¶9. Proof of T6001.

Recall that a set u is <u>transitive</u> $\longleftrightarrow$ $\bigwedge v', w' (v' \in w' \in u \longrightarrow v' \in u)$; equivalently, that $u \subseteq S(u)$.

D 6900 $\quad C(u) = \bigcap \{ v \mid u \subseteq v \subseteq S(v) \}$.

$C(u)$ is the <u>transitive closure</u> of u; and is a set, as $u \subseteq V_{\rho(u)} \subseteq S(V_{\rho(u)}) \in V$; $C(u)$ is the smallest transitive set containing u.

T6901 $\quad ZF \vdash$ (i) $\quad v \subseteq w \longrightarrow C(v) \subseteq C(w)$

(ii) $\quad v \in w \longrightarrow C(v) \subseteq C(w)$

(iii) $\quad C(\bigcup v) = \bigcup \{ C(w) \mid w \in v \}$

(iv) $\quad C(v) = v \cup \bigcup \{ C(w) \mid w \in v \}$

<u>Proof.</u> (i), (ii) are easily checked from the definitions.

To prove (iii); write $A =_{df} \bigcup \{ C(w) \mid w \in v \}$;

then $v' \in w' \in A \longrightarrow v' \in w' \in C(w)$ some $w \in v$

$\longrightarrow v' \in C(w)$ (as $C(w)$ is transitive)

$\longrightarrow v' \in A$. So A is transitive.

$w \in v \longrightarrow w \subseteq C(w)$ by (ii); so $\bigcup v \subseteq A$;

so $C(\bigcup v) \subseteq A$.

$w \in v \longrightarrow w \subseteq \bigcup v \longrightarrow C(w) \subseteq C(\bigcup v)$ by (i);

$\longrightarrow A \subseteq C(\bigcup v)$.

(iv) is proved similarly.

QED

D6902 $\quad HROD(u) \longleftrightarrow ROD(u) \wedge \bigwedge_{v \in C(u)} ROD(v)$.

Write $u \in HROD$, $u \in ROD$ for $HROD(u)$, $ROD(u)$.

HROD is the class of sets u s.t. u and all its elements, and the elements of its elements ---- are in ROD.

T6903 $ZF \vdash HROD = \{u \in ROD \mid u \subseteq HROD\}$.

<u>Proof</u>: from the definition using T6901.

Note that $\alpha \in On \rightarrow (\alpha = C(\alpha) \wedge \alpha \in ROD)$,

so $On \subseteq HROD$. Further, every real is in HROD.

Let OD be the class of sets definable with ordinal (but <u>not</u> real) parameters. OD, HOD may be defined in set theory by the reflection principle. In the celebrated paper of Myhill and Scott [8], to which people have referred for many years and which has now actually been written and will appear in the Proceedings of the Summer Institute on Set Theory, UCLA 1967, it is shown that for every axiom Ω of ZF+AC,

$$ZF \vdash \Omega^{HOD}.$$

The verification of all axioms save AC is straightforward, using T6903 for HOD. Their proof adapts to the present case, <u>except for AC</u>. Further it is clear that an ROD set of reals is HROD, and so

T6904 $ZF \vdash$ no SF is RoD $\longrightarrow$ (there are no SFs)^{HRoD}.

and $ZF \vdash \Omega^{HRoD}$, for every axiom Ω of ZF.

T6905 (McAloon) $ZF + DC \vdash DC^{HRoD}$.

It follows from T6905 that if ZF+AC+"no SF is RoD" is consistent, then so is ZF + DC + "there are no SFs", thus proving T6001: for if

$$ZF + DC + \text{"there are no SFs"} \vdash 0 = 1,$$

then for some finite subset of these axioms, $\Omega_1' \ldots \Omega_k'$

$$\Omega_1' \wedge \ldots \wedge \Omega_k' \longrightarrow 0 = 1$$

is provable in the predicate calculus; and so as

$$ZF + AC + \text{"no SF is RoD"} \vdash \Omega_1'^{HRoD} \wedge \ldots \Omega_k'^{HRoD},$$

$$ZF + AC + \text{"no SF is RoD"} \vdash (0=1)^{HRoD},$$

and therefore $ZF + AC + \text{"no SF is RoD"} \vdash 0 = 1$, as 0 is the empty set in HRoD too, and 1 is its unit set: $\{0\}$.

It only remains therefore to prove T6905.

(This method of extracting theorems like T6001 from theorems like T6002 was suggested by McAloon: cf T3307.)

<u>Proof of T6905.</u>

The following functions will be defined at least on RoD:

D6906 $\psi_1(u)$ = the least α s.t. u is ROD in V_α. (cf. 6803)

D6907. Let $\dot{\Omega}^0, \dot{\Omega}^1, \ldots$ be a recursive enumeration of all ZF formulae of the object language for ZF with three free variables

D 6908. $\psi_2(u)$ = the least n s.t.

$$\forall \beta \in On, x \subseteq \omega \quad u = \{v \in V_{\psi_1(u)} \mid V_{\psi_1(u)} \models \dot{\Omega}^n(v, \beta, x)\}.$$

D 6909 $\psi_3(u)$ = the least β s.t.

$$\forall x \subseteq \omega \quad u = \{v \in V_{\psi_1(u)} \mid V_{\psi_1(u)} \models \dot{\Omega}^{\psi_2(u)}(v, \psi_3(u), x)\}$$

D 6910 $\psi(u) = \langle \psi_1(u), \psi_2(u), \psi_3(u) \rangle$.

Give the class of $\psi(u)$'s the lexicographical ordering: that will be a well ordering; denote it by $\leq$.

D 6911 $K(u) = \{x \subseteq \omega \mid u = \{v \in V_{\psi_1(u)} \mid V_{\psi_1(u)} \models \dot{\Omega}^{\psi_2(u)}(v, \psi_3(u), x)\}\}$.

Note that if $x \in K(u)$, u is uniformly definable from $\langle \psi(u), x \rangle$: so write then $u = \chi(\langle \psi(u), x \rangle)$.

Now let $R \in HROD$ be a relation on a set $Q \in HROD$, such that $Q \neq 0$ and $\bigwedge u \in Q \bigvee v \in Q \; uRv$.

Set α_0 = least α $\bigvee u \in Q \; \psi(u) = \alpha$.

Pick $x_0 \in \bigcup \{K(u) \mid u \in Q \wedge \psi(u) = \alpha\}$.

set α_1 = least α $\bigvee u \in Q \; (\psi(u) = \alpha \wedge uR\chi(\langle \alpha_0, x_0 \rangle))$

Pick $x_1 \in \bigcup \{K(u) \mid u \in Q \wedge \psi(u) = \alpha_1\}$

Set α_2 = least α $\bigvee u \in Q (\psi(u) = \alpha \wedge uR\chi(\langle \alpha_1, x_1 \rangle))$.

. . . . . .

Using DC, get a sequence $x_0, x_1 \ldots$. Define $\mathbb{S} = \{2^n \cdot 3^m \mid n \in x_m\}$. Define $y = \{2^n \cdot 3^{f(n)} \mid n < \omega\}$, where f is the recursive enumeration of D6907.

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Pick $\bar{\alpha}, z$ such that $R = \chi(\langle \bar{\alpha}, z \rangle)$.

Then the sequence $\langle u_i \mid i < \omega \rangle$ is definable from the sequence $\langle \langle \alpha_i\ x \rangle \mid i < \omega \rangle$, $\bar{\alpha}$, y and z.

But the sequence $\langle x_i \mid i < \omega \rangle$ is definable from x; Q is definable from R, and therefore from $\bar{\alpha}, z$; and the sequence $\langle \alpha_i \mid i < \omega \rangle$ is definable from Q, y and the sequence $\langle x_i \mid i < \omega \rangle$. Hence the sequence

$$\langle u_i \mid i < \omega \rangle$$

is definable from $\bar{\alpha}$, x, y and z, and is therefore in RoD. But all its elements are in HRoD, and therefore it is.

Choice is only used in this proof to pick the sequence $\langle x_i \mid i < \omega \rangle$, and for that, DC is enough. By construction,

$$\bigwedge i\ u_i R u_{i+1}$$

and so DC holds in HRoD. Q.E.D.

The proof of T6001 is now complete.

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