable n ranges over the finite numbers, which are defined as equivalence classes of (Russell) finite sets.

T 5106 (Hailperin) NF is finitely axiomatizable.

An extension of Hailperin's argument proves.

T 5107 (Jensen) NFU is finitely axiomatizable.

By T 5107 and a theorem of Kreisel it follows that (+) is independent of NFU.

P 5108 (Jensen) Is there a reasonable set theory S such that the equiconsistency of S and NFU^+ is provable in elementary number theory?

T 5109 (Jensen) $ZF \vdash Con(NFU^+)$.

T 5109 is a consequence of the relativization to L of the following theorem.

T 5110 (Jensen) $ZF + AC \vdash$ If α is any ordinal, there is a model of NF containing V_α as an initial segment.

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