Slim models of Zermelo set theory

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Abstract Working in $Z+KP$, we give a new proof that the class of hereditarily finite sets cannot be proved to be a set in Zermelo set theory, extend the method to establish other failures of replacement, and exhibit a formula $\Phi(\lambda, a)$ such that for any sequence $\langle A_\lambda \mid \lambda \text{ a limit ordinal } \rangle$ where for each λ, $A_\lambda \subseteq {}^\lambda 2$, there is a supertransitive inner model of Zermelo containing all ordinals in which for every λ $A_\lambda = \{a \mid \Phi(\lambda, a)\}$.

Preliminaries This paper explores the weakness of Zermelo set theory, Z, as a vehicle for recursive definitions. We work in the system $Z+KP$, which adds to the axioms of Zermelo those of Kripke–Platek set theory KP. $Z+KP$ is of course a subsystem of the familiar system ZF of Zermelo–Fraenkel. Mention is made of the axiom of choice, but our constructions do not rely on that Axiom. It is known that $Z+KP+AC$ is consistent relative to Z: see [M2], to appear as [M3], which describes *inter alia* a method of extending models of $Z+AC$ to models of $Z+AC+KP$.

We begin by reviewing the axioms of the two systems Z and KP. In the formulation of KP, we shall use the familiar Lévy classification of formulæ: Δ_0 formulæ are those in which every quantifier is restricted, $\forall x(x \in y \implies \ldots)$ or $\exists x(x \in y \ \& \ \ldots)$, which we write as $\forall x{:}{\in} y \ \ldots$ and $\exists x{:}{\in} y \ \ldots$ respectively. In all such cases x and y must be distinct variables. Π_1 and Σ_1 formulæ are those respectively of the form $\forall x \mathfrak{B}$ and $\exists x \mathfrak{B}$, where $\mathfrak{B}$ is a Δ_0 formula.

0·0 The axioms of the system Z are Extensionality $\forall z(z \in x \iff z \in y) \implies x = y$, Empty Set $\varnothing \in V$, Pairing $\{x, y\} \in V$, Union $\bigcup x \in V$, Power Set $\mathcal{P}(x) \in V$, Foundation $\forall x\, [x \neq \varnothing \implies \exists y{:}{\in} x\ (x \cap y = \varnothing)]$, Infinity $\omega \in V$, and for each class A the axiom

$$\forall x\ x \cap A \in V.$$

That last scheme of axioms is known in German as the Aussonderungsschema and in English variously as the Comprehension or the Separation scheme.

A set or class A is *transitive* if $x \in y \in A \implies x \in A$. We call A *supertransitive* if it is transitive and if $x \subseteq y \in A \implies x \in A$. For example in ZF the set $V_{\omega+\omega}$ is supertransitive.

0·1 Here are some easy facts about transitive sets which will be used repeatedly:

if x is transitive then $x \cup \{x\}$ is transitive;
if x is transitive and $a \in x$ then $a \subseteq x$;
if x transitive and $\forall a{:}{\in} A\ a \subseteq x$ then $x \cup A$ is transitive.

By TC or *Transitive Containment* we mean the statement that each set is a member of a transitive set. Given the above facts and the Axiom of Pairing, that is equivalent to saying that each set is a subset of a transitive set.

In Zermelo + TC the axiom of foundation may be strengthened to the scheme of class Foundation, so that for each class A it is provable that $\exists y(y \in A) \implies \exists y(y \in A \ \& \ y \cap A = \varnothing)$. Jensen and Schröder [JS] and Boffa [B1], [B2] have shown that there are instances of the scheme of class Foundation which are not theorems of Zermelo set theory, and consequently that Transitive Containment is not provable in Zermelo.

0·2 We take Kripke–Platek set theory to have as its axioms Extensionality, Empty Set, Pairing, Union, Foundation for sets and for Π_1 classes A, Infinity (in the form "there exists a limit ordinal"), the Aussonderungsschema in the reduced form

$$x \cap A \in V \qquad \text{for } A \text{ a class defined by a } \Delta_0 \text{ formula,}$$

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and the Δ_0 scheme of collection

$$\forall x \exists y\, \mathfrak{A} \implies \forall u \exists v \forall x :\in u\; \exists y :\in v\; \mathfrak{A}$$

for each Δ_0 formula $\mathfrak{A}$ in which the variables u, v have no occurrence.

As pointed out to me by Professor Jané, we may derive TC in KP, starting from the observation that since the class of sets which are not members of transitive sets is Π_1, by Π_1 Foundation we may find an $\in$-minimal counterexample.

KP is a system in which recursive constructions are natural; for example we may define the set-theoretic rank and the transitive closure $tcl(x)$ of a set $\varrho : V \longrightarrow ON$ by the recursions

$$\varrho(x) = \bigcup\{\varrho(y) + 1 \mid y \in x\}; \qquad tcl(x) = x \cup \bigcup\{tcl(y) \mid y \in x\}.$$

0·3 We write $\mathcal{P}(a)$ for the power set $\{x \mid x \subseteq a\}$ of the set a. For a formula Φ of the language of set theory and a transitive class $\mathcal{M}$ we write $(\Phi)^{\mathcal{M}}$ for the relativisation of Φ to $\mathcal{M}$, namely the result of rewriting Φ so as to bind all variables to range over members of $\mathcal{M}$.

When discussing functions $f, g : \omega \to \omega$, we shall write $f \leqslant g$ for $\forall n :\in \omega\ f(n) \leqslant g(n)$, $f \equiv g$ for $\forall n :\in \omega\ f(n) = g(n)$, and so on, and write $f + 1$ for the function $n \mapsto f(n) + 1$ and $f + g$ for the function $n \mapsto f(n) + g(n)$.

0·4 Definition As usual, we write for each ordinal α $V_\alpha = \{x \mid \varrho(x) < \alpha\}$.

0·5 Remark For each $\alpha \leqslant \omega$, $KP + \omega \in V$ proves V_α to be a set. Each V_n is finite, and V_ω is countably infinite, and is the class of hereditarily finite sets.

We write WO for the well-ordering principle, that every set has a well-ordering.

The following proofs were inspired by arguments of, and with, Dr Thomas Forster and Professor Akihiro Kanamori, who also kindly supplied some bibliographical references. Their contributions are described in more detail in the final section.

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1: A general method for constructing models of Zermelo set theory.

1·0 Definition A class $\mathcal{T}$ is *fruitful* if

(1·0·0) $x \in \mathcal{T} \implies \bigcup x \subseteq x$
(1·0·1) $ON \subseteq \mathcal{T}$
(1·0·2) $x \in \mathcal{T}\ \&\ y \in \mathcal{T} \implies x \cup y \in \mathcal{T}$
(1·0·3) $x \in \mathcal{T}\ \&\ A \subseteq \mathcal{P}(x) \implies x \cup A \in \mathcal{T}$.

1·1 Remark Were we not concentrating in this paper on building models containing all ordinals, we could weaken (1·0·1) to (1·0·1$'$) : $\omega + 1 \in \mathcal{T}$, and many of our results would still hold. Indeed, in proving Theorem 5·6, we shall use a set $\mathcal{T}$ that satisfies (1·0·0), (1·0·2) and (1·0·3), but only the very weak assertion (1·0·1$''$) that $0 \in \mathcal{T}$, so the reader should notice as he verifies the various clauses leading to a proof of the next proposition that (1·0·1) is not used for the verification of any axiom other than those of nullset and infinity, and that (1·0·1$''$) suffices for the first and (1·0·1$'$) for the second.

1·2 Proposition *Let $\mathcal{T}$ be a fruitful class and let $\mathcal{M} = \bigcup \mathcal{T}$. Then $\mathcal{M}$ is a supertransitive model of Zermelo set theory plus the hypothesis of Transitive Containment. Further if WO holds then $(WO)^{\mathcal{M}}$ holds.*

The proof is straightforward.

1·3 Remark Note that properties (1·0·0) – (1·0·3) are such that if for each i in some indexing class I $\mathcal{T}_i$ is fruitful then so is $\bigcap_{i \in I} \mathcal{T}_i$. We shall in later sections design for certain sets x that can be constructed using Δ_0 replacement a fruitful class $\mathcal{T}_x$ so that for fruitful $\mathcal{T} \subseteq \mathcal{T}_x$, $x \notin \bigcup \mathcal{T}$. We shall find large families of such x which are independent in the sense that for $x \neq y$, $\mathcal{T}_x$ does not exclude y from $\bigcup \mathcal{T}$ where $\mathcal{T} \subseteq \mathcal{T}_x$.

2: Families of functions

2·0 Definition Suppose that we have a function $Q : \omega \to V$ where each $Q(n)$ is a finite set.
We associate to each set x a function $f^Q_x : \omega \to \omega$ defined thus:

$$f^Q_x(n) = \overline{\overline{x \cap Q(n)}} \tag{2·0·0}$$

Given Q and a family $\mathcal{G}$ of functions from ω to ω, we may then define the class $\mathcal{T}^{Q,\mathcal{G}}$:

$$\mathcal{T}^{Q,\mathcal{G}} = \{x \mid \bigcup x \subseteq x \ \&\ f^Q_x \in \mathcal{G}\}. \tag{2·0·1}$$

Usually the function $n \mapsto \overline{\overline{Q(n)}}$ itself will not be in $\mathcal{G}$, from which we shall infer that certain sets are excluded from our models.

2·1 Proposition *Suppose that $\mathcal{G}$ has the following four properties:*
(2·1·0) For each ordinal ζ, $f^Q_\zeta \in \mathcal{G}$; *(2·1·1) $f \leqslant g \in \mathcal{G} \implies f \in \mathcal{G}$;*
(2·1·2) $f \in \mathcal{G} \ \&\ g \in \mathcal{G} \implies f + g \in \mathcal{G}$; *(2·1·3) $\bigcup x \subseteq x \ \&\ f^Q_x \in \mathcal{G} \implies f^Q_{\mathcal{P}(x)} \in \mathcal{G}$.*
Then $\mathcal{T}^{Q,\mathcal{G}}$ is fruitful.

Proof : note that $f^Q_{x\cup y} \leqslant f^Q_x + f^Q_y$ and that if $A \subseteq \mathcal{P}(x)$ and $x \in \mathcal{T}^{Q,\mathcal{G}}$, then $f^Q_{x\cup A} \leqslant f^Q_x + f^Q_{\mathcal{P}(x)}$. ⊣ (2·1)

2·2 Remark In the final section an example is given where $Q(n)$ is not required to be finite. In such a case the definition of f^Q_x and of a family $\mathcal{G}$ uses cardinal arithmetic without choice, rather than finite arithmetic.

3: An example

3·0 Definition For each $k \in \omega$ we define the function $b_k : \omega \longrightarrow \omega$ by $b_0(n) = n$, $b_{k+1}(n) = 2^{b_k(n)}$.
Thus $b_1(n) \equiv 2^n$, $b_2(n) \equiv 2^{2^n}$, $b_3(n) \equiv 2^{2^{2^n}}$, and so on.

3·1 Definition We also define a function $h : \omega \to \omega$ by $h(0) = 0$, $h(n+1) = 2^{h(n)}$

3·2 Definition For $f : \omega \to \omega$ we define $f^* : \omega \to \omega$ by $f^*(0) = 0$, $f^*(n+1) = 2^{f(n)}$

3·3 Definition $\mathcal{F} =_{\mathrm{df}} \{f : \omega \to \omega \mid \exists k{:}{\in}\omega\ f \leqslant b_k\}$.

3·4 Proposition *(i) $b_0 \in \mathcal{F}$; (ii) $f \in \mathcal{F} \implies f + 1 \in \mathcal{F}$; (iii) $f \in \mathcal{F} \ \&\ g \in \mathcal{F} \implies f + g \in \mathcal{F}$;*
(iv) $f \in \mathcal{F} \implies f^* \in \mathcal{F}$; (v) $g \leqslant f \in \mathcal{F} \implies g \in \mathcal{F}$; (vi) $h \notin \mathcal{F}$.

3·5 Remark As noted by the referee, the idea at work here, that of a function h growing faster than each in the sequence b_k goes back to du Bois Reymond's Freiburg study [BR] of rates of growth.

3·6 Definition In this section, let $R(n) = V_n$; write f_x for f^R_x, so that for any set x $f_x : \omega \longrightarrow \omega$ is defined by $f_x(n) = \overline{\overline{x \cap V_n}}$; write $\mathcal{T} = \mathcal{T}^{R,\mathcal{F}}$; and take $\mathcal{M} = \bigcup \mathcal{T}$.

So f_x counts, for each n, the number of elements of x of rank less than n; as $V_n \subseteq V_{n+1}$, f_x is weakly increasing.

3·7 Proposition *For each n, $h(n) = \overline{\overline{V_n}}$. Hence for any x, $f_x \leqslant h$.*

3·8 Theorem *The class $\mathcal{M}$ is supertransitive and contains all ordinals. Further for each axiom Φ of Zermelo set theory, the relativisation $(\Phi)^{\mathcal{M}}$ of Φ holds. If we assume the axiom of choice, the same will be true of that. $\mathcal{M}$ does not think that V_ω is a set, but it does believe that every set is a member of a transitive set.*

By the results of the opening section, it is enough for the first part of the theorem to verify that $\mathcal{T}$ is fruitful, and therefore enough to verify that $\mathcal{F}$ has properties (2·1·0) – (2·1·3). In view of Proposition 3·4, the following two lemmata suffice.

3·9 Lemma *If ζ is an ordinal, $f_\zeta \leqslant b_0 \in \mathcal{F}$.*

Proof : $\zeta \cap V_n = \zeta \cap n$. ⊣ (3·9)

3·10 Lemma $f_x \in \mathcal{F} \implies f_{\mathcal{P}(x)} \in \mathcal{F}$.

Proof : $f_{\mathcal{P}(x)}(0) = 0$ and $\mathcal{P}(x) \cap V_{n+1} = \mathcal{P}(x \cap V_n)$; hence $f_{\mathcal{P}(x)}(n+1) = 2^{f_x(n)}$ and so $f_{\mathcal{P}(x)} = f^*_x$. ⊣ (3·10)

For the last part,

3·11 LEMMA $V_\omega \notin \mathcal{M}$.

Proof : Suppose that $V_\omega \in z$ and that z is transitive. Then $f_z \equiv h \notin \mathcal{F}$, so $z \notin \mathcal{T}$. ⊣ (3·11)

The proof of Theorem 3·8 is complete.

3·12 REMARK The family $\mathcal{F}$ used above is the least that would satisfy all the clauses of Proposition 2·1, but one can find larger families $\mathcal{G}$ of functions with the properties 3·4 (*i – vi*); and hence we have a method for obtaining fatter models than those given above from which nevertheless V_ω is excluded.

3·13 REMARK If V_ω is not a set, then no set can be closed under the power set operation, or any other operation that will produce the empty set and then in turn the members of the various V_n's: for V_ω would be a subclass of that closed set, and hence would itself be a set by the Separation scheme.

3·14 REMARK It is tempting for the theorem to consider instead of the model $\mathcal{M} = \bigcup \mathcal{T}$ the class $\mathcal{N} =_{\mathrm{df}} \{a \mid f_a \in \mathcal{F}\}$, arguing that we are working in a set theory which proves that every set is a subset, or indeed a member, of a transitive set. But that gets into trouble for there is a set with small growth but whose transitive closure has large growth. Indeed, let $a = \{V_n \mid n \in \omega\}$. That is a set in $Z + KP$. $f_a \equiv b_0$, so $f_a \in \mathcal{F}$, but for any transitive x with $a \subseteq x$, $f_x \equiv h$, and so no such x is in $\mathcal{N}$, and that a is not in M. Even worse, let $b = a \cup \{V_\omega\}$. $f_b \equiv f_a$, and so $b \in \mathcal{N}$ but $V_\omega \notin \mathcal{N}$, and so since $V_\omega \in b$, $\mathcal{N}$ is not transitive.

4: Zermelo towers

4·0 DEFINITION For any set a we define the *Zermelo tower over a* as follows.

$$Z_0(a) = \varnothing, \quad Z_1(a) = \{a\}, \quad Z_{n+1}(a) = \{a\} \cup \left[\mathcal{P}(Z_n(a)) \setminus \{\varnothing\}\right], \quad Z(a) = \bigcup_{n<\omega} Z_n(a).$$

Thus $V_\omega = Z(\varnothing)$, and $Z(a)$ is the result of looking at V_ω as a collection of words in {, } and $\varnothing$ and replacing all occurrences of $\varnothing$ by a.

As the computations of the present section will show, the Zermelo tower $Z(\omega)$ is in the model $\mathcal{M}$ of the previous section, for its transitive closure is $\omega \cup Z(\omega)$, but $Z(\varnothing)$, to which it is isomorphic, is not.

4·1 LEMMA *For each a and $n \in \omega$, $Z_n(a) \subseteq Z_{n+1}(a)$.*

Proof : trivially true for $n = 0, 1$. If true for $n = k > 0$, $\mathcal{P}(Z_k(a)) \subseteq \mathcal{P}(Z_{k+1}(a))$, and that inclusion will remain true when $\{a\}$ is added and $\varnothing$ removed from both sides. ⊣ (4·1)

4·2 LEMMA *If $b \in Z(a)$ then $Z(b) \subseteq Z(a)$.*

Proof : let $b \in Z_{k+1}(a)$. Then $Z_1(b) = \{b\} \subseteq Z_{k+1}(a)$. An easy induction now shows that for each $n > 0$, $Z_n(b) \subseteq Z_{k+n}(a)$. ⊣ (4·2)

4·3 LEMMA *If $\zeta < \varrho(x)$ where x is any set, then for some $y \in TC(x)$, $\varrho(y) = \zeta$.*

Proof : by an easy induction on the rank of x. ⊣ (4·3)

4·4 LEMMA *If $w \in Z(a)$ then $\varrho(a) \leqslant \varrho(w) < \varrho(a) + \omega$ and $\varrho(w) = \varrho(a) \iff w = a$. Further, if $\varrho(w) > \varrho(a)$ then a is the unique member of $TC(w)$ of rank $\varrho(a)$.*

Most of the members of $Z(a)$ are finite non-empty sets and thus of successor rank: we shall utilise this fact to prove that for a and b distinct sets both of non-successor rank, the presence or absence of one of $Z(a)$ and $Z(b)$ in a model $\mathcal{M}$ does not imply the presence or absence of the other.

We write $T(b)$ for $TC(\{Z(b)\})$: thus $T(b) = TC(b) \cup Z(b) \cup \{Z(b)\}$.

4·5 LEMMA *If $w \in T(b)$ and $\varrho(b) = \beta$, a limit ordinal, then exactly one of the following holds:*

- *(4·5·0)* $\varrho(w) = \beta + \omega$ *and* $w = Z(b)$
- *(4·5·1)* $\beta < \varrho(w) < \beta + \omega$, *$w$ is a finite set, and b is the unique member of $TC(w)$ of rank β*
- *(4·5·2)* $\beta = \varrho(w)$ *and* $w = b$
- *(4·5·3)* $\beta > \varrho(w)$ *and* $w \in TC(b)$.

4·6 LEMMA *Let $Z(a) \cap T(b)$ be non-empty, where both $\varrho(a)$ and $\varrho(b)$ are limit ordinals. Then precisely one of the following holds: $a = Z(b)$, $a = b$, $a \in TC(b)$.*

Proof : we write $\alpha = \varrho(a)$, $\beta = \varrho(b)$. Let $w \in Z(a) \cap T(b)$. We argue by cases on $\varrho(w)$, bearing in mind that $\alpha \leqslant \varrho(w) < \alpha + \omega$. If $\varrho(w) = \beta + \omega$, then $w = a = Z(b)$, by 4·4 and 4·5. If $\beta < \varrho(w) < \beta + \omega$, then $\alpha = \beta$, and so $a = b$, each being the only member of $TC(w)$ of rank α or β accordingly. If $\varrho(w) = \beta$, again $\alpha = \beta$ and $a = w = b$. Finally, if $\varrho(w) < \beta$, $w \in TC(b)$, and so $a \in TC(b)$. ⊣ (4·6)

Now, to simplify matters, we consider only sets a and b which are functions from some limit ordinal to $2 = \{0, 1\}$. Thus each member of a or b is an ordered pair $\langle \alpha, \zeta \rangle$ where $\alpha = 0$ or 1. In these circumstances clause (4·5·3) may be strengthened to read $\beta > \varrho(w)$ *and for some* $i = 0$ *or 1, and* $\nu < \beta$, $w \in TC(\{\langle i, \nu \rangle\})$.

4·7 Proposition *Let α and β be (possibly equal) limit ordinals. Let $a : \alpha \to 2$ and $b : \beta \to 2$. Then if $a \neq b$, $Z(a) \cap T(b) = \emptyset$.*

We emphasize that the assumptions permit $\alpha < \beta$, $\alpha = \beta$ or $\beta < \alpha$.

Proof : in these circumstances, $a \neq Z(b)$ and $a \notin TC(b)$. ⊣ (4·7)

4·8 Theorem *For each limit ordinal λ let $A_\lambda \subseteq \{a \mid a : \lambda \longrightarrow 2\}$. Then there is a supertransitive class $\mathcal{M}$, thus containing all ordinals, sequences of ordinals and sets of sequences of ordinals, which is a model of Zermelo set theory, and is such that for every limit λ*

$$A_\lambda = (\{a \in {}^{\lambda}2 \mid Z(a) \textit{ is a set } \})^{\mathcal{M}}.$$

Proof : let $\mathcal{S} = \{c : \lambda \to 2 \mid 0 < \lambda = \bigcup \lambda \in ON\}$. For each $a \in \mathcal{S}$ and each set x, we set $f^a_x(n) = \overline{\overline{x \cap Z_n(a)}}$. With $\mathcal{F}$ as above, we define $\mathcal{T}^a = \{x \mid \bigcup x \subseteq x \;\&\; f^a_x \in \mathcal{F}\}$ and verify that if $f^a_x \leqslant b_k$, $f^a_{\mathcal{P}(x)} \leqslant b_{k+1}$. Hence each class $\mathcal{T}^a$ is fruitful. Proposition 4·7 shows that for distinct a, b in $\mathcal{S}$, $f^a_{T(b)} \equiv 0$, and so $T(b) \in \mathcal{T}^a$. Thus $Z(b) \notin \bigcup \mathcal{T}^b$ but $Z(b) \in \bigcap\{\mathcal{T}^a \mid a \in \mathcal{S} \;\&\; \neq b\}$.

Hence if we put $\mathcal{I} = \{c : \lambda \to 2 \mid \lambda$ is a limit ordinal and $c \notin A_\lambda\}$, $\mathcal{T} = \bigcap\{\mathcal{T}^c \mid c \in \mathcal{I}\}$ and $\mathcal{M} = \bigcup \mathcal{T}$, then $\mathcal{T}$ will be fruitful, and $\mathcal{M}$ will be a supertransitive model of Zermelo containing all ordinals. Further, for each limit λ and $a : \lambda \to 2$, $Z(a) \in \mathcal{M} \iff a \notin \mathcal{I} \iff a \in A_\lambda$, as required. ⊣ (4·8)

4·9 Problem Let M be a countable transitive model of $Z + KP$. Is it true that M will have $2^{\aleph_0}$ distinct transitive subsets, each modelling $Z + TC$ and no two with the same theory ?

The difficulty there is that our proof that $\mathcal{M}$ is a model of Zermelo relies on the scheme $x \cap C \in V$ being true in the outer model when C is a class defined by a formula which might mention the fruitful class $\mathcal{T}$.

One case where M will have numerous transitive submodels each modelling Zermelo set theory is this: let N be a countable transitive model such that some countable ordinal κ is a measurable cardinal in N, and let M be $V_\kappa \cap N$. Fix in N a normal measure $\mathcal{U}$ on κ, and let A_0 be a Prikry sequence over N with respect to $\mathcal{U}$. Thus A_0 is a subset of κ cofinal in κ and of order type ω, such that for each $B \in \mathcal{U}$, $A_0 \setminus B$ is finite. Every infinite subset A of A_0 is also Prikry generic over N, (Mathias [M1]) and for each such A, $N[A] \cap V_\kappa = N \cap V_\kappa$: hence we may treat A as a class over M, and thus form a transitive $\mathcal{M}^A \subseteq M$ which is supertransitive in the sense of M, so that $y \subseteq x \in \mathcal{M}^A \implies y \in \mathcal{M}^A$, and such that

$$\forall \lambda \big[0 < \lambda = \bigcup \lambda < \kappa \implies (Z(\lambda) \in \mathcal{M}^A \iff \lambda \in A) \big].$$

Thus the $\mathcal{M}^A$'s are distinct for different A and there are $2^{\aleph_0}$ infinite subsets A of A_0. However, the pure theory of each $\mathcal{M}^A$, with a predicate for A but no name for individual ordinals, will be, under reasonable coding, a member of $N[A] \cap V_{\omega+1}$ and therefore of M and so the continuum many models $\mathcal{M}^A$ will only realise countably many theories.

Thus the problem is to find ways of adding classes to M without adding sets, so that new inner models of M become definable: in short, the search is on for "generic inner models".

4·10 Remark One can define the sequence of sets V_n in Zermelo, and also the rank function restricted to $\bigcup_n V_n$, without assuming the set-hood of V_ω. Similarly one may in Zermelo define $\varrho \restriction Z(a)$ for each a, as a class but not as a set.

For the theorem to have maximum impact it was necessary to start from a universe in which each $Z(a)$ exists: hence we assumed the axioms of Kripke–Platek in addition to those of Zermelo. If we started from a universe in which some $Z(a)$'s did not exist, our arguments would yield methods of proceeding to submodels from which selected further $Z(a)$'s were excluded.

4·11 Problem Lemma 4·2 shows that if $Z(a)$ is in some supertransitive model $\mathcal{M}$ and $b = \{a\}$, then $Z(b) \in \mathcal{M}$. Is it true that if $b = \{a\}$ and $Z(b)$ exists then so does $Z(a)$?

5: Concluding remarks

5·0 Models where V_ω is not a set have been known for some time: e.g., define

$$\mathcal{K}_0 = \omega, \quad \mathcal{K}_{n+1} = \mathcal{P}(\mathcal{K}_n), \quad \mathcal{K} = \bigcup_{n<\omega} \mathcal{K}_n.$$

Then $\mathcal{K}$ is a transitive model of Zermelo + Transitive Containment + V_ω is not a set. That model differs from ours in two ways: first, our $\mathcal{M}$ contains all ordinals, whereas $\mathcal{K} \cap ON = \omega + \omega$; and secondly, the set $D =_{\rm df} \{0, \{0\}, \{\{0\}\}, \{\{\{0\}\}\}, \ldots\}$ is a member of $\mathcal{M}$ but not a member of $\mathcal{K}$. The first point of disagreement would disappear if one varied the above definition by starting from $\mathcal{K}_0 = ON$; but the second is a substantial difference.

Moschovakis [Mo], page 177 onwards, gives a similar construction, but starting from $\mathcal{K}_0 = D$; there is an earlier mention in Enderton [E], page 178. Perhaps the earliest argument of this type is that given by Skolem in his 1922 critique [Sk] of Zermelo set theory, when in illustration of the fourth of his eight points he shows that $\{D, \mathcal{P}(D), \mathcal{P}(\mathcal{P}(D)), \ldots\}$ cannot be proved in Zermelo to be a set.

Repeated application of Lemma 1·4 (*iv*) and the argument of Lemma 2·14 shows that $\mathcal{K} \subseteq \mathcal{M}$.

5·1 V_ω is a set of the model given in section 4. To see that, go back to Proposition 4·6: if $Z(a) \cap T(b)$ is nonempty, then $a = Z(b)$, or $a = b$, or $a \in TC(b)$. Now if $b = \varnothing$, and $a \in \mathcal{S}$, that rapidly tells us that $Z(a) \cap T(b) = \varnothing$, and so $Z(\varnothing) \in \mathcal{M}$.

We could arrange to exclude V_ω, if we wanted, by intersecting the $\mathcal{T}$ of section 4 with the $\mathcal{T}$ of section 2. To see that the existence of other $Z(b)$ is unaffected, we must check that for $b \in \mathcal{S}$, $f^{\varnothing}_{T(b)} \in \mathcal{F}$. Thus, we must estimate the size of $TC(b) \cap V_n$. Inspection shows that for $n \geqslant 4$, and $b : \lambda \to 2$, $\overline{\overline{TC(b) \cap V_n}} \leqslant 3n + 1$, which certainly gives $f^{\varnothing}_{T(b)} \in \mathcal{F}$.

In particular the existences of $Z(0)$ and of $Z(\omega)$ are independent of each other in Zermelo set theory, yet they are both isomorphic to Ackermann's coding [Ack] of HF as a subset of ω !

5·2 We have assumed $Z + KP$ throughout, but we need much less, and could carry out our arguments in the theory $MAC + KP$, discussed in Mathias [M3]. What we would get is that our $\mathcal{M}$'s are always models of MAC; if the classes $\mathcal{T}$ are defined by Σ_1 formulæ, and Σ_1 separation is assumed in the starting universe, then each of the models $\mathcal{M}$ will also be models of Σ_1 separation.

5·3 For $\mathcal{M}$ a model of the kind constructed in §4,

(5·3·0) $\mathcal{M}$ thinks that every well-ordering is isomorphic to an ordinal, but

(5·3·1) $\mathcal{M}$ does not think that every well-founded extensional relation is isomorphic to a transitive set;

so these models distinguish the "orders of difficulty" of various set-theoretic recursive constructions. Further,

(5·3·2) $\mathcal{M}$ is closed under each of Jensen's rudimentary functions, but

(5·3·3) no member of $\mathcal{M}$ is rudimentarily closed;

so Zermelo set theory does not even begin to supply a foundation for the detailed study of constructibility initated by Jensen thirty years ago.

5·4 The construction of sections 2 and 3 grew out of an attempt to modify a model proposed by Thomas Forster. His model is essentially the result of taking $\mathcal{G}$ to be the set of bounded functions from ω to ω and Q to be the function $n \mapsto \{\{m\} \mid 1 \leqslant m < n{-}1\}$. It satisfies all the axioms of Zermelo + Transitive Containment except the power set axiom, and does not contain V_ω.

The construction of section 4 was inspired by Kanamori's suggestion that my idea of using the rates of growth of functions to control entry into the model would have further applications. In particular he suggested considering for each set A the function $Q^A : \omega \to V$ defined with the help of a modified power-set operation $\mathcal{Q}$, where $\mathcal{Q}(x) =_{\rm df}$ the set of infinite subsets of x, by setting

$$Q^A(0) = A, \qquad Q^A(n+1) = \mathcal{Q}(Q^A(n)).$$

One would then define for any set A, bounding functions b^A_k with values possibly infinite cardinals rather than members of ω, and thence a family $\mathcal{F}^A$ being those functions bounded by some b^A_k, thus:

$$b^A_0(n) = \overline{\overline{A}} + n; \quad b^A_{k+1}(n) = 2^{b^A_k(n)}; \quad \mathcal{F}^A = \{f : \omega \to CARD \mid \exists k {:\in} \omega \; f \leqslant b^A_k\}.$$

Here, for A infinite, the formulæ must be interpreted according to the usual rules of naïve cardinal arithmetic, but the previous proofs will go through without the Axiom of Choice. If we are in a universe where both Foundation and AC fail, we will have to paraphrase certain definitions in terms of injections so as to avoid mention of $CARD$, the class of cardinals.

Finally, one would define for each x, the function g_x^A to be $n \mapsto \overline{\overline{x \cap Q^A(n)}}$, $\mathcal{T}^A$ to be the class of those transitive x with $g_x^A \in \mathcal{T}^A$; then one could select certain $\mathcal{T}^A$'s, or indeed form $\mathcal{T}^\infty =_{\rm df} \bigcap_{A \in V} \mathcal{T}^A$, and $\mathcal{M}^\infty =_{\rm df} \bigcup \mathcal{T}^\infty$: this model will contain all ordinals and will be slimmer than the model of §3, as in it for every x and every A, $g_x^A \in \mathcal{F}^A$. For example it will contain no set closed under the operation $\mathcal{Q}$.

Plainly there are boundless possibilities here for building models of various degrees of fatness.

5·5 Lastly we consider the modification of Zermelo where we replace the specific axiom $\omega \in V$ by the more general "there is a Dedekind-infinite set". Call that theory Z'.

5·6 THEOREM *There are two transitive models of Z', $\mathcal{J}$ and $\mathcal{K}$, with $\mathcal{J} \cap \mathcal{K} = V_\omega$.*

Let $\mathcal{J}$ be the model obtained as follows. Define $z(0) = 0$, $z(n+1) = \{z(n)\}$, and $D = \{z(n) \mid n \in \omega\}$, a more formal definition of the set $\{0, \{0\}, \{\{0\}\}, ...\}$ introduced in 5·0. Notice that D is transitive. Set $\mathcal{J}(0) = D$, $\mathcal{J}(n+1) = \mathcal{P}(\mathcal{J}(n))$ and let $\mathcal{J} = \bigcup_{n<\omega} \mathcal{J}(n)$. Then ω is not a member of $\mathcal{J}$, by our first

5·7 LEMMA *$x \cap \omega = n$ implies $\mathcal{P}(x) \cap \omega = n+1$*

which is readily proved by induction on n and since $D \cap \omega = 2$ has the

5·8 COROLLARY *$\mathcal{P}^k(D) \cap \omega = k+2$.*

$\mathcal{J}$ is a model of the rest of Zermelo, as the set of $\mathcal{J}(n)$'s is fruitful except for (1·0·1); D is a member of $\mathcal{J}$, and is Dedekind infinite under the map $z \mapsto \{z\}$; and indeed $\mathcal{J}$ contains a relation on D which well-orders it in order type ω.

Let $\mathcal{K}$ be as defined in 5·0 by setting $\mathcal{K}(0) = \omega$, $\mathcal{K}(n+1) = \mathcal{P}(\mathcal{K}(n))$ and $\mathcal{K} = \bigcup_{n<\omega} \mathcal{K}(n)$. $\mathcal{K}$ is a model of Zermelo containing ω but not containing D. Define $s(n) = \{z(m) \mid m < n\}$.

5·9 LEMMA *$0 = s(0)$, $1 = s(1)$; $2 = s(2) = \omega \cap D$; $x \cap D = s(n)$ implies $\mathcal{P}(x) \cap D = s(n+1)$.*

5·10 COROLLARY *$\mathcal{P}^k(\omega) \cap D = s(k+2)$.*

Proof of the theorem: Both $\mathcal{J}$ and $\mathcal{K}$ are models of Z', so we need only show that $\mathcal{J} \cap \mathcal{K} = V_\omega$. Suppose therefore that $x \in \mathcal{P}^k(D) \cap \mathcal{P}^m(\omega)$: we must show that $x \in V_\omega$. If $k \geq m$, then $\bigcup^m x \subseteq \mathcal{P}^{k-m}(D) \cap \omega = k - m + 2$; if $k < m$, then $\bigcup^k x \subseteq D \cap \mathcal{P}^{m-k}(\omega) = s(m-k+2)$. In either case, $\bigcup^j x$ for some finite j is a subset of a hereditarily finite set, and therefore x is hereditarily finite. ⊣ (5·6)

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