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A REMARK ON RARE FILTERS

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Definition. A filter F on ω is *rare* if it contains all cofinite subsets of ω and for any function π from ω onto ω such that $\pi^{-1}\{i\}$ is finite for each $i \in \omega$, F contains a subset A of ω on which π is $1-1$.

Theorem. *No* Σ^1_1 *subset of* 2^ω *can be a rare filter.*

First, notation. A, B, C will denote infinite subsets of ω, and $\langle a_n \mid n < \omega \rangle$, $\langle b_n \mid n < \omega \rangle$, $\langle c_n \mid n < \omega \rangle$ the enumerations of their elements in increasing order. F will always denote a filter on ω containing the Fréchet filter of all cofinite sets.

A family P of infinite subsets of ω is called a *Scott family* if $\forall A\, \exists B \subseteq A (A \in P \Leftrightarrow B \notin P)$; if, in other words, every infinite subset of ω has an infinite subset in P and an infinite subset not in P. It is a theorem of Silver [4] that no Σ^1_1 subset of 2^ω can be a Scott family; and in [2] it was shown that in Solovay's model in which all sets of reals are Lebesgue measurable, there is no Scott family; or, equivalently, that the partition relation $\omega \to (\omega)^\omega$ holds. Proofs of Silver's theorem that eschew forcing are known: one due to the author, uses Ramsey ultrafilters and another, recently given by Ellentuck [5], proceeds by reduction to a classical result of analytic topology.

The proof presented at Keszthely of the theorem, which was announced in [3, page 209], used forcing and, though not without its charms, was long. The argument below is inspired by a recent letter of Baumgartner for which the author here records his gratitude.

Given $A = \{a_n \mid n < \omega\}$, define $O(A) = \{m \mid m \leqslant a_0\} \cup \cup \{m \mid \exists n (a_{2n+1} < m \leqslant a_{2n+2})\}$ and $E(A) = \{m \mid \exists n (a_{2n} < m \leqslant a_{2n+1})\}$. Then $E(A)$ is the complement of $O(A)$ in ω. Now given F define $P^{(F)} = \{A \mid O(A) \in F\}$ and $Q^{(F)} = \{A \mid E(A) \in F\}$. Then $P^{(F)}$ and $Q^{(F)}$ are disjoint; further if $A \in P^{(F)}$ and $n \in A$, then $A - \{n\} \in Q^{(F)}$, so that every infinite A has an infinite subset not in $P^{(F)}$.

Lemma. *If F is rare then $P^{(F)}$ is a Scott family.*

The theorem is an immediate consequence of the lemma and Silver's theorem, for if F is Σ^1_1 so is $P^{(F)}$; similarly the lemma yields a new proof of the author's result that there is no rare filter in Solovay's model, which was originally established using forcing and some absoluteness arguments from [1]. More generally, an adequate class in the sense of [1] which contains no Scott family contains no rare filter.

To prove the lemma we have to show that if F is rare, then every infinite subset A of ω has an infinite subset in $P^{(F)}$. Define $\pi\colon \omega \to \omega$ by $\pi(i) =$ the least n with $i \leqslant a_{2n}$. As F is rare it contains a B on which F is $1-1$. Then $\{n \mid \neg\exists m (m \in B \text{ and } a_n < m \leqslant a_{n+1})\}$ is infinite, so there is an infinite subset C of A such that $E(C) \cap B = O$, whence $O(C) \in F$ and $C \in P^{(F)}$ as required.

In fact F's having a much weaker property than rarity is sufficient for $P^{(F)}$ to be a Scott family, as examination of the argument will show: call F *feeble* if there is a weakly monotonic function $\psi\colon \omega \to \omega$ such that $\{B \mid \psi^{-1}\text{``}B \in F\}$ is the Fréchet filter. Then $P^{(F)}$ is a Scott family when and only when F is not feeble; no ultrafilter is feeble; and hence the

Proposition. *If* $\omega \to (\omega)^\omega$ *then every filter containing the cofinite sets is feeble and every ultrafilter principal.*

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