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THE REAL LINE AND THE UNIVERSE

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Part One reviews the development of Zermelo-Fraenkel set theory and is written for the general reader; Part Two continues with a list, largely collected at the meeting, of open problems in set theory, and is for the specialist; Part Three summarises the philosophical speculations with which the author began his Oxford address.

PART ONE

The universe, V, is the collection $\{x|x=x\}$ of all sets. To give some structure to it, we arrange it in a hierarchy by defining, by induction through ths class On of all ordinals,

$$V_0 = 0, \text{ the empty set}$$

$$V_{\zeta+1} = P(V_\zeta) \text{ the power set } \{x|\ x \subseteq V_\zeta\} \text{ of } V_\zeta$$

for limit λ, $V_\lambda = \bigcup\{V_\zeta|\ \zeta < \lambda\}$;

and as a consequence of the axiom of foundation, we have

$$V = \bigcup\{V_\zeta|\ \zeta \in On\}.$$

A natural question to ask, once its use has been observed in mathematics, is this:

Is AC, the axiom of choice, true?

Gödel in 1938 gave a partial answer: he defined a collection L of sets by another recursion on the ordinals:

$$
\begin{aligned}
L_0 &= 0\\
L_{\zeta+1} &= \mathrm{Def}(L_\zeta)\\
L_\lambda &= \bigcup\{L_\zeta \mid \zeta < \lambda\} \quad \text{for limit } \lambda,\\
\text{and}\quad L &= \bigcup\{L_\zeta \mid \zeta \in \mathrm{On}\},
\end{aligned}
$$

where $\mathrm{Def}(L_\zeta)$ is the set not of all subsets of L_ζ, as in the recursive definition of the V_ζ's, but of all definable subsets of L_ζ, definable meaning of the form

$$\{x \in L_\zeta \mid \phi(x,\vec{y}) \quad \text{is true in } L_\zeta\}$$

for some finite sequence $\vec{y}$ of elements of L_ζ and some formula ϕ of the language of set theory. We write '$L_\zeta \models \phi$' to mean 'ϕ is true in L_ζ'.

The class L contains all ordinals, and is such that $x \in y \in L \implies x \in L$; further all axioms of Zermelo-Fraenkel set theory are true when interpreted in L. Such classes are called inner models and L is the smallest such.

The hypothesis that V = L is known as the axiom of constructibility; Gödel showed that if ZF is consistent, that is, is free of contradiction, so is ZF+V = L, and that both the axiom of choice and the generalised continuum hypothesis (GCH) follow from it. Subsequent investigations, summarised in Mathias [1], have shown the axiom of constructibility decides many set-theoretical hypotheses known to be undecided by ZF+AC alone, and applications of it have been made in other branches of mathematics. For example, each of the following statements is a consequence of V = L:

every Whitehead group is free (Shelah [2],[3])

every locally compact normal Moore space is metrisable (Fleissner [4])

there is a counter-example to Souslin's hypothesis (Jensen [5])

for each n there is a space of (covering) dimension n the one-point compactification of which has dimension 0 (Ostaszewski [6])

the gap-n conjecture in model theory (Jensen [7]).

We may therefore ask instead:

Is V = L?

Sad to say, Cohen in 1963 showed that if ZF is consistent so is ZF + GCH + V ≠ L; moreover so are ZF + not AC and ZF + AC + not GCH. His method as later reformulated by set theorists, consists

of an expansion of the universe. Start with a complete Boolean algebra $\mathbb{B} = \langle B,+,\cdot,-,\mathbb{0},\mathbb{1}\rangle$; and define the B-valued universe $V^{\mathbb{B}}$ by induction on the ordinals:

$$
\begin{aligned}
V_0^{\mathbb{B}} &= 0\\
V_{\zeta+1}^{\mathbb{B}} &= \{f \mid \exists u (u \subseteq V_\zeta^{\mathbb{B}} \text{ and } f\colon u \to B)\}\\
V_\lambda^{\mathbb{B}} &= \bigcup\{V_\zeta^{\mathbb{B}} \mid \zeta < \lambda\}\\
V^{\mathbb{B}} &= \bigcup\{V_\zeta^{\mathbb{B}} \mid \zeta \in \mathrm{On}\}.
\end{aligned}
$$

For u in the domain of f, f(u) is, approximately, the Boolean truth-value of the statement "$u \in f$". Definitions of the truth values $[\![g \in f]\!]^{\mathbb{B}}$ and $[\![g = f]\!]^{\mathbb{B}}$ may be given by recursion, for all pairs $g,f \in V^{\mathbb{B}}$.

Consider the case of the two-element algebra $\mathbb{2} = \{\mathbb{0},\mathbb{1}\}$; $V^{\mathbb{2}}$ is, once factored by the equivalence relation induced by identifying f and g whenever $[\![f = g]\!]^{\mathbb{2}} = 1$, isomorphic to the original universe V. As $\mathbb{2}$ is a complete subalgebra of any complete $\mathbb{B}$, $V^{\mathbb{2}} \subseteq V^{\mathbb{B}}$, and thus $V^{\mathbb{B}}$ may be regarded as an extension of the original universe V. Such extensions are called Boolean extensions. If we were discussing instead of V a countable transitive model M, with $\mathbb{B}$ a member of M and a complete Boolean algebra in the sense of M, then we would be able, by factoring $\mathbb{B}$ and hence $M^{\mathbb{B}}$ by a suitable ultrafilter, to convert $M^{\mathbb{B}}$ to a countable transitive model $N \supseteq M$ having the same ordinals as M. We could then express within N the statement that N is a (factored) Boolean extension of M.

Being unable to decide whether V = L, let us reformulate the question thus:

Is V a Boolean extension of L?

One case where the answer is known to be negative is when there exist measurable cardinals. Originally, a cardinal κ was defined to be measurable if there was a measure taking only the values 0 and 1, defined on all subsets of κ, assigning measure 1 to κ, measure 0 to each singleton $\{\zeta\}$ $(\zeta<\kappa)$, and countably additive. It was noticed that the least such cardinal κ_0 had the further property that for each λ less than κ_0, the union of λ sets of measure 0 was again of measure 0; so with a view to generalisation that property of κ-additivity of the measure was added to the definition of the measureability of κ.

In fact if there is a measurable cardinal then the constructible universe, L, becomes very small in relation to V: let us denote by

Φ the following hypothesis

for every uncountable cardinal λ there is a closed unbounded subset X_λ of λ such that

(1) for any positive integer n any formula $\phi(x_1,\ldots,x_n)$ of the language of set theory and any two ascending n-tuples $\xi_1 < \ldots < \xi_n$, $\zeta_1 < \ldots < \zeta_n$, from X_λ,

$$L_\lambda \models \phi(\xi_1,\ldots,\xi_n) \longleftrightarrow L_\lambda \models \phi(\zeta_1,\ldots,\zeta_n).$$

(2) for each member x of L_λ there is a ϕ and $\zeta_1,\ldots,\zeta_n$ in X_λ such that for all y in L_λ,

$$L_\lambda \models \phi(y,\zeta_1,\ldots,\zeta_n) \text{ iff } y = x.$$

Moreover, if $\lambda < \kappa$, and λ,κ are uncountable cardinals, $X_\lambda = \lambda \cap X_\kappa$, and L_λ is an elementary submodel of L_κ in the sense that for any $\phi(\ldots\ldots)$ and $x_1,\ldots,x_n \in L_\lambda$.

$$L_\lambda \models \phi(x_1,\ldots,x_n) \text{ iff } L_\kappa \models \phi(x_1,\ldots,x_n).$$

There is a property $\Psi(\alpha)$ of real numbers α, such that it is provable in ZF that at most one α has the property Ψ; if V = L, no α has the property Ψ; and that $\exists\alpha\Psi(\alpha)$ iff the hypothesis Φ above holds. Moreover, Ψ is expressible purely in terms of integers, real numbers and recursive functions thereof.

The unique α of $\Psi(\alpha)$ is known as $0^\#$: it may be proved that

$0^\#$ is in no Boolean extension of L.

The assertion that $0^\#$ does not exist is denoted by $\neg 0^\#$.

A measure of the significance of $0^\#$ is given by a theorem of Jensen; before it can be stated, some comments on constructibility and Boolean algebras are necessary.

First the concept of constructibility may be relativised: if A is a set of ordinals, we define

$$L_0[A] = \{A\} \cup \bigcup\{\zeta+1 \mid \zeta\in A\}$$
$$L_{\zeta+1}[A] = \mathrm{Def}(L_\zeta[A])$$
$$L_\lambda[A] = \bigcup\{L_\zeta[A] \mid \zeta<\lambda\}$$
$$L[A] = \bigcup\{L_\zeta[A] \mid \zeta\in On\}.$$

Then L[A] is the smallest inner model of which A is a member. If A is a class of ordinals, we define

$$L[A] = \bigcup_\zeta\{L[A\cap\zeta] \mid \zeta\in On\}.$$

L is thus L[O], and we may generalise the concept of $0^\#$ by defining $A^\#$, as is done in Boos [8].

Second, the Boolean algebras occurring in applications of Cohen's method are usually given in terms of a partial ordering, called the set of conditions, which generates the algebra. A natural generalisation of that is to consider partial orderings which are a proper class, though caution must be exercised as the Boolean extension corresponding to such a class need not be a model of ZF.

With those remarks in mind, let us enunciate Jensen's theorem:

Let M be a countable transitive model of ZF + GCH + $\neg 0^\#$; and let θ be the least ordinal not in M. Then there is an $a \subseteq \omega = \{0,1,2,\ldots\}$ such that, writing N for $L_\theta[a]$,

$$M \subseteq N;$$
$$N \models ZF + GCH + \neg 0^\# ;$$
$$N \models V = L[a];$$

the cardinals of N are exactly those of L, and in the passage from M to N cofinalities are preserved, (as are the large cardinal properties of being Mahlo, weakly compact and ineffable). Moreover N may also be construed as a Boolean extension of M with respect to a collection of conditions that is a class of M.

Since any model of ZF + AC has a Boolean extension, via a class of conditions, which satisfies GCH, the theorem shows, loosely, that if $0^\#$ does not exist, the universe may be coded by a single real, in the sense of being contained in some class-generic extension of the form L[a] with $a \subseteq \omega$.

To return to the question "Is V a Boolean extension of L?", Jensen's coding theorem shows a connection between that question and the existence of $0^\#$. In fact Jensen's paper [ ] of which the above is Theorem 1, contains further information. Call a set $X \subseteq On$ set generic over L if L[X] is a Boolean extension of L with respect to a Boolean algebra which is a set in L. To quote from Jensen [ ]:

"A well-known conjecture of Solovay is this:

(SC) If $a \subseteq On$ is a set such that $L[a] \models \neg 0^\#$, then a is set-generic.

"Using Theorem 1 we can construct a model in which SC fails... There is a weaker form of SC which reads:

(WSC) If $a \in L[0^\#]$ and $0^\# \notin L[a]$ then a is set generic.

"Theorem 2 Assume that $0^\#$ exists. Then there is $a \subseteq \omega$ such that (a) a is not set generic

(b) L[a] has the preservation properties of Theorem 1 with respect to L (hence $0^{\#} \notin L[a]$)

(c) $0^{\#}$ and $a^{\#}$ are recursive in each other.

"We have thus shown that SC is not provable in ZFC, even from $\neg 0^{\#}$, and that WSC + "$0^{\#}$ exists" is provably false."

Thus $\neg 0^{\#}$ emerges as an axiom that constrains the universe severely. Here is one consequence; others will be mentioned in Part II.

Jensen's Covering Lemma. If $0^{\#}$ does not exist, then given any uncountable set $X \subseteq On$, there is a Y in L, such that $X \subseteq Y \subseteq On$ and X and Y have the same cardinal.

If $\neg 0^{\#}$ were provable, the effect on set theory would be very marked: in the light of Jensen's theorem the notion of relative constructibility would emerge as the central concept of set theory, and the union of set theory and abstract recursion theory would be almost complete.

For a treatment of Gödel's work see Devlin [10]; for Cohen's method see Shoenfield [11] and Jech [12]; for sharps and large cardinals see Drake [13]; and for an exposition of Shelah's work on Abelian groups, see Eklof [14],[15].

PART TWO

Blass [16] has shown that the existence of measurable cardinals is equivalent to the existence of a non-trivial exact functor from Sets to Sets; two problems suggest themselves:

Problem 1 Find a category-theoretic formulation of $\neg 0^{\#}$, or of '$\forall a \subseteq \omega$ $a^{\#}$ exists'.

Problem 2 Find a proof of the equivalence of the two following statements that makes no mention of ultrafilters:

$\exists$ non-trivial elementary embedding of V into some inner model

$\exists$ non-trivial exact functor from Sets to Sets.

A third version of Solovay's is this: call a Π^1_2 predicate $\phi(a)$ singular if '$\exists^{\leq 1} a\phi(a)$' and '$V=L \to \neg\exists a\phi(a)$' are both theorems of ZF. The predicate '$a = 0^{\#}$' is singular.

Problem 3 Let $\phi(a)$ be singular. Is '$\exists a\phi(a) \to 0^{\#}$ exists' a theorem of ZF?

Some progress on this has been made by Jensen using the core model, K, discussed in Jensen and Dodd [17].

Jensen's coding theorem suggests that an attempt might now be made to classify inner models. There is work by Vopěnka and Hajek, Balcar [18] and Grigorieff [19] in this direction.

Boolean algebras supply extensions of the universe; the ultrapower method of Scott (see Kunen [20] and Gaifman [21]) provides a means, given measurable cardinals, of shrinking it. The recent theorem of Dehornoy [22] that the intersection of the first ω iterated ultrapowers of V is a Prikry generic extension of the ω-th, supplies one link between the two. A second means of combining Boolean-valued models and ultrapowers is known: the following definition is due to Prikry and Jech.

Let I be a κ-complete ideal on K. Let B be the regular minimal completion of the Boolean algebra $P(\kappa)/I$. In V^B there is an ultrafilter U in the algebra $P(\kappa)$ of standard subsets of κ that extends $\{X \subseteq \kappa \mid \kappa \setminus X \in I\}$, and with which the ultrapower V^{κ}/U of equivalence classes mod U of standard functions $f: \kappa \to V$ may be formed. I is termed precipitous if, with truth value 1, that ultrapower is well-founded. For an elementary alternative formulation and for applications, see the text of Jech's talk at this meeting.

Problem 4 (Jech) If there is a precipitous ideal, does there necessarily exist a normal one?

Theorem (Magidor) If Con($\exists \kappa$(κ is κ^+-supercompact)) then Con(the ideal of nonstationary subsets of ω_1 is precipitous).

Precipitous ideals have been used to obtain cardinality bounds, in the spirit of the proof and statement of the following

Theorem (Silver [23]) If $2^{\aleph_\nu} = \aleph_{\nu+1}$ for a stationary set of countable ordinals ν, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

Thus Jech and Prikry have proved from the assumption that there is a precipitous ideal on ω_1 the following

(*) If $\aleph_{\omega_1}$ is a strong limit cardinal, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.

The same conclusion has been derived by Magidor from the consistency-wise weaker assumption known as Chang's conjecture:

Any structure $\mathcal{M} = \langle\omega_2, \omega_1, R\ldots\rangle$ (where '$\in \omega_1$' is here regarded as an unary predicate) has an elementary substructure $\langle A, A\cap\omega_1, R', \ldots\rangle$ with $\bar{\bar{A}} = \aleph_1$ and $\overline{\overline{A \cap \omega_1}} = \aleph_0$.

But Magidor's work has itself been superseded by results of Shelah [24].

Problem 5 Is (*) provable in ZFC?

Problem 6 (Jech): Let $\phi(\zeta)$ be the ζ^{th} fixed point of the $\aleph$ function. Show in ZFC that if $\phi(\aleph_1)$ is a strong limit cardinal, then $2^{\phi(\aleph_1)} < \phi((2^{\aleph_1})^+)$.

Chang's conjecture is equivalent to

(CC_1) given $f: [\omega_2]^{<\omega} \to \omega_2$, $\exists A(\bar{\bar{A}} = \aleph_1$, $A \cap \omega_1$ countable, and A closed under f).

An ostensibly weaker form is

(CC_2) for each $f: [\omega_2]^{<\omega} \to \omega_1$ there is an A with $\bar{\bar{A}} = \aleph_1$ and $f''[A]^{<\omega}$ countable.

(CC_2) implies that $\forall \alpha\ \alpha^{\#}$ exists. A still weaker form is

(CC_3) for each $f: [\omega_2]^2 \to \omega_1$ there is an uncountable A with $f''[A]^2$ countable.

(CC_3) implies that there is no Kurepa $\aleph_1$-tree.

Problem 7 Is CC_3 equivalent to CC_1?

Problem 8 If not, is (CC_3) true in some Boolean extension of L?

Silver's consistency proof for CC_1, expounded in Devlin [25], proceeds via an intermediate extension in which Martin's axiom $+ \neg$ CH holds. If we write CC_1 in the notation $(\omega_2,\omega_1) \twoheadrightarrow_e (\omega_1,\omega)$ then

Problem 9 Is $(\omega_3,\omega_2) \twoheadrightarrow_e (\omega_2,\omega_1)$ consistent?

The natural generalisation of Silver's proof leads to a question concerning a version of MA for higher cardinals, for which see Devlin [26].

Problem 10 (Magidor) Refute the following version of MA:

if B does not collapse cardinals, any $\aleph_1$ dense sets can be diagonalised.

Problem 11 (Erdős, Hajnal) Establish the consistency of $\omega_2 \to (\omega_1+\omega)^2$

Problem 12 Establish the consistency of $2^{\aleph_0} = \aleph_1 + SH_{\aleph_2}$, or show that that implies the existence of $0^{\#}$.

Magidor shows in his paper [27] that if CC is true, then there is, in a certain Boolean extension, an ultrapower V^{κ}/U of the standard universe such that if $j: V \to V^{\kappa}/U$ is the canonical elementary embedding, $j(\omega_1)$ is well founded and isomorphic to ω_2.

Problem 13 Does that statement imply CC?

The problem seems similar to Kunen's question whether if there is an $\aleph_2$-saturated ideal on $\aleph_1$, $\aleph_1$ is huge in some inner model.

Another property of ultrafilters which turns out to be connected with sharps is that of regularity: an ultrafilter U is (κ,λ) regular if there is a family of λ elements of U the intersection of any κ of which is empty. An ultrafilter U on I is uniform if $\forall X \in U\ \bar{\bar{X}} = \bar{\bar{I}}$, and is regular if it is $(\omega,\bar{\bar{I}})$ regular. For recent work on this topic see Ketonen [28], and Jensen and Koppelberg [29].

Problem 14 If V = L, or, more generally, if $\neg 0^{\#}$, is every uniform ultrafilter on κ regular?

The case $\kappa = \aleph_n$ is known if V = L or if κ is not ineffable in L.

Problem 15 That every uniform ultrafilter on ω_1 is regular is a consequence of V = L. Is it a theorem of ZFC?

An easier form of Problem 15 is

Problem 16 If there is an irregular ultrafilter over ω_1, is there an inner model with a measurable cardinal? Is Chang's conjecture true?

Recall that a cardinal κ is huge if there is an elementary embedding j of the universe into some inner model M such that κ is the first ordinal moved by j, and every sequence of length $j(\kappa)$ of elements of M lies in M. For information on these and other large cardinals see [30].

Theorem (Magidor) If the existence of a huge cardinal is compatible with ZFC, so is the existence of an irregular uniform ultrafilter on ω_2.

Problem 17 Is it consistent to have a uniform ultrafilter U on ω_1 such that the ultrapower ω^{ω_1}/U is of cardinality $\aleph_1$?

Kunen proved that the consistency of the existence of a huge cardinal implies that of the existence of an $\aleph_2$-saturated ideal on $\aleph_1$. The following is still open.

Problem 18 Can the ideal of non-stationary subsets of ω_1 be $\aleph_2$-saturated?

Problem 19 (Kanamori) Does the existence of $0^\#$ follow from the existance of an indecomposable ultrafilter on some singular cardinal?

For a discussion of the following problem see Shelah [31].

Problem 20 Is there a Jónsson algebra in every successor cardinal?

Problem 21 Is there a Jónsson algebra of cardinality $\aleph_\omega$?

Silver proved some years ago that if not, $\aleph_\omega$ is measurable in an inner model.

Some further problems on ultrafilters.

Let $\mathcal{D}$ be an ultrafilter over I. Define $\ell cf(\kappa,\mathcal{D})$ to be the coinitiality of $\{a \in K^I/_\mathcal{D} \mid \zeta < a \text{ for each } \zeta < \kappa\}$.

Problem 22 (Shelah) Can $\ell cf(\kappa,\mathcal{D}) \leq \kappa$ for regular $\mathcal{D}$?

Problem 23 (Shelah) Can $\ell cf(\omega_1,\mathcal{D}) = \omega$ for $I = \omega_1$?

Call a free filter F on ω feeble if for some strictly monotonic $f:\omega \to \omega$ $\{X \subseteq \omega \mid f^{-1}\text{"}X \in F\}$ is the Fréchet filter. Jalali-Naini and, independently, Talagrand have proved that a filter is feeble if and only if, considered as a subset of 2^ω, it has the property of Baire.

Call a filter F a p-filter if given $X_i \in F$ $(i<\omega)$ $\exists\, Y \in F$ $\forall i (Y \setminus X_i$ is finite). A p-point is a p-filter which is also an ultrafilter.

Problem 24 Is there a p-point?

Such may readily be constructed using Martin's axiom; for other constructions see Ketonen [32]. As no ultrafilter is feeble, a weaker question is

Problem 25 (Kanamori) Is there a p-filter which is not feeble? Equivalently, is there a filter G on ω that is coherent in Kanamori's sense that whenever $X \in G$ and $A \subseteq G$ are such that $\forall n < \omega\ \{A \in A \mid A \cap n = X \cap n\}$ is infinite, there is an infinite $B \subseteq A$ with $\bigcap B \in G$?

The author has recently proved that if $\neg 0^\#$, a coherent filter on ω exists.

Problem 26 Is it provable in arithmetic that the consistency of "There is a strongly compact cardinal" implies that of "There is a supercompact cardinal".?

The next four problems are connected with strong assumptions such as the axiom of determinacy, AD, or huge cardinals.

Magidor [33] has proved that if "There is a huge cardinal" is consistent, so is

(†) "$\forall n < \omega\ 2^{\aleph_n} = \aleph_{n+1}\ \&\ 2^{\aleph_\omega} > \aleph_{\omega+1}$"

Problem 27 How strong is (†)?

Mitchell has shown that (†) implies that $\aleph_\omega$ is measurable in some inner model.

Problem 28 Prove "$\forall \lambda 2^\lambda > \lambda^{+}$" consistent.

Problem 29 Prove from a large cardinal axiom that all Σ^1_5 sets are Lebesgue measurable.

Problem 30 Show that AD is inconsistent but that "$\forall n$ Determinacy (Σ^1_n)" is not.

Problem 31 Find a consistency proof for "$\neg$ AC + all limit ordinals have cofinality ω".

It follows from Jensen's covering lemma that if $cf(\omega_1)=cf(\omega_2)=\omega$, $0^\#$ exists. Magidor can get the first α cardinals to be singular, starting from $\alpha+1$ supercompacts.

Problem 32 Obtain consistency proofs for the following consequences of AD:

the closed unbounded subsets of ω_1 generate an ultrafilter;
for some uncountable κ, $\kappa \to (\kappa)^\omega$;
for some uncountable κ, $\kappa \to (\kappa)^{\omega+\omega}$.

Some more problems concerned with the negation of the axiom of choice:

Problem 33 (Blass) Does AC follow in ZF from "$\forall x \exists y$ (y can be mapped onto x and y-indexed choice holds)"?

Problem 34 (Pincus) Find a model of $\neg$ AC + $\forall \alpha$ AC^α which satisfies AC_n, the ordering theorem or the prime ideal theorem.

Problem 35 (Pincus) Find ZF models for the following, which are all known to have Fraenkel-Mostowski models:

The Hahn-Banach theorem + The Krein-Mil'man theorem + $\neg$ AC;
Urysohn's Lemma + $\neg$ AC^ω;
The uniqueness of algebraic closures, where they exist, but not the existence of algebraic closures.

Here AC_n means AC for families of sets of power n; AC^ω means AC for countable families.

Problem 36 (Truss) Consider the three statements

(P) every uncountable set of reals has a perfect subset

(L) every set of reals is Lebesgue measurable

(B) every set of reals has the property of Baire.

Prove in ZF + DC that (P) implies (L) and that (L) is equivalent to (B).

Two odd problems:

Problem 37 (Yates) Is there an initial segment of the Turing degrees of order type ω_1?

Problem 38 (Blass) Let $\Pi = \{f | f:\omega \to \omega$ and f strictly increasing$\}$. Is there a family $\{A_f | f \in \Pi\}$ of subsets of ω such that given distinct members $f_1 \dots f_n, g_1 \dots g_m$, of Π,

$$A_{f_1} \cap \dots \cap A_{f_n} \cap (\omega \setminus A_{g_1}) \cap \dots \cap (\omega \setminus A_{g_m}) \neq 0,$$

and such that each interval $\{x | f(n) \le x < f(n+1)\}$ $(f \in \Pi,\ n \in \omega)$ is included in or disjoint from A_f?

By considering Cohen reals it may be seen that the answer to that is yes if $\mathbb{R}$ is not the union of fewer than $2^{\aleph_0}$ nowhere dense sets.

Finally, some problems connected with L.

Problem 39 (Jech) If V = L, there is a complete Boolean algebra with no proper infinite regular / complete subalgebra. Is the existence of such provable in ZFC?

Call $\langle S_\zeta | \zeta < \omega_1 \rangle$ a λ- $\Diamond$ -witness if $\forall \zeta\ S_\zeta \subseteq P(\zeta)$, $\bar{\bar{S}}_\zeta < \lambda$, and $\forall X \subseteq \omega_1 \{\zeta | X \cap \zeta \in S_\zeta\}$ is stationary. It was shown by Kunen that there is an $\aleph_1$- $\Diamond$ - witness iff there is an 2- $\Diamond$ -witness. Let $\Theta(\kappa,\lambda)$ be the statement that whenever $\langle S_\zeta \rangle$ is a κ- $\Diamond$ -witness, there is a λ- $\Diamond$ - witness $\langle T_\zeta \rangle$ with $\forall \zeta\ T_\zeta \subseteq S_\zeta$. The following are readily checked:

1. Fleissner's principle, in his paper [4], is equivalent to $\Theta((2^\omega)^+, 2)$.

2. $\Diamond^+ \to \Theta((2^\omega)^+, \aleph_1)$.

3. $V{=}L \to \Theta(\aleph_1, 2)$.

Problem 40 Is $\Theta(\aleph_1, 2)$ a theorem of ZFC, or of ZFC + $\Diamond^+$?

Problem 41 (Jech) Find a class of statements including $\Diamond$ and Silver's W such that if such a statement can be shown consistent via forcing countable conditions then it is true in L.

Problem 42 (Juhasz) Find a set-theoretical assertion which follows in ZFC from each of the continuum hypothesis, the negation of Souslin's hypothesis, and the assertion that the universe is the result of adding a Cohen real to some inner model, and which implies some of their common consequences.

Perhaps a solution to Problem 42 lies in the forthcoming papers [34] and [35], which offer a solution to the problem of axiomatising Jensen's consistency proof for GCH + Souslin's hypothesis in the sense that MA + $2^{\aleph_0} > \aleph_1$ axiomatises the original Solovay-Tennenbaum proof consistency proof for SH.

Problem 43 Does the existence of a morass imply Jensen's $\Box$?

PART THREE

The author began his Oxonian oration with an outline of the steps by which he has come to believe that the debate between the various philosophies of mathematics is a particularisation of the debate between various accounts of the world. Put succinctly, his thesis is that one's view of life determines one's view of mathematics; though in that form it is often found to be false, as people happily believe inconsistent things. The author considers that parallels may be drwan between Platonism and Catholicism, which are both concerned with what is true; between intuitionism and Protestant presentations of Christianity, which are concerned with the behavious of mathematicians and the morality of individuals; between formalism and atheism, which deny any need for postulating external entities; and between category theory and dialectical materialism. The author reached this last parallel through contemplating the Hegelian overtones of category theory, and he was gratified to find it supported in lectures of Lawvere.

The author is most grateful for the stimulating criticism he received of his talk from many mathematicians and philosophers, which he hopes to put to good use in due course.

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