\[ \textbf{§9. Proof of T 6001.} \]

Recall that a set \( u \) is transitive if
\[ \forall u', w \ (u' \in w \wedge u' \in u \implies u' \in u) \],
equivalently, \[ u \in S(u). \]

By 6900 \[ C(u) = \bigcap \{ v \mid u \subseteq v \subseteq S(v) \}. \]

\( C(u) \) is the transitive closure of \( u \); and is a set, so \( u \subseteq V_{p(u)} \subseteq S(V_{p(u)}) \in V \); \( C(u) \) is the smallest transitive set containing \( u \).

\[ \text{T 6901 \textbf{ZF} \quad (i) \quad u \subseteq w \implies C(u) \subseteq C(w) \]
\[ (ii) \quad u \in w \implies C(u) \subseteq C(w) \]
\[ (iii) \quad C(U_u) = U \{ C(w) \mid w \in u \} \]
\[ (iv) \quad C(u) = u \cup U \{ C(w) \mid w \in u \} \]

\[ \text{Proof.} \quad (1), (ii) \text{ are easily checked from the definition.} \]

To prove (iii) \(; \) write \( A = U \{ C(w) \mid w \in u \} \); then \( u' \in v' \in A \implies u' \in C(w) \) some \( w \in v' \)
\[ \implies u' \in C(w) \quad \text{(as } C(w) \text{ is transitive)} \]
\[ \implies u' \in A. \quad \text{So } A \text{ is transitive.} \]

\[ w \in u \implies w \subseteq C(w) \text{ by (ii); } \Rightarrow U_u \subseteq A; \]
\[ \Rightarrow C(U_u) \subseteq A. \]

\[ w \in v \implies w \subseteq U_u \implies C(w) \subseteq C(U_u) \text{ by (i);} \]
\[ \Rightarrow A \subseteq C(U_u). \]

(iv) \( \) is proved similarly, \[ \text{QED.} \]
D6902 \[ H_{\text{OD}}(u) \iff \text{ROD}(u) \land \forall \alpha \in \text{C}(u) \text{ROD}(\alpha). \]

where \( u \in H_{\text{OD}}, u \in \text{ROD} \) for \( H_{\text{OD}}(u), \text{ROD}(u) \).

\( H_{\text{OD}} \) is the class of sets \( u \) and all its elements, and the elements of its elements, and so on in \( \text{ROD} \).

T6903ZF-1 \[ H_{\text{OD}} = \{ u \in \text{ROD} \mid u \in H_{\text{OD}} \}. \]

Proof : from the definition using T6901.

Note that \( \forall \alpha \in \text{On} \rightarrow (\alpha = \text{C}(\alpha) \land \alpha \in \text{ROD}) \),

so \( \text{On} \subseteq H_{\text{OD}} \). Further, every real is in \( H_{\text{OD}} \).

Let \( OD \) be the class of all definable non-trivial (but not real) parameters. \( OD \), \( H_{\text{OD}} \) may be defined in set theory by the reflection principle. In the celebrated paper of Kechris and Scott [8], to which people have referred for many years and which has now actually been written and will appear in the Proceedings of the Summer Institute on Set Theory, UCLA 1967, it is shown that for every axiom \( \text{O2} \) or \( \text{ZF} + \text{AC} \),

\[ \text{ZF} \vdash \forall \text{OD}. \]

The verification of all axioms, save \( \text{AC} \) is straightforward, using T6903 for \( H_{\text{OD}} \). Their proof adapts to the present case, except for \( \text{AC} \). Further it is clear that an \( \text{ROD} \) set \( \alpha \) leads to \( H_{\text{OD}} \), and so
\[ T6904 \quad \text{ZF} \vdash \text{no SF is Rod} \to (\text{there are no SFs})^{HROD} \]
and \[ \text{ZF} \vdash \text{DC}^{HROD} \], for every axiom DC of ZF.

\[ T6905 \quad (\text{McMoor}) \quad \text{ZF + DC} \vdash \text{DC}^{HROD} \]

It follows from T6905 that if \( \text{ZF + AC + "no SF is Rod"} \) is consistent, then so is \( \text{ZF + DC + "there are no SFs"} \), thus proving T6001: for if \( \text{ZF + DC + "there are no SFs"} \vdash 0 = 1 \), then for some finite subset of these axioms, \( \forall \ell \in \text{\# finite} \quad \ell_1 \land \ldots \land \ell_k \to 0 = 1 \), a premise in the predicate calculus, and so as

\[ \text{ZF + AC + "no SF is Rod"} \vdash (0 = 1)^{HROD} \]

and therefore \( \text{ZF + AC + "no SF is Rod"} \vdash 0 = 1 \), as \( 0 \in \text{the empty set \# HROD} \) too, and \( 1 \in \text{its unit set: } \{0\} \).

It only remains, therefore, to prove T6905.

(As a method of extracting theorem like T6001 from theorem like T6002 was suggested by McMoor; cf. T3307.)

Proof of T6905:

The following function will be defined at least on Rods:
D6906 $\psi_1(n) =$ the least $\alpha$ s.t. $\alpha \in \mathbb{R}0^\alpha$ in $V_{\omega_1}$ (cf. D6803).

D6907 Let $\psi_0, \psi_1, \ldots$ be a recursive enumeration of all ZF formulas of the object language for ZF with linear free variables.

D6908 $\psi_2(n) =$ the least $\alpha$ s.t.

$$\forall \beta \in \mathbb{R}_n, \alpha \leq \omega \quad \forall \beta \in \mathbb{R}_n, \alpha \leq \omega \quad \psi_2(n) = \{ \psi \in \mathcal{V}_{\psi_1}(\psi) \mid \mathcal{V}_{\psi_2}(\psi) = \beta \}.$$

D6909 $\psi_3(n) =$ the least $\beta$ s.t.

$$\forall \alpha \in \mathbb{R}_n, \alpha \leq \omega \quad \psi_3(n) = \{ \psi \in \mathcal{V}_{\psi_1}(\psi) \mid \mathcal{V}_{\psi_3}(\psi) = \beta \}.$$

D6910 $\psi(n) = (\psi_1(n), \psi_2(n), \psi_3(n)).$

Give the class $\psi(n)$ the lexicographical ordering; let this be a well ordering; denote it by $\leq.$

D6911 $\mathcal{K}(n) = \{ \alpha \in \mathbb{R}_n \mid \forall \psi \in \mathcal{V}_{\psi_1}(\psi) \mid \mathcal{V}_{\psi_2}(\psi) = \beta \psi(n), \psi(\mathcal{K}(n)) \psi(n) = \beta \}.$

Note that if $\alpha \in \mathcal{K}(n),$ $\alpha$ is uniformly definable from $\psi(n), \alpha,$ so write then $\alpha = \mathcal{K}(\psi(n), \alpha).$

Now let $R \in \text{HRDO}$ be a relation on a set $Q \in \text{HRDO},$ such that $Q \neq \emptyset$ and $\forall \psi \in \mathcal{V}_{\psi_1}(\psi) \forall \psi \in \mathcal{V}_{\psi_2}(\psi) \forall \psi \in \mathcal{V}_{\psi_3}(\psi).$

Set $d_0 =$ least $\alpha$ $\forall \psi \in \mathcal{V}_{\psi_1}(\psi) \psi(n) = \alpha.$

Pick $x_0 \in \mathcal{U} \{ \mathcal{K}(n) \mid \psi(n) = \alpha \}.$

Set $d_1 =$ least $\alpha$ $\forall \psi \in \mathcal{V}_{\psi_1}(\psi) \psi(n) = \alpha \land \mathcal{R}(x_0, \alpha).$

Pick $x_1 \in \mathcal{U} \{ \mathcal{K}(n) \mid \psi(n) = \alpha \}.$

Set $d_2 =$ least $\alpha$ $\forall \psi \in \mathcal{V}_{\psi_1}(\psi) \psi(n) = \alpha \land \mathcal{R}(x_1, \alpha).$

Using DC, get a sequence $x_0, x_1, \ldots$. Define $s_n = \{ 2^n, 3^n \mid n \in \mathbb{N} \}.$

Define $y_1 = \{ 2^n, 3^n \mid n \in \mathbb{N} \},$ where $f$ is the recursive enumeration of D6907.
Pick \( \overline{a}, \overline{y} \) such that \( R = \chi (\overline{a}, \overline{y}) \).

Then the sequence \( \langle a_i | i < \omega \rangle \) is definable from the sequence \( \langle \langle x_i | x_i \rangle | i < \omega \rangle \). \( \overline{a}, \overline{x}, \overline{y} \) and \( \overline{y} \).

But the sequence \( \langle x_i | i < \omega \rangle \) is definable from \( \overline{a} \).

\( \overline{a} \) is definable from \( R \), and therefore from \( \overline{a}, \overline{y} \) and \( \overline{x} \).

The sequence \( \langle x_i | i < \omega \rangle \) is definable from \( \overline{a}, \overline{y} \) and \( \overline{x} \).

Hence the sequence \( \langle a_i | i < \omega \rangle \)

is definable from \( \overline{a}, \overline{x}, \overline{y} \) and \( \overline{y} \), and is therefore \( \text{HR} \).

But all its elements are \( \in \text{HR} \), and therefore it is.

Choice is only used

in this proof to pick the sequence \( \langle z_i | i < \omega \rangle \), and for that, \( \text{DC} \) is enough. By construction,

\[
\forall i \quad a_i R u_{i+1}
\]

and so \( \text{DC} \) holds in \( \text{HR} \). \( \text{R} \). \( \text{D} \).

The proof of \( T_{6001} \) is now complete.
References


