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[6800]

¶ 8. Proof of T 6002.

First, some remarks on definability.

D 6800 (ZF). For $l < \omega$, $r_1 \subseteq \omega, \dots, r_l \subseteq \omega$, define

$\langle r_1, \dots, r_l \rangle$ as the real r such that

for all $k < \omega$, $m < l$,

$$\ell k + m \in r \iff k \in r_{m+1}.$$

T6801 Let $\Omega(x, y_1, \dots, y_k, z_1, \dots, z_\ell)$ be a ZF-formula with the $k+l+1$ free variables shown.

Then there is a ZF formula $\Omega'(x, y, z)$ with 3 free variables such that

$$ZF \vdash \forall d_1, \dots, d_k \in On \ \forall r_1, \dots, r_\ell \subseteq \omega [d_1 < d_2 < \dots < d_k,$$

and if $d_1 < d_2 < \dots < d_k$, then there is an $a \in On$ and an $r \subseteq \omega$ such that

$$\Lambda u (\Omega(u, d_1, \dots, d_k, r_1, \dots, r_\ell)) \leftrightarrow \Omega'(u, a, r).$$

Proof (2) By Cantor's normal form theorem, if d_1, \dots, d_k are ordinals, and $d_1 < \dots < d_k$, then each d_i is definable from the ordinal

$$\omega =_{df} \omega^{\alpha_k} + \dots + \omega^{\alpha_1}.$$

That is, the function f_k defined by $f_k(d_1, \dots, d_k) = \omega^{\alpha_k} + \dots + \omega^{\alpha_1}$ for ordinals d_1, \dots, d_k ($d_1 < \dots < d_k$) is definable by a ZF-formula

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[6802]

$F(\alpha_1, \dots, \alpha_k, \alpha)$ and there are formulae $G_1(\cdot, \cdot), \dots, G_k(\cdot, \cdot)$ such that

$$ZF \vdash \Lambda \alpha_1, \dots, \alpha_k (\alpha_1 < \dots < \alpha_k, \alpha \in \text{On}, \alpha_1, \dots, \alpha_k \in \text{On} \rightarrow$$

$$\forall \alpha \Lambda \alpha' (F(\alpha_1, \dots, \alpha_k, \alpha') \leftrightarrow \alpha = \alpha')$$

$$ZF \vdash \Lambda \alpha_1, \dots, \alpha_k, \alpha \in \text{On} (F(\alpha_1, \dots, \alpha_k, \alpha) \rightarrow$$

$$\Lambda r (G_1(\alpha, r) \leftrightarrow r = \alpha_1 \wedge \dots \wedge G_k(\alpha, r) \leftrightarrow r = \alpha_k)).$$

(2) The operation $\langle \cdot, \dots \rangle_k$ is definable in set theory,

say by $H_k(r_1, \dots, r_k, r)$ and there are formulae

K_1, \dots, K_k such that

$$ZF \vdash \Lambda r_1, \dots, r_k, r \in \omega [H(r_1, \dots, r_k, r) \rightarrow:$$

$$\Lambda s \in \omega [K_1(r, s) \leftrightarrow s = r_1 \wedge \dots \wedge K_k(r, s) \leftrightarrow s = r_k]).$$

Using F, G_i, H, K_i , Ω' may now be written down.

T6801 says that if a set is definable with parameters for ordinals and reals, then it is definable using only one parameter of each sort. Suppose u is such a set : that is, there is an ordinal α , and a real r , and a ZF formula $\Omega(x, y, z)$ such that

$$u = \{v \mid \Omega(v, \alpha, r)\} :$$

that is,

$$6802 \quad \Lambda v (v \in u \longleftrightarrow \Omega(v, \alpha, r)).$$

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[6803]

Call the formula of line 6802 $\text{fr}(u, \alpha, r)$.

Using the reflection principle T6513, pick a $\beta \in \text{On}$ greater than $\max\{\rho(u), \rho(r), \alpha\}$ such that

$$\text{fr}(u, \alpha, r) \longleftrightarrow \text{fr}^{\mathcal{V}_\beta}(u, \alpha, r).$$

Then as $\text{fr}(u, \alpha, r)$ holds, $\text{fr}^{\mathcal{V}_\beta}(u, \alpha, r)$ holds:

that is, $\forall v \in \mathcal{V}_\beta (v \in u \longleftrightarrow \dot{\sigma}^{\mathcal{V}_\beta}(v, \alpha, r))$,

so that

$$6803 \quad u = \{v \mid \langle \mathcal{V}_\beta, \in \rangle \models \dot{\sigma}[v, \alpha, r]\};$$

as in \mathcal{V}_β , α is an ordinal and r a real, u is "ordinally definable from a real" in \mathcal{V}_β . 6803 shows that the property of being definable with ordinal and real parameters may be defined in set theory by

$$\text{D 6804} \quad \text{ROD}(u) \longleftrightarrow \forall \beta \in \text{On} \forall \alpha^{\text{ZF}} \text{formula } \dot{\sigma}(e, y, z)$$

of the object language for ZF $\forall x \in \mathcal{V}_\beta \forall n \forall r \in \mathcal{V}_\beta \exists w S(w)$

$$\left[u = \{v \in \mathcal{V}_\beta \mid \langle \mathcal{V}_\beta, \in \rangle \models \dot{\sigma}[v, \alpha, r]\} \right];$$

(or, conversely to the argument just given, any set u of the form 6803 is definable with the parameters α, β and r).

$\text{ROD}(u)$ is commonly read " u is ordinally definable [or ordinal-definable] from a real".

Let SI be the ZF sentence "There is a strongly inaccessible cardinal."

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T 6002 may now be restated:

Con(ZF + AC + SI) implies Con(ZF + AC + no SF is ROD).

Let $\mathcal{L}(x)$ be a ZF formula with one free variable.

D 6805 $\mathcal{L}(x)$ is presentable iff:

$$\text{ZF+AC} \vdash \lambda u \lambda v (\mathcal{L}(u) \wedge \mathcal{L}(v) \rightarrow u = v).$$

and $\text{ZF+AC} \vdash \lambda u (\mathcal{L}(u) \rightarrow u \text{ is a homogeneous CBA}).$

Let L_M be the language of set theory enriched by adding a one-place predicate letter M.

D 6806 Let \mathcal{D}_L be the following infinite set of sentences of L_M :

1. all axioms of ZF + AC.
2. all axioms of the schema "M is an inner model of ZF + AC"
(that is: On $\subseteq M$, M is transitive, and Ω^M for every axiom Ω of ZF + AC)
3. the sentence $\lambda x \in \omega \forall y \in \omega (y \text{ is } \mathbb{P}\text{-generic over } M[x]).$
4. the sentence

$\lambda x \in \omega \forall F \forall u (\mathcal{L}(u) \text{ is true in } M[x] \text{ and } u \in M[x]$
and F is an $M[x]$ -complete ultrafilter on u
and $V = M[x][F]).$

Groups 1 - 4 are the axioms of the theory \mathcal{D}_L .

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[6807]

The heart of the proof of T 6002 will be presented in two steps:

T 6807 $\exists F + AC + SI \vdash$ Let γ be the first strongly inaccessible cardinal. Let Ω be any axiom of \mathcal{D}_{Ω_2} . Then

$$[\dot{\alpha}]^{C_\gamma} = 1,$$

(where $\dot{\alpha}$ is obtained from α by writing Ω in L^{C_γ} and replacing M by V , and Ω_2 is the formula of D6838).

T 6808 (Schema) For any presentable \mathbb{L} ,

$$\mathcal{D}_{\mathbb{L}} \vdash \text{no SF is ROD.}$$

Proof of T 6807.

Remark first that by T 6726 Ω_2 is presentable.

That all axioms of groups 1 and 2 of \mathcal{D}_{Ω_2} have C_γ -value 1 is immediate from T 6102, T 6107. Sentence 3 is a consequence of T 6746, and the theorem of Rasiowa and Sikorski, and Sentence 4 is T 6747. QED

Proof of T 6808.

Let \mathbb{L} be presentable. Reason now in $\mathcal{D}_{\mathbb{L}}$.

Let $\Omega, P, r \subseteq \omega$ and $a \in On$ be such that

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$$P = \{x \subseteq \omega \mid Q(x, r, \dot{D})\}$$

(where Q is a ZF formula with 3 free variables).

Consider P -generic extensions of $M[r]$. To simplify the notation, write P^r , B^r , \dot{x}^r for $(P^{M(r)}, B^{M(r)})$ and $\dot{x}^{M(r)}$ (D 6409).

First, remark that if $x \subseteq \omega$, then $M[r](x) = M[\langle r, x \rangle_2]$ (where $\langle \cdot, \cdot \rangle_2$ is the function of D6800).

Secondly, note that by sentence 3 of \exists_L , and by T6431, for any $\langle s, S \rangle \in |P^r|$, there is an F , an $M[r]$ -generic filter on P^r containing $\langle s, S \rangle$.

Therefore, $[(V \dot{\wedge} \dot{L}(\dot{D}))]^{B^r} = \mathbb{I}$; for if for some $\langle s, S \rangle$, $\langle s, S \rangle \Vdash \neg V \dot{\wedge} \dot{L}(\dot{D})$, then take an $M[r]$ -generic F containing $\langle 0, s \rangle$: let $F \xrightarrow{M(r)} x$. Then $M[r](x) = M[\langle r, x \rangle_2]$, $\neg V \dot{\wedge} \dot{L}(\dot{D})$, contradicting sentence 4 of \exists_L .

Pick by the maximum principle a $\dot{D} \in M[r]^{B^r}$ such that

$$[\dot{L}(\dot{D})]^{B^r} = \mathbb{I};$$

and consider the following sentence χ of L^{B^r} :

$$[\dot{Q}(\dot{x}^r, \dot{F}, \dot{\chi})]^{B^r} = \mathbb{I}.$$

(Explanation: $\dot{F} \in M[r]^{B^r}$, and is the image of r under the canonical embedding $\dot{\vee}$. χ makes an assertion about a Boolean valued universe constructed in $M[r]^{B^r}$ with respect to \dot{D} , and $\dot{\chi}$ is the image of χ in the embedding $\dot{\vee}$ of

(153) \dot{V} in $\dot{V}^{\mathbb{D}}$ (in the sense of $M[r]^{\mathbb{B}^r}$).

By T6429, there is an $\langle 0, s \rangle \in \mathbb{P}^r$ such that

$$\langle 0, s \rangle \Vdash \chi \circ \langle 0, s \rangle \Vdash \neg \chi.$$

The first case is argued like the second, but is easier so I consider the second. By the presentability of \mathbb{L} ,

$$\langle 0, \omega \rangle \Vdash [\dot{\alpha}(\dot{x}, \dot{r}, \dot{\chi})]^{\mathbb{D}} = \emptyset \circ \dot{1},$$

$$\stackrel{\text{so}}{\langle 0, s \rangle \Vdash [\dot{\alpha}(\dot{x}, \dot{r}, \dot{\chi})]^{\mathbb{D}} = \emptyset}.$$

Let F be a $M[r]$ -generic filter on \mathbb{P}^r containing $\langle 0, s \rangle$, and let $F \xrightarrow[M[r]]{} x$. I assert that $2_\infty^x \cap P = \emptyset$.

For let $y \subseteq x$, y infinite. Then by T6017, y is P -generic over $M[r]$; let $F_y \xrightarrow[M[r]]{} y$. Then as $0 \subseteq y \subseteq x \subseteq 0 \cup s$, $\langle 0, s \rangle \in F_y$, and so in $M[r][y]$,

$$[\dot{\alpha}(y, r, \dot{\chi})]^{\mathbb{D}} = \emptyset,$$

where $\mathbb{D} = \phi_{F_y}(\mathbb{D})$. Now $M[r][y] = M[\langle r, y \rangle_2]$.

Pick, by $\exists_{\mathbb{L}}$, sentence 4, an F' that is $M[\langle r, y \rangle_2]$ -generic on \mathbb{D} such that $\dot{V} = M[\langle r, y \rangle_2][F']$.

$$\text{As } [\dot{\alpha}(y, r, \dot{\chi})]^{\mathbb{D}} = \emptyset,$$

in \dot{V} $\neg \dot{\alpha}(y, r, \dot{\chi})$: that is, $y \notin P$.

Hence P is not an SF, and T6808 is proved.

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Proof of T6002.

It follows from T6807 and T6808, that

$ZF + AC + SI \vdash$ Let \beth be the first inaccessible cardinal,

Then $[\text{no } SF \in ROD]^{C_\beth} = \mathbb{I}$.

Suppose that the theory $ZF + AC + \text{"no } SF \in ROD"$ is inconsistent. Then there are ZF sentences $\alpha_1^*, \dots, \alpha_k^*$, each axioms of $ZF + AC + \text{no } SF \in ROD$, such that the following is provable in the first order predicate calculus:

$$\alpha_1^* \wedge \dots \wedge \alpha_k^* \rightarrow (0=1).$$

But in the theory $ZF + AC + SI$, the following are provable:

If \beth is the first strongly inaccessible cardinal, then

$$[\alpha_1^*]^{C_\beth} = \dots = [\alpha_k^*]^{C_\beth} = \mathbb{I}.$$

Therefore

$$[0 = 1]^{C_\beth} = \mathbb{I}.$$

But $[\neg(0 = 1)]^{C_\beth} = \mathbb{I}$, as $0 \neq 1$ is a theorem of ZF , and so $[\neg(0 = 1)]^{C_\beth} = 0$.

Hence $0 = 1$, a contradiction.

Thus if $ZF + AC + SI$ is consistent, so is
 $ZF + AC + \text{"no } SF \in ROD."$
