

¶ 7. Collapsing algebras.

In ¶ 5, T 6017 and absoluteness arguments were used to prove in certain theories that certain sets were not SFs; the basic method was this: start from a transitive model M of $ZF + DC$; consider P -generic extensions of M . Then for an appropriate x P -generic over M , show that if x has a certain property in $M[x]$, then every $y \in x$ has that property in $M[y]$. An absoluteness argument uses the fact to conclude that all $y \in x$ have some property in V . Now until the absoluteness argument is applied, the relation of M to V is irrelevant: the first part uses only T 6017 and general properties of forcing. The problem is therefore to find some way of using knowledge about M or $M[x]$ to obtain facts about V . Now one situation in which the universe is in some sense described in an inner model $M[x]$ is when $V = M[x][F]$ where F is an $M[x]$ -complete ultrafilter on an BA B that is, in $M[x]$, complete. Then if $y \in M$ and $[\Phi(x, y)]^B \in F$, for some ZF -formula $\Phi(x, y)$, then $\Phi(x, y)$ holds in V . If B is definable in $M[x]$, say by $\delta(x)$, and is homogeneous, then the fact that $\Phi(x, y)$ holds in V can be expressed in M , in part, by the sentence $\langle s, S \rangle \Vdash \delta(B) \wedge [\Phi(\check{x}, \check{y})]^B \equiv 1$, where $\langle s, S \rangle \in$ the filter F_x over M such that $F_x \xrightarrow{M} x$. If x' is another real P -generic over M , and $\langle s, S' \rangle \in F_{x'} \xrightarrow{M} x'$, then in $M[x']$, $[\Phi(\check{x}', \check{y})]^{B'} = 1$, where B' is the algebra in $M[x']$ satisfying the definition δ . If now there is an ultrafilter F' on B' that is $M[x']$ -complete and such that $V = M[x'][F']$, then in V , $\Phi(x', y)$. That (with "P-generic" replaced by "random") is Solovay's basic idea for an absoluteness argument using forcing; and it is worked out in the present paragraph, using

[115]

[6700]

Theorem of Jensen. The first milestone is T 6724; the sequence of definitions and theorems starting from D 6727 leads to the second, T 6742; T 6747 sums up.

AC is assumed throughout the paragraph, and in the second half, the existence of at least one strongly inaccessible cardinal.

Let $\alpha \in \text{On}$, and suppose $\alpha \geq w$.

$$\text{D 6700 } X_\alpha = \{ \langle r, \langle \beta, n \rangle \rangle \mid \beta < \alpha \wedge n < w \wedge r < \beta \}.$$

Define a partial ordering P_α by

$$\text{D 6701 (i)} \quad |P_\alpha| = \{ p \mid p \subseteq X_\alpha, p \text{ is finite, and}$$

$$\langle \beta, n, r, r' \rangle \langle r, \langle \beta, n \rangle \rangle \in p \wedge \langle r', \langle \beta, n \rangle \rangle \in p \rightarrow r = r' \}.$$

For $p, p' \in |P_\alpha|$,

$$\text{(ii)} \quad p \leq_\alpha p' \iff p' \subseteq p.$$

The suffix α will usually be omitted.

$$\text{(iii)} \quad P_\alpha = \langle |P_\alpha|, \leq_\alpha \rangle.$$

That is, a condition is a finite map p with domain a subset of $\alpha \times w$ and range a subset of α such that if $p(\beta, n)$ is defined, then $p(\beta, n) < \beta$. A condition is stronger than another if it includes it.

$$\text{D 6702 } \mathbb{C}_\alpha =_{\text{def}} \text{the algebra over } P_\alpha.$$

The notion of the product of two CBAs will also be used. Let B, C be CBAs; let $b, b', \dots, c, c', \dots$ be variables ranging over $B \setminus \{0\}, C \setminus \{0\}$ respectively.

[116]

[6703]

Define a partial ordering $\mathbb{P}_{B,C}$ by

D6703 (i) $|\mathbb{P}_{B,C}| = \{(b,c) \mid b \in |B| \setminus \{\emptyset\}, c \in |C| \setminus \{\emptyset\}\}.$

(ii) $\langle b,c \rangle \leq_{\mathbb{P}_{B,C}} \langle b',c' \rangle \iff b \leq^B b' \wedge c \leq^C c'.$

(iii) $\mathbb{P}_{B,C} = \langle |\mathbb{P}_{B,C}|, \leq_{\mathbb{P}_{B,C}} \rangle.$

D6704 The (Boolean complete) product $B \times C$ of B and C is defined as the algebra over $\mathbb{P}_{B,C}$.

The next two theorems establish that this definition coincides with (say) Sikorski's definition of the complete product of two cBAs. [12].

T6705 (i) The maps i_1, i_2 defined by

$$i_1(b) = 0_{(b,1)}^{B \times C} \quad i_1(\emptyset) = 0$$

$$i_2(c) = 0_{(1,c)}^{B \times C} \quad i_2(\emptyset) = 0$$

are regular embeddings of $B \otimes C$ in $B \times C$.

$$(ii) \quad i_1(b) \circ i_2(c) \neq 0.$$

(iii) the set A of all elements of the form $i_1(b) \circ i_2(c)$ is dense in $B \times C$

(iv) $B \times C$ is an r.m.c. of the algebra generated by A .

Proof. (i) is straightforward checking; as $\langle b,c \rangle \leq \langle b',1 \rangle \leftrightarrow \langle b,1 \rangle \leq \langle b',1 \rangle$

[17]

[6706]

$$\sum_{\sum \mathcal{X}}^{\mathcal{B} \times \mathcal{C}} \{0_{(b,c)} \mid b \in \mathcal{X}\} = 0_{\sum \mathcal{X}}, \text{ etc.}$$

(ii) and (iii) because $i_1(b) \cdot i_2(c) = 0_{(b,c)}$.

(iv) from (i'') and T 6117. qed

Throughout this paragraph I write \hookrightarrow to denote a regular embedding; $\overleftarrow{\hookrightarrow}$ an isomorphism.

The appearance of a diagram with ≥ 3 arrows means that I assert that it commutes.

T 6706 Let \mathbb{D} be a cBA for which there are regular embeddings $i_1: \mathbb{B} \rightarrow \mathbb{D}$, $i_2: \mathbb{C} \rightarrow \mathbb{D}$ satisfying

T 6705 (ii) (iii) with " $\mathcal{B} \times \mathcal{C}$ " replaced by \mathbb{D} .

Then there is an isomorphism $\pi: \mathcal{B} \times \mathcal{C} \overleftarrow{\hookrightarrow} \mathbb{D}$ for which

$$\begin{array}{ccccc} & \mathcal{B} \times \mathcal{C} & & & \\ i_1^{\mathcal{B} \times \mathcal{C}} \nearrow & & \downarrow \pi & \searrow i_2^{\mathcal{B} \times \mathcal{C}} & \\ \mathcal{B} & & \mathbb{D} & & \mathbb{C} \\ \searrow i_1^{\mathbb{D}} & & \uparrow \pi & & \swarrow i_2^{\mathbb{D}} \\ & \mathbb{D} & & & \end{array}.$$

Remark that superscripts are added to the various i 's when needed.

Proof. Without much loss of generality suppose $i_1^{\mathbb{D}}, i_2^{\mathbb{D}}$ are the identity. It is enough to show that

$$b \cdot c \leq b' \cdot c' \iff b \leq b' \wedge c \leq c'$$

for then $b \cdot c = b' \cdot c' \implies b = b' \wedge c = c'$, and so the partial orderings $\langle \{b \cdot c \mid (b \neq 0, c \neq 0)\} \leq \rangle$, $\mathbb{P}_{\mathcal{B} \times \mathcal{C}}$ as isomorphic, or the result follows by T 6705 (iii) and T 6117.

[118]

[6707]

Suppose therefore that $b.c \leq b'.c'$ and wlog
 that $b \neq b'$. Then $b'' =_w b - b' \neq 0$.

$$\therefore c.b'' \leq c.b \leq b.c' \leq b'$$

$$\text{and } c.b'' \leq b'' \leq -b'; \text{ so } c.b'' = 0, \text{ } *6705(\text{ii}).$$

π is now defined following T6117 on $P_{B \times C}$ and extended
 to $B \times C$. QED.

T6707. Let B be a cBA; let $\omega_1 < \beta < \gamma$. Then

(i) there is a canonical regular embedding

$$i_{\alpha\beta}^B : B \times C_\alpha \hookrightarrow B \times C_\beta;$$

(ii)

$$\begin{array}{ccc} B \times C_\alpha & \xrightarrow{i_{\alpha\gamma}^B} & B \times C_\gamma \\ i_{\gamma\beta}^{B,\alpha} \nearrow & \downarrow i_{\beta\gamma}^\gamma & \searrow i_{\beta\gamma}^B \\ B & \xrightarrow{i_{B,C_\beta}^{\gamma}} & B \times C_\beta \end{array}$$

(iii) if λ is a limit ordinal then

$$B \times C_\lambda = \text{r.m.c. of } \bigcup_{\alpha < \lambda} i_{\alpha\lambda}^B((B \times C_\alpha)).$$

(iv) if λ a regular cardinal (i.e. $\lambda = cf(\lambda)$) and $\overline{|B|} < \lambda$

then $B \times C_\lambda$ satisfies the λ -c-c.

(v) $(\lambda = cf(\lambda) \text{ and } \overline{|B|} < \lambda)$ implies that

$$B \times C_\lambda = \bigcup_{\alpha < \lambda} i_{\alpha\lambda}^B((B \times C_\alpha)).$$

[119]

[6708]

During the proof the letters $p_\alpha, p'_\alpha, q_\alpha \dots; p_\beta, p'_\beta, q_\beta \dots$
 will be used as variables ranging over $|P_\alpha|, |P_\beta|$ resp. b, b'
 in usual range over $|B| \setminus \{0\}$.

It will be convenient to work with certain isomorphic copies of the $B \times C_\alpha$.

D6708 (i) Let $|\Pi_\alpha| =$ the set of all $\langle b, p_\alpha \rangle$.

$$(ii) \quad \langle b, p_\alpha \rangle \lesssim_{\Pi_\alpha} \langle b', p'_\alpha \rangle \iff b \leq b', p_\alpha \leq p'_\alpha.$$

$$(iii) \quad \text{Write } \Pi_\alpha = \langle |\Pi_\alpha|, \lesssim_{\Pi_\alpha} \rangle.$$

D6709 $\mathbb{D}_\alpha =$ the algebra over Π_α .

$$O_{\langle b, p_\alpha \rangle}^\alpha = \{ \langle b', p'_\alpha \rangle \mid \langle b', p'_\alpha \rangle \lesssim \langle b, p_\alpha \rangle \}$$

$$\text{note that } O_{\langle b, p_\alpha \rangle}^\beta = \{ \langle b', p'_\beta \rangle \mid b' \leq b, p'_\beta \leq p'_\alpha \}:$$

straight $\alpha < \beta$ (and this notation is not used elsewhere),

$$|P_\alpha| \subseteq |P_\beta|.$$

Now the mapping $\pi: \langle b, p_\alpha \rangle \mapsto O_{\langle b, O_{p_\alpha}^{\alpha} \rangle}^{\alpha \times C_\alpha}$

embeds Π_α as a dense subset of $B \times C_\alpha$ ($\alpha P_\alpha \subseteq \text{dense in } C_\alpha$):

so the mapping $\pi'_\alpha: O_{\langle b, p_\alpha \rangle}^\alpha \mapsto O_{\langle b, O_{p_\alpha}^{\alpha} \rangle}^{\alpha \times C_\alpha}$

extends to an isomorphism $\pi_\alpha: \mathbb{D}_\alpha \xrightarrow{\sim} B \times C_\alpha$. Thus it suffices to prove the theorem for the \mathbb{D}_α 's.

[120]

[6711]

Proof of T6707 for the D_α 's.

T6707 (i). Let $\omega \leq \alpha < \beta$. The idea is to apply the lemma T6117 with D_β as the B of its statement. Set

$$P = \{ q_\beta^B \mid b \in |B| \setminus \{0\}, p_a \in P_\alpha \}.$$

I assert that P satisfies conditions (i) - (iv) of T6117.

First, note that

$$6711 \quad q_\beta \leq p_a \iff (q_\beta \cap X_\alpha) \leq p_a,$$

which is true as \leq is reverse inclusion and $p_a \subseteq X_\alpha$. It follows that

$$\begin{aligned} 6712 \quad q_\beta \text{ is compatible with } p_a (\in P_\beta) &\iff \\ q_\beta \cap X_\alpha \text{ is compatible with } p_a (\in P_\beta) &\iff \\ q_\beta \cap X_\alpha \text{ is compatible with } p_a (\in P_\alpha). \end{aligned}$$

For q_β compatible with p_a ($\in P_\beta$) \rightarrow

$$\text{for some } q'_\beta, \quad q'_\beta \leq q_\beta \wedge q'_\beta \leq p_a$$

$$\rightarrow q'_\beta \cap X_\alpha \leq q_\beta \cap X_\alpha \wedge q'_\beta \cap X_\alpha \leq p_a$$

$\rightarrow q_\beta \cap X_\alpha$ is compatible with p_a in P_β .

$$\rightarrow \forall q'_\beta \quad q'_\beta \cap X_\alpha \wedge q'_\beta \leq p_a$$

$$\rightarrow q'_\beta \cap X_\alpha \leq q_\beta \cap X_\alpha \wedge q'_\beta \cap X_\alpha \leq p_a;$$

$\rightarrow (\neg q'_\beta \cap X_\alpha \in P_\alpha) \quad q_\beta \cap X_\alpha \text{ is compatible with } p_a$
 $(\in P_\alpha);$

[121]

and that implies that $V p_2' \leq p_2 \leq q_{\beta} \cap X_{\alpha} \cap p_2' \leq p_2$

$$\rightarrow p_2' \cup (q_{\beta} \cap X_{\alpha}) \leq q_{\beta} \cap p_2' \cup (q_{\beta} \cap X_{\alpha}) \leq p_2' \leq p_2,$$

$\rightarrow q_{\beta}$ and p_2 are compatible in P_{β} .

Now T6II7 (ii) is trivially verified. Let now X, Y dense nonvoid subsets of P

T6II7 (iv): $O_{(b, p_2)}^{\beta} \cdot \sum^D \exists \neq \emptyset \rightarrow V(b', q_{\beta})$ s.t.

$$O_{(b', q_{\beta})}^{\beta} \subseteq O_{(b, p_2)}^{\beta} \cap \sum^D \exists, \text{ as that is}$$

an open set and " $Y \cdot Z = Y \cap Z$ ".

Then $b' \leq b$ and $q_{\beta} \leq p_2$; set $p_2' = q_{\beta} \cap X_{\alpha}$.

By (711), $O_{(b, p_2)}^{\beta} \subseteq O_{(b, p_2')}^{\beta}$;

By 6712 $O_{(b, p_2')}^{\beta} \subseteq \sum^D \exists$.

T6II7 (iii) $\sum^D \exists \neq \emptyset \rightarrow V(b, q_{\beta})$ s.t.

$$O_{(b, q_{\beta})}^{\beta} \cap \sum^D \exists = \emptyset.$$

But then $\langle b, q_{\beta} \rangle$ is incompatible with each $\langle b', p_2 \rangle$

such that $O_{(b', p_2)}^{\beta} \cap Y \neq \emptyset$;

and so $\langle b, q_{\beta} \cap X_{\alpha} \rangle$ is also incompatible with them;

$$\rightarrow O_{(b, q_{\beta} \cap X_{\alpha})}^{\beta} \cap \sum^D \exists = \emptyset, \text{ and 6II7 (iii) follows.}$$

[122]

[6712]

$$\text{T 6117 (ii)} \quad \sum \mathbb{D}_\beta X - \sum \mathbb{D}_\beta Y \neq 0 \rightarrow$$

$$\forall \langle b, q_\beta \rangle \text{ s.t. } O_{\langle b, q_\beta \rangle}^\beta \subseteq \sum \mathbb{D}_\beta X$$

$$\text{and } O_{\langle b, q_\beta \rangle}^\beta \cap \sum \mathbb{D}_\beta Y = \emptyset$$

Pick now $a \langle b', q'_\beta \rangle \leq \langle b, q_\beta \rangle$ and an

$$O_{\langle b_0, p_\alpha \rangle}^\alpha \text{ s.t. } O_{\langle b', q'_\beta \rangle}^\beta \subseteq O_{\langle b_0, p_\alpha \rangle}^\alpha \subseteq \sum \mathbb{D}_\alpha X$$

and set $q'_\alpha = q'_\beta \cap X_\alpha$.

$$\text{Then } O_{\langle b', q'_\alpha \rangle}^\alpha \subseteq O_{\langle b_0, p_\alpha \rangle}^\alpha \subseteq \sum \mathbb{D}_\alpha X,$$

$$\text{and } O_{\langle b', q'_\alpha \rangle}^\alpha \cap \sum \mathbb{D}_\beta Y = \emptyset.$$

Thus $O_{\langle b', q'_\alpha \rangle}^\alpha \in P$ and $O_{\langle b', q'_\alpha \rangle}^\alpha \leq \sum \mathbb{D}_\alpha X - \sum \mathbb{D}_\beta Y$,

so 6117 (ii) is verified. There is therefore, by T 6117 and suitable composing of maps a regular embedding

$$6712 \quad \pi_{df} : D_\alpha \xrightarrow{\sim} D_\beta$$

which extends the 1-1 map $\pi'_{df} : O_{\langle b, p_\alpha \rangle}^\alpha \mapsto O_{\langle b, p_\beta \rangle}^\beta$ as required. Set

$$6713 \quad i_{df} = \pi_\alpha^{-1} \pi_{df} \pi_\beta \quad (\text{algebraists' order}).$$

T 6707 (ii) is immediate from the definitions of $\pi_{df}, \pi_{fr}, \pi_{dfr}$.

[123]

T6707 (iii) : that $D_2 = \text{r.m.c. } \bigcup_{\alpha < \lambda} \pi''_{\alpha \alpha} D_\alpha$ follows from T6117 $\Rightarrow T_2 \subseteq$ densely embedded in D_2 , and

$$|T_2| = \bigcup_{\alpha < \lambda} |T_\alpha| \text{ which is embedded in}$$

$$\bigcup_{\alpha < \lambda} \pi''_{\alpha \alpha} D_\alpha, \text{ all by embeddings which commute.}$$

(iv) : Suppose λ regular and $X \subseteq |D_2|$ a set of pairwise disjoint non-0 elements of $|D_2|$. $|T_2|$ is densely embedded in $|D_2|$, so pick to each $x \in X$ a $\langle b_x, p_x \rangle \in |T_2|$ s.t. $O_{\langle b_x, p_x \rangle}^\lambda \leq x$. Then

$\{\langle b_x, p_x \rangle \mid x \in X\}$ is a set of pairwise incompatible conditions of T_2 ; and the cardinality of this set is \bar{X} . It is enough to show $\bar{X} \neq \lambda$. Suppose then that $\bar{X} = \lambda$; as $|\bar{B}| < \lambda$

there is, by the regularity of λ , an $X_2 \subseteq X$, ($\bar{X}_2 = \lambda$) s.t.

$$x, x' \in X_2 \rightarrow b_x = b_{x'},$$

and $\langle b, p_x \rangle$ is compatible with $\langle b, p_{x'} \rangle \iff$

p_1 is compatible with p_2 .

So $\{p_x \mid x \in X_2\}$ is a set of pairwise incompatible conditions of cardinality λ . Each p_x is finite; so that by the regularity of λ , there is a $k < \omega$ such that

$$X' = \{p_x \mid \bar{p}_x = k\}$$

is of cardinality λ . The proof now proceeds by induction on k .

Suppose $k = 1$. Then each p_x is of the form $\{(r_x, \langle p_x, m_x \rangle)\}$ and so $x \neq x' \rightarrow p_x = p_{x'}, m_x = m_{x'} \wedge r_x = r_{x'}$. But $r_x < p_x < \lambda$, and so $\bar{p}_x < \lambda$, so $\bar{X}' < \lambda$. *

[124]

So let X be such that k is least possible. Then $k \neq 1$. Pick $p_x \in X'$. To each $p' \in X'$ there is a $\langle r, \langle \beta, m \rangle \rangle \in p_x$ such that for some $r \neq r'$, $\langle r', \langle \beta, m \rangle \rangle \in p'$. p_x is finite; so again $\langle r_0, \langle \beta_0, m_0 \rangle \rangle \in p_x$ can be chosen so that

$$X'' = \{p' \in X' \mid \forall r_p \neq r_0 \langle r_p, \langle \beta_0, m_0 \rangle \rangle \in p'\}$$

has cardinality 2. Now for each $p' \in X'$, and each such

$r_{p'}, \quad r_{p'} < \beta_0; \quad \bar{\beta}_0 < 2$, so by the regularity of 2, there is a r_1 such that

$$X''' = \{p'' \in X'' \mid \langle r_1, \langle \beta_0, m_0 \rangle \rangle \in p''\}$$

has cardinality 2. But then the set

$$\{p - \{\langle r_1, \langle \beta_0, m_0 \rangle \rangle\} \mid p \in X''\}$$

is a set of cardinality 2 of pairwise incompatible conditions each of cardinality $k-1$, contradicting the choice of X . \square_{ED}

(v) It suffices to show that $\mathbb{D} = \bigcup_{\alpha < \lambda} \pi_{\alpha}^{\mathbb{D}}(\mathbb{D}_{\alpha})$ is complete, (by (iii)). By an earlier observation \mathbb{D} is dense in \mathbb{D}_{λ} and is therefore a regular subalgebra. Suppose \mathbb{D} is not complete, and let X be a subset of \mathbb{D} of minimum cardinality, κ , such that $\sum X$ does not exist in \mathbb{D} . Enumerate $X \cong \langle x_v \rangle_{v < \kappa}$. Define $y_v = x_v \setminus \sum_{\mu < v} x_{\mu}$: by the minimality of κ , for every $v < \kappa$, $\sum_{\mu < v} x_{\mu}$ exists in \mathbb{D}_{λ} and equals $\sum_{\mu < v} \mathbb{D}_{\lambda} x_{\mu}$.

[125]

[6714]

The y_ν are pairwise disjoint, and so the set

$$Y = \{y_\nu \mid y_\nu \neq 0\} \subseteq \text{by (iv) of cardinality } < \lambda.$$

But each $y_\nu \in |\mathbb{D}|$; by the regularity of λ , there is an $\alpha < \lambda$ s.t. $Y \subseteq i_{\alpha\lambda}'' \mathbb{D}_\alpha$. But $i_{\alpha\lambda}'' \mathbb{D}_\alpha$ is complete, and so $\bigvee b \in i_{\alpha\lambda}'' \mathbb{D}_\alpha$ s.t. $b = \sum i_{\alpha\lambda}'' \mathbb{D}_\alpha y$; by (i)

$$b = \sum^{\mathbb{D}} y = \sum^{\mathbb{D}_\alpha} y = \sum^{\mathbb{D}_\alpha} X; b \in B(\mathbb{D})$$

and so $b = \sum^{\mathbb{D}} X$. \star . T 6707 is now proved.

T 6714 (i) There are regular embeddings for $\omega \leq \alpha < \beta < \gamma$

$$i_{\alpha\beta} : \mathbb{C}_\alpha \xrightarrow{\sim} \mathbb{C}_\beta$$

such that

$$\begin{array}{ccc} \mathbb{C}_\alpha & \xrightarrow{i_{\alpha\beta}} & \mathbb{C}_\beta \\ & \downarrow i_{\alpha\beta} & \downarrow i_{\beta\gamma} \\ & \mathbb{C}_\beta & \end{array}$$

$$(ii) \text{ If } B \subseteq \text{a CBA, then } B \times \mathbb{C}_\alpha \xrightarrow{i_{\alpha\beta}^B} B \times \mathbb{C}_\beta$$

$$\begin{array}{ccc} i_{\alpha\beta}^B & \uparrow & i_{\beta\gamma}^B \\ \uparrow & & \uparrow \\ \mathbb{C}_\alpha & \xrightarrow{\sim} & \mathbb{C}_\beta \end{array}$$

$$(iii) \lim(\lambda) \rightarrow \mathbb{C}_\lambda = \text{r.m.c. } \bigcup_{\alpha < \lambda} i_{\alpha\lambda}'' \mathbb{C}_\alpha.$$

$$(iv) \lambda = cf(\lambda) \rightarrow \mathbb{C}_\lambda \text{ satisfies the } \lambda-\text{c.c.}$$

$$(v) \lambda = cf(\lambda) \rightarrow \mathbb{C}_\lambda = \bigcup_{\alpha < \lambda} i_{\alpha\lambda}'' \mathbb{C}_\alpha.$$

Proof: For (i), (iii) - (v), set $B = \mathbb{D}$ in T 6707; as $|P_{\mathbb{D}, \mathbb{C}}| =$

[626]

[6715]

$\{\langle 1, c \rangle | c \neq 0\}$, $2 \times \mathbb{C}$ is "nicely" isomorphic to \mathbb{C} . (ii) from the definitions.

QED.

$$T6715 \quad \lambda = cf(\lambda) \rightarrow [\dot{\lambda} \in \mathbb{C}_\lambda]^{C_\lambda} = 1.$$

(That $\dot{\lambda}$ is in V^{C_λ} , λ is the first uncountable ordinal).

Proof. (i) $[\dot{\lambda} \text{ is a cardinal}]^{C_\lambda} = 1$ as C_λ satisfies the λ -c.c. (T6129). $(\alpha \geq \omega)$

(ii) Let $\alpha < \lambda$. Consider the function $f_\alpha \in V^{C_\lambda}$ defined

6716 by $[\dot{f}_\alpha : \dot{\omega} \rightarrow \dot{\lambda}]^{C_\lambda} = 1$ and

$$\text{for } n < \omega, \beta < \alpha : [\dot{f}_\alpha(n) = \dot{\beta}]^{C_\lambda} = O_{\beta, \dot{f}_\alpha, n}^\lambda.$$

Let $\beta < \alpha$, $p \in (\dot{P}_\lambda)$: pick an n greater than any m s.t. $\forall \alpha', \beta' \langle \beta', \langle \alpha', m \rangle \rangle \in p$.

Then $\beta' =_q p \cup \{\langle \beta, \langle \alpha, n \rangle \rangle\} \in (\dot{P}_\lambda)$ and $\beta' \leq p$; so that

$$O_{\beta'}^\lambda \leq O_p^\lambda, \text{ and } O_{\beta'}^\lambda \leq [\dot{f}_\alpha(n) = \dot{\beta}]^{C_\lambda}$$

$$\leq [\forall n < \dot{\omega} \dot{f}_\alpha(n) = \dot{\beta}]^{C_\lambda};$$

so that $\{O_p^\lambda / O_{\beta'}^\lambda \leq [\forall n < \dot{\omega} \dot{f}_\alpha(n) = \dot{\beta}]^{C_\lambda}\}$ is dense in C_λ , and so $[\forall n < \dot{\omega} \dot{f}_\alpha(n) = \dot{\beta}]^{C_\lambda} = 1$. As that is true for all $\beta < \alpha$,

$$[\forall \beta < \lambda \forall n < \dot{\omega} \dot{f}_\alpha(n) = \dot{\beta}]^{C_\lambda} = 1, \text{ so that}$$

$$[\dot{\lambda} < \dot{\omega}]^{C_\lambda} = 1. ([\dot{\omega} = \dot{\omega}]^D = 1 \text{ for every } D). (i), (ii) \text{ prove the theorem.}$$

[127]

[6717]

+ 6717 (Jensen) Let \beth be strongly inaccessible; i.e. $\omega < \beth$, $\beth = \bar{\beth} < \beth \rightarrow 2^\beth < \beth$, and \beth is regular. Let $C_\beth \subseteq_{\text{reg}} B$ where B is a cBA . Then B is in the following sense completely saturated w.r.t. cBA 's of cardinality $< \beth$:

If $A \subseteq_{\text{reg}} B$ and $A \subseteq_{\text{reg}} A'$, A, A' complete and $|A'| < \beth$, then there is a regular embedding

π s.t. $A \subseteq B$

$$\begin{array}{ccc} & \nearrow \pi & \\ A & & A' \end{array}$$

(that is, $\pi \upharpoonright |A|$ is the identity $|A|$).

Proof (Jensen) Let $\dot{F} \in \dot{\mathcal{V}}^B$ be the canonical $\dot{\mathcal{V}}$ -complete ultrafilter on B , and $\dot{F}_A \in \dot{\mathcal{V}}^A$ that on A . [Then $\dot{F}_A \in \dot{\mathcal{V}}^B$ and $[\dot{F}_A \equiv \dot{F} \upharpoonright |A|]_B = \mathbb{I}$].

By T 6220 there is an $\dot{F}^+ \in \dot{\mathcal{V}}^A$

such that $[\dot{F}^+ \text{ is a } \dot{\mathcal{V}}\text{-complete filter on } A']_A$
 $= [\dot{F}^+ \text{ is a } \dot{\mathcal{V}}\text{-complete filter on } A']_B = \mathbb{I}$,

and $[\dot{F}^+ = \{\alpha' \in |A'| \mid \forall a \in \dot{F}_A \ a \leq \alpha'\}]_A$

$= [\dot{F}^+ = \{\alpha' \in |A'| \mid \forall a \in \dot{F}_A \ a \leq \alpha'\}]_B$
 $= \mathbb{I}$.

[vide D 6222].

[67.8]

[67.18]

and for $a' \in |A'|$, $[\dot{\alpha}' \in \dot{F}^+]^B = [\forall a \in \dot{F}_A \ a \leq \dot{\alpha}']^B$;

so that $[\dot{\alpha}' \in \dot{F}^+ \iff \forall a \in \dot{F}_A \ a \leq \dot{\alpha}']^B = 1$

and

$[\dot{F}_A \text{ is a basis for } \dot{F}^+]^B = 1$.

67.18 Now let $X = S(|A'|)$: as $\vartheta < \omega_1$ strongly inaccessible, $\dot{X} < \vartheta$. By T 67.15

$[\dot{X} \text{ is countable}]^{C_{\vartheta}} = 1$; $C_{\vartheta} \subseteq {}_{\vartheta} B$,
 so $V^{C_{\vartheta}} \subseteq V^B$, and therefore $[\dot{X} \text{ is countable}]^B = 1$.

The idea now is to use T 62.14 to construct in V^B a
 \dot{V} -complete ultrafilter \dot{F}' on A' which extends \dot{F}^+ ;
 but we want an \dot{F}' s.t. $a' \in |A'| \wedge a' \neq 0 \rightarrow [\dot{\alpha}' \in \dot{F}']^B \neq 0$.

Now as $V^{C_{\vartheta}} \subseteq V^B$, the functions f_a ($a < \vartheta$) defined

(67.16) during the proof of T 67.15 are in V^B . For
 $\beta < a < \vartheta$, $n < \omega$,

$$[\dot{f}_a(n) \neq \dot{\beta}]^B = [\dot{f}_a(n) \neq \dot{\beta}]^{C_{\vartheta}} \in C_{\vartheta},$$

and indeed

$$[\dot{f}_a(n) \neq \dot{\beta}] = \text{int cl } \{p \in |P_{\vartheta}| \mid Vp' \neq \beta, \\ \langle \beta' \langle a, n \rangle \rangle \in p\}$$

$$= \{p \in |P_{\vartheta}| \mid Vp' \neq \beta \langle \beta' \langle a, n \rangle \rangle \in p\},$$

so if p' is not in that set, then $p'' = p' \cup \{\beta' \langle a, n \rangle\} \in |P_{\vartheta}|$, $p'' \leq p'$ and $O_{p''}^{\vartheta} \leq [\dot{f}_a(n) \neq \dot{\beta}]$. Therefore
 for any $b \in |B| \setminus \{0\}$, the set

[129]

$$\{ \langle a, n \rangle \mid \forall \beta \quad b \in [f_n(\beta) \neq \beta] \},$$

is a subset of the domain of some $p \in \dot{\mathbb{P}}_{\alpha_1}$ and therefore finite. Hence $\dot{\mathbb{P}}_{\alpha_1} \leq \dot{\mathbb{P}}_A < \aleph_0$, the set

$$\{ \dot{\alpha} \mid \forall \beta \quad \forall a \in |A| \quad a \cdot [f_\alpha(\beta) = \beta] = 0 \}$$

is of cardinality $< \aleph_0$.

Choose therefore a cardinal $\kappa < \aleph_0$ such that $\dot{\mathbb{X}} \leq \kappa$ (6718) and that

$$\forall a \in |A| \setminus \{0\} \forall \beta < \aleph_0, \quad a \cdot [f_\kappa(\beta) = \beta] \neq 0.$$

Let $\sigma : \kappa \xrightarrow{\text{onto}} |A'|$ in V . By T6715, $[\kappa \text{ is countable}]^B = 1$, so as $\dot{\sigma} \in V^B$, \dot{F}' is a \dot{V} -complete filter on $|A|$, the method of Rasiowa and Sikorski, T6214, may be applied $\overset{\text{to}}{\underset{(\text{in } V^B)}{\text{to}}}$ construct an $\dot{F}' \in V^B$ s.t.

$[\dot{F}' \text{ is a } \dot{V}\text{-complete ultrafilter on } |A'| \text{ which extends } \dot{F}']^B = 1$,

and $1 = [\dot{\sigma}^\vee(f_\kappa(\beta)) \in \dot{F}' \leftrightarrow -\dot{\sigma}^\vee(f_\kappa(\beta)) \notin \dot{F}']$: the second is possible as unless $-\dot{\sigma}^\vee(f_\kappa(\beta)) \in \dot{F}'$ (when necessarily $-\dot{\sigma}^\vee(f_\kappa(\beta)) \in \dot{F}'$), we can by the lemma of Rasiowa and Sikorski require that $\dot{\sigma}^\vee(f_\kappa(\beta)) \in \dot{F}'$.

[130]

[6719]

Define $\pi : |A'| \longrightarrow |B|$ by

$$6719 \quad \pi(a') =_{\exists} [\check{a}' \in \dot{F}']^B.$$

Then as in the proof of T 6603, π is seen to be a \check{V} -complete homomorphism; by the choice of \dot{F}' , π is 1-1; for if $a' \in |A'| \setminus \{0\}$, then

$$[\neg a' \notin \dot{F}^+]^B \in |A|, \text{ as}$$

$\dot{F}^+ \in V^A$; by T 6224, $[\neg \check{a}' \notin \dot{F}^+]^B = h(a') \neq \emptyset$, so let $a' = \sigma(x)$ where $x < \omega$. Then

$$\begin{aligned} 0 \neq [f_k(\check{0}) = \check{v}]^B \cdot [\neg a' \notin \dot{F}^+]^B \\ \leq [\check{a}' \in \dot{F}']^B = \pi(a'). \end{aligned}$$

Finally, $\pi \cdot \uparrow |A| = \text{id} \uparrow |A|$ (id is the identity).

Now $[\dot{F}' \text{ extends } \dot{F}_A]^B = \mathbb{I}$;

$[\dot{F}_A \text{ is an ultrafilter on } A]^B = \mathbb{I}$;

and so for $a \in |A|$, as $[\check{a}' \in \dot{F}_A \leftrightarrow \neg a \notin \dot{F}_A]^B = \mathbb{I}$,

$$[\check{a} \in \dot{F}_A]^B = -[\neg \check{a} \in \dot{F}_A]^B;$$

$$[\check{a} \in \dot{F}_A]^B \leq [\check{a}' \in \dot{F}']^B;$$

$$[-\check{a} \in \dot{F}_A]^B \leq [-\check{a}' \in \dot{F}']^B;$$

$$\text{and } \pi(a) = [\check{a}' \in \dot{F}']^B = [\check{a} \in \dot{F}_A]^B = a.$$

Q.E.D.

An immediate generalisation is

[631]

[6720]

T 6720 (Jensen) Let \beth , C_\beth , \underline{B} be as before. Then if

A, A' are cbAs, $|A'| < \beth$,

and $\pi_1: A \xrightarrow{\sim} A'$, $\pi_2: A \xrightarrow{\sim} B$ then there is a π s.t.

$$\begin{array}{ccc} A & \xrightarrow{\pi_2} & B \\ \pi_1 \searrow & \nearrow \pi & \\ A' & & \end{array}$$

A particular consequence of T 6717 is that every cBA of cardinality $< \beth$ is regularly embedded in C_\beth .

DIGRESSION

(Jensen) The proof of T 6717 shows that if \beth is regular (but not necessarily strongly inaccessible)

if $C_\beth \subseteq_{\text{reg}} B$, $A \subseteq_{\text{reg}} B$, $A \leq_{\text{reg}} A'$, $|A'| < \beth$,

A complete and S a set of suprema and infima in A' s.t. $\overline{S} < \beth$, then there is a π preserving all sups and inf in S s.t. $A \subseteq_{\text{reg}} B$.

$$\begin{array}{c} \overline{S} \\ \subseteq_{\text{reg}} \\ A' \xrightarrow{\pi} B \end{array}$$

END OF DIGRESSION.

T 6719 leads by a standard argument to the first goal of this paragraph, T 6724. First a definition.

Let \beth be strongly inaccessible.

D 6721 A cBA B is \beth -sublime iff the following hold:

[132]

[6722]

(A) $|\overline{B}| = \beth$

(B) $C_\beta \subseteq_{\text{sg}} B$

(C) there is a sequence $\langle B_\alpha | \alpha < \beth \rangle$ of CBA's s.t. $\alpha \leq \beta \rightarrow B_\alpha \subseteq_{\text{sg}} B_\beta \subseteq_{\text{sg}} C_\beta$;

$$\lim(\lambda) \rightarrow B_\lambda = \text{r.m.c. of } \bigcup_{\alpha < \lambda} |\overline{B_\alpha}|;$$

$$|B| = \bigcup_{\alpha < \beth} |\overline{B_\alpha}|; \text{ and } \alpha < \beth \rightarrow \overline{|\overline{B_\alpha}|} < \beth.$$

T6722 C_\beth is \beth -sublime.

Proof. Set for $\alpha \geq \omega$, $A_\alpha = \bigcup_{\gamma < \beth} C_\alpha$; for $\alpha < \omega$,

$$A_\alpha = A_\omega.$$

$$(A) \alpha \geq \omega \rightarrow \overline{|\overline{B_\alpha}|} = \beth; \text{ so } \beth \leq \overline{|\overline{C_\alpha}|} \leq \overline{\beth}$$

(as the elements of $|\overline{C_\alpha}|$ are subsets of $|\overline{B_\alpha}|$) and
 $\overline{\beth} < \beth$, if $\alpha < \beth$; so $\alpha < \beth \rightarrow \overline{|\overline{C_\alpha}|} < \beth$.

By T6714(v),

$$\beth = \overline{|\overline{B_\omega}|} \leq \overline{|\overline{C_\beth}|} = \sup_{\alpha < \beth} \overline{|\overline{C_\alpha}|} \leq \sup_{\alpha < \beth} \overline{\beth} = \beth,$$

$$\therefore \overline{|\overline{C_\beth}|} = \beth.$$

(B) is totally trivial

(C) : the sequence $\langle A_\alpha | \alpha < \beth \rangle$ will do by T6714;

that $\alpha < \beth \rightarrow \overline{|\overline{A_\alpha}|} < \beth$ has really been shown while checking (A). Q.E.D.

That any two \beth -sublime algebras are isomorphic follows from

[133]

[6724]

T6724 (Jensen) Let B be ω -subline, $D \subseteq_{\text{reg}} B$, $D' \subseteq_{\text{reg}} C_\alpha$, $|D| < \omega$ and $\sigma : D \xrightarrow{\sim} D'$; then there is a τ such that

$$\begin{array}{ccc} D & \xrightarrow{\sim} & D' \\ \uparrow \sigma & & \downarrow \tau \\ B & \xrightarrow{\sim} & C_\alpha \end{array}$$

Proof By repeated application of T6720 and the subline nature of B and C_α .

Let $\langle B_\alpha | \alpha < \omega \rangle$ be a sequence for B as in the definition of "subline".

6725 Observe that if $E \subseteq_{\text{reg}} B$ and $|E| < \omega$, then there is an $\alpha < \omega$ such that $E \subseteq_{\text{reg}} B_\alpha$; for α (B_α) = $|B|$, $\forall \alpha |E| \subseteq |B_\alpha|$; therefore $0 \neq E \subseteq |E|$,

$$\begin{aligned} \sum^B E \times t &= \sum^B B_\alpha \times t \quad (\text{as } E \subseteq_{\text{reg}} B) \\ &= \sum^{B_\alpha} (E \times t) \quad (\text{as } B_\alpha \subseteq_{\text{reg}} B). \end{aligned}$$

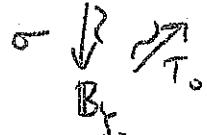
The same holds for C_α and the sequence $\langle A_\alpha | \alpha < \omega \rangle$.

Using this remark, define a sequence of isomorphisms $\langle \tau_v | v < \omega \rangle$ between regular subalgebras of B and C_α , and simultaneously a monotone continuous sequence $\langle \beta_v | v < \omega \rangle$ of ordinals $< \omega$ so that for $v > 0$,

$$\tau_{2v} : B_{\frac{v}{2v}} \xrightarrow{\sim} A_{\frac{v}{2v+1}} \quad \text{and} \quad \tau_{2v+1}^{-1} : A_{\frac{v}{2v+1}} \xrightarrow{\sim} B_{\frac{v}{2v+2}}$$

(134)

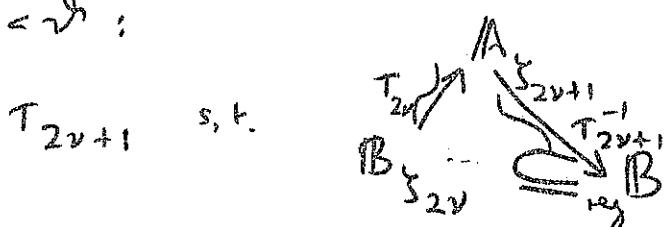
Pick ξ_0 s.t. $D \subseteq_{\text{sg}} B_{\xi_0}$; T_0 s.t. $D' \subseteq C_{T_0}$;



$\xi_1 > \xi_0$ s.t. $T_0^{-1}|B_{\xi_0}| \subseteq |A_{\xi_1}|$;

T_1 s.t. $\begin{matrix} A_{\xi_1} \\ \nearrow T_0 \quad \searrow T_1^{-1} \\ B_{\xi_0} \subseteq B \end{matrix}$; $\xi_2 > \xi_1$ s.t. $T_1^{-1}|A_{\xi_1}| \subseteq |B_{\xi_2}|$;

for $0 < v < \lambda$:



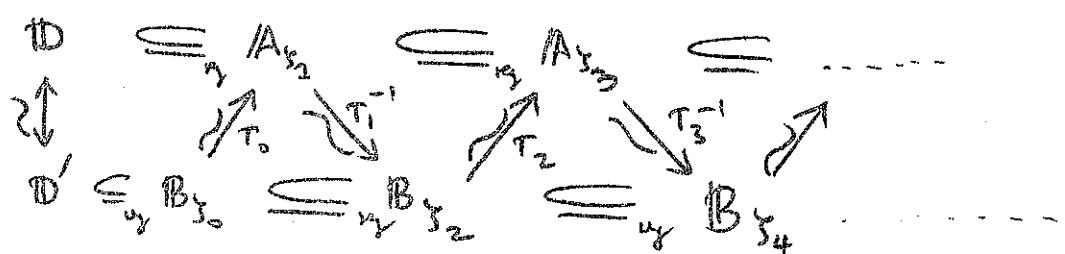
ξ_{2v+2} s.t. $T_{2v+1}^{-1}|A_{\xi_{2v+1}}| \subseteq |B_{\xi_{2v+2}}|$;

T_{2v+2} s.t. $\begin{matrix} A_{\xi_{2v+2}} \\ \nearrow T_{2v+1} \quad \searrow T_{2v+2} \\ B_{\xi_{2v+2}} \subseteq_{\text{sg}} C_{T_{2v+2}} \end{matrix}$;

ξ_{2v+3} s.t. $T_{2v+2}^{-1}|B_{\xi_{2v+2}}| \subseteq A_{\xi_{2v+3}}$.

At limit number λ , set $\xi_\lambda = \sup \{\xi_v \mid v < \lambda\}$ and $\xi_{\lambda+1} = \xi_\lambda + 1$.

Set $\tilde{T}_\lambda = \bigcup_{v < \lambda} T_v$: on the whole tower



[135]

[6726]

commutes, $\tilde{\tau}_2 : \bigcup_{v < \lambda} B_v \xrightarrow{\sim} \bigcup_{v < \lambda} A_v$; that is

$\tilde{\tau}_2 : \bigcup_{v < \lambda_2} B_v \xrightarrow{\sim} \bigcup_{v < \lambda_2} A_v$. Now by the

subline nature of C_λ and B , $B_{\lambda_2} = \text{r.m.e. of } \bigcup_{v < \lambda_2} B_v$ and $A_{\lambda_2} = \text{r.m.e. of } \bigcup_{v < \lambda_2} A_v$; so $\tilde{\tau}_2$ extends uniquely to an isomorphism (defined to be τ_2) of B_{λ_2} and A_{λ_2} . So

$\tau_2 : B_{\lambda_2} \xrightarrow{\sim} A_{\lambda_2}$. Set $\tau = \bigcup_{v < \lambda} \tau_v$. Then as

$$C_\lambda = \bigcup_{v < \lambda} A_v; B = \bigcup_{v < \lambda} B_v, \quad \tau : B \xrightarrow{\sim} C_\lambda \text{ and}$$

$$D \xleftarrow{\cong} D'$$

$$\begin{matrix} \text{In} \\ \text{B} \end{matrix} \xleftarrow[\tau]{\cong} \begin{matrix} \text{In} \\ C_\lambda \end{matrix}$$

Q.E.D.

T6726 C_λ is homogeneous.

Proof Let $b, b' \in (C_\lambda) \setminus \{0, 1\}$. Set $D = \{0, b, -b, 1\}$, (the 4-element algebra generated by b); $D' = \{0, b', -b', 1\}$, and σ the map $\{(0, 0), (b', b), (-b', -b), (1, 1)\}$. Then $\sigma : D \xrightarrow{\cong} D'$; as $D \subseteq_{\text{sgn}} C_\lambda$ and $D' \subseteq_{\text{sgn}} C_\lambda$, there is by T6724 an automorphism τ of C_λ such that

$$\begin{matrix} D & \xleftarrow{\cong} & D' \\ \text{In} & & \text{In} \\ C_\lambda & \xleftarrow[\tau]{\cong} & C_\lambda \end{matrix}$$

$$\tau(b) = b'.$$

Q.E.D.

[36]

[6727)

For the rest of the paragraph, suppose that there is at least one strongly inaccessible cardinal and that θ is the first. It is therefore definable.

From now till D6745 suppose that $x \in V^{\mathbb{C}_\theta}$ and that $[\exists n \in \omega]^{C_\theta} = 1$.

D6727. Define $C_x \subseteq_{\text{rg}} C_\theta$ as the intersection of all regular complete subalgebras of C_θ including

$$\{[\exists n \in \omega]^{C_\theta} \mid n < \omega\}.$$

Now that set is countable and $C_\theta = \bigcup_{\alpha < \theta} A_\alpha$; by the regularity of θ ,

$$\forall \alpha < \theta \quad \{[\exists n \in \omega]^{C_\theta} \mid n < \omega\} \subseteq |A_\alpha|;$$

as A_α is complete and a regular subalgebra, $|C_x| \leq |A_\alpha|$; so

T6728 C_x is a regular complete subalgebra of C_θ and

$$|\overline{\overline{C_x}}| < \theta.$$

Consider now $C_x \times C_\theta$: by T6707, it is the union of a θ -sequence of smaller algebras $C_x \times C_\alpha$, each of cardinality $< \theta$ (as $|\overline{\overline{C_x \times C_\alpha}}| \leq 2^{\max(|\overline{\overline{C_x}}|, |\overline{\overline{C_\alpha}}|)} < \theta$); so it is of cardinality θ , and $i_2 : C_\theta \xrightarrow{\sim} C_x \times C_\theta$. Hence

T6729 $C_x \times C_\theta$ is sublime.

By T6724 there is a $\pi_x : C_\theta \xleftarrow{\sim} C_x \times C_\theta$ such that

[137]

6730

$$\begin{array}{c} C_x \subseteq_{\text{reg}} C_\varphi \\ i_2 \swarrow \quad \searrow \pi_x \\ C_x \times C_\varphi \end{array}$$

[6730]

Set

D6731 $C^\varphi = \pi_x^{-1}((i_2''(C_\varphi))).$

Then $C_\varphi = C_x \times C^\varphi$ (or to make a finer distinction, C_φ is the internal Boolean product of C_x and C^φ , whereas $C_x \times C^\varphi$ is the external, defined in D6704). That is,

$$C_x \subseteq_{\text{reg}} C_\varphi; \quad C^\varphi \subseteq_{\text{reg}} C_\varphi;$$

$$c \in C_x \setminus \{0\}, c' \in C^\varphi \setminus \{0\} \rightarrow c \cdot c' \neq 0,$$

and the set of such $c \cdot c'$ is dense in C_φ .)

6732 It is significant that $C^\varphi \cong C_\varphi$.

T6733 Let $\dot{F} \in V^{C_\varphi}$ be the canonical V -complete ultrafilter on C_φ , and $\dot{F}_x \in V^{C_x}$ that on C_x . (So $[\dot{F}_x \equiv \dot{F} \cap [C_x]]^{C_\varphi} = 1$).

Then

$$[\dot{\bigvee}^V[C_x] \equiv \dot{\bigvee}^V[\dot{F}_x]]^{C_\varphi} = 1.$$

Post $\lambda n < \omega [\dot{\forall}^V \dot{\varepsilon} x \longleftrightarrow \dot{\forall}^V \dot{\varepsilon} x]^{C_\varphi} \in \dot{F}_x]]^{C_\varphi} = 1$,

for let $b = [\dot{\forall}^V \dot{\varepsilon} x]^{C_\varphi}$; then $b \in C_x$, and

$$b = [\dot{b} \in \dot{F}]^{C_\varphi} = [\dot{b} \in \dot{F}_x]^{C_\varphi}.$$

[38]

consequently,

$$[\check{V}[\dot{x}] \in \check{V}[F_{\dot{x}}]]^{C_\alpha} = 1.$$

Conversely, consider the way C_x is formed from

$$Z_0 = \{ [x \in \dot{x}]^{C_\alpha} \mid n < \omega \};$$

$$\text{set } Z_{2n+1} = \{ \sum^{C_\alpha} x \mid 0 \neq x \in Z_{2n} \},$$

$$Z_{2n+2} = \{ -b \mid b \in Z_{2n+1} \} \cup Z_{2n+1},$$

$$Z_\lambda = \bigcup_{\alpha < \lambda} Z_\alpha \quad \text{for limit numbers } \lambda.$$

$$\text{Then } \alpha < f \rightarrow Z_\alpha \subseteq Z_f, \text{ and } C_x = \bigcup_{\alpha < f} Z_\alpha.$$

Now in $\check{V}[\dot{x}]$ we can attempt to define a filter F'_x from x by induction on α :

if $b \in Z_{2n+1} \setminus Z_{2n}$, and $b = \sum^{C_\alpha} x$ where

$$0 \neq x \in Z_{2n},$$

then

$$[b' \in F'_x]^{C_\alpha} = \sum \{ [c' \in F'_x] \mid c \in x \},$$

if $b \in Z_{2n+2} \setminus Z_{2n+1}$, then

$$[b' \in F'_x]^{C_\alpha} = - [b \in F'_x]^{C_\alpha},$$

but evidently for all b , $[b' \in F'_x]^{C_\alpha} = [b \in F'_x]^{C_\alpha}$,

so that $[F'_x \in \check{V}[\dot{x}]]^{C_\alpha} = 1$ and $[\check{V}[F'_x] \subseteq \check{V}(x)]^{C_\alpha} = 1$.

QED

[139]

[6734]

T 6734 Let $\dot{\alpha} \in \mathcal{C}_x$. Then

$$[\dot{\alpha} \text{ is true in } \check{V}[x]]^{\mathcal{C}_x} = [\dot{\alpha}]^{\mathcal{C}_x}.$$

Proof: $[\dot{\alpha} \text{ is true in } \check{V}[x]]^{\mathcal{C}_x} = [\dot{\alpha} \text{ is true in } \check{V}[F_x]]^{\mathcal{C}_x}$

$$= \left[\begin{array}{c} \swarrow \\ [[\dot{\alpha}]]^{\mathcal{C}_x} \in F_x \end{array} \right]^{\mathcal{C}_x}$$

$$= [\dot{\alpha}]^{\mathcal{C}_x}, \text{ as } [F_x \subseteq F]^{\mathcal{C}_x} = 1$$

$\Rightarrow [\dot{b} \in F_x]^{\mathcal{C}_x} = b$ for elements of \mathcal{C}_x . $\boxed{\dot{b} \in F}$ Q.E.D.

T 6735 $[\dot{\aleph} \text{ is the first strongly inaccessible}]^{\mathcal{C}_x} = 1$.

Proof. Let $\overline{|\mathcal{C}_x|} = \kappa$. Let λ, μ be cardinals s.t. $\kappa < \lambda, \mu < \aleph$ and $2^\lambda = \mu$. By T 6129 and T 6130, $[\dot{\lambda} \text{ and } \dot{\mu} \text{ are cardinals and } 2^\lambda = \mu]^{\mathcal{C}_x} = 1$.

Hence $[\forall \lambda (\lambda \text{ a cardinal and } \lambda < \aleph \rightarrow 2^\lambda < \aleph)]^{\mathcal{C}_x} = 1$.

By T 6129, $[\dot{\aleph} \text{ is regular}]^{\mathcal{C}_x} = 1$, and so \aleph is strongly inaccessible in $\check{V}^{\mathcal{C}_x}$, and is the first, and is the first in V . Q.E.D.

By T 6734 and T 6735,

T 6736 $1 = [\dot{\aleph} \text{ is the first strongly inaccessible cardinal in } \check{V}[x]]^{\mathcal{C}_x}$.

Let now $\Omega_1(x), \Omega_2(x)$ be ZF formulae defining P_\aleph and C_\aleph : viz.

[140]

[6737]

D6737 $\Omega_1(\kappa)$ says " κ is the canonical collapsing partial ordering for the first strongly inaccessible cardinal"

and

D6738 $\Omega_2(\kappa)$ says " κ is the canonical complete Boolean algebra over the \dot{P}_κ such that $\Omega_1(\kappa)$ ".

$$T6739 \quad [\dot{\Omega}_1(\dot{P}_\kappa)]^{\dot{C}_\kappa} = 1.$$

Proof. By T6735 and the maximum principle, there is a $\dot{P}_\kappa \in V^{\dot{C}_\kappa}$ such that $[\dot{\Omega}_1(\dot{P}_\kappa)]^{\dot{C}_\kappa} = 1$. Argue in $V^{\dot{C}_\kappa}$: an element of $|\dot{P}_\kappa|$ is a finite map p from $\dot{I} \times \dot{\omega}$ to \dot{I} (such that $p(v, n) < v$ if defined) and is therefore $\in V$;

so for $\dot{y} \in V^{\dot{C}_\kappa}$,

$$[\dot{y} \in |\dot{P}_\kappa|]^{\dot{C}_\kappa} = \sum_{p \in |\dot{P}_\kappa|} [\dot{y} = p]^{\dot{C}_\kappa}$$

$$\text{and so } [|\dot{P}_\kappa| = |\dot{P}_{\dot{P}_\kappa}|]^{\dot{C}_\kappa} = 1.$$

The partial ordering is reverse inclusion, an absolute notion, and so $[\dot{P} = \dot{P}_{\dot{P}_\kappa}] = 1$. Q.E.D.

Now $[\forall \kappa \dot{\Omega}_2(\kappa)]^{\dot{C}_\kappa} = 1$ by T6735, and so there is an $\dot{E}_\kappa \in V^{\dot{C}_\kappa}$ such that

$$6740 \quad [\dot{\Omega}_2(\dot{E}_\kappa)]^{\dot{C}_\kappa} = 1.$$

[141]

[6741]

Let $\dot{F}_x^+ \in \dot{V}^{C_x}$ be the \dot{V} -complete filter on C_x , generated by \dot{F}_x^+ , whose existence is established by T6220. Then in \dot{V}^{C_x} the quotient algebra \dot{C}_x / \dot{F}_x^+ exists. There is therefore a $\dot{D}_x \in \dot{V}^{C_x}$ such that

$$6741 \quad [\dot{D}_x \text{ is a cBA}]^{C_x} = 1$$

and $[\dot{D}_x \text{ is an r.m.c. of } \dot{C}_x / \dot{F}_x^+]^{C_x} = 1$.

DIGRESSION It can be shown that

$$[\dot{C}_x / \dot{F}_x^+ \text{ is complete}]^{C_x} = 1,$$

but I shall not use that. END OF DIGRESSION.

I have now reached the second goal of the paragraph,

$$T6742. \quad [\dot{D}_x \cong \dot{E}_x]^{C_x} = 1.$$

Proof By T6117, it is enough to show that in \dot{V}^{C_x} , $\dot{P}_{\dot{x}}$ can be embedded as a dense subset of \dot{D} .

First of all, in \dot{V} : C_x is the internal Boolean product $C_x \times C^*$; that is, $C_x \subseteq_{\text{reg}} C_x$, $C^* \subseteq_{\text{reg}} C_x$, and the set of all $b \cdot c$ ($b \in C_x \setminus \{0\}$, $c \in C^* \setminus \{0\}$) is dense in C_x and does not contain $\{0\}$.

[142]

(6743)

b743 Let $H = \{b.c \mid b \in \overline{\mathcal{C}_x} \setminus \{\bar{0}\}, c \in \overline{\mathcal{C}^2} \setminus \{\bar{0}\}\}$.

Let $X = S(H)$.

I shall write the typical member \mathfrak{X} of $S(H)$ as
 $\mathfrak{X} = \{b_x.c_x \mid x \in \mathfrak{X}\}$.

The remarks above about \mathcal{C}_x translate to give

T6744 $\boxed{\mathcal{C}_x}$ is a Boolean algebra and $\overline{\mathcal{C}_x}$ and $\overline{\mathcal{C}^2}$
 are subalgebras of $\overline{\mathcal{C}_{28}}$; $\bar{H} = \{b.c \mid b \in \overline{\mathcal{C}_x} \setminus \{\bar{0}\}$
 and $c \in \overline{\mathcal{C}^2} \setminus \{\bar{0}\}\}$; $\bar{0} \notin \bar{H}$ and \bar{H} is
 dense in $\overline{\mathcal{C}_{28}}$ and for all $d \in \overline{\mathcal{C}_{28}}$ there is
 an $\mathfrak{X} \in X$ such that $d = \sum_{\mathfrak{X}} \mathfrak{X} \prod \mathcal{C}_x = \bar{1}$.

In the rest of the proof, which is reasoned in
 $\overline{\mathcal{C}_x}$, b is a variable ranging over $\overline{\mathcal{C}_x} \setminus \{\bar{0}\}$,
 and c one over $\overline{\mathcal{C}^2} \setminus \{\bar{0}\}$.

Let $\dot{\psi}_x$ be the quotient map $\overline{\mathcal{C}_{28}} \rightarrow \overline{\mathcal{C}_{28}/\dot{F}_x^+}$.

(1) Let $b \notin \dot{F}_x^+$, then as \dot{F}_x^+ is an ultrafilter on
 $\overline{\mathcal{C}_x}$, $-b \in \dot{F}_x^+$, and so $b \in$ kernel of $\dot{\psi}_x$;

therefore for every c , $\dot{\psi}_x(b.c) \leq \dot{\psi}_x(b) = \bar{0}'$,
 where $\bar{0}'$ is the $\bar{0}$ of $\overline{\mathcal{C}_{28}/\dot{F}_x^+}$.

[143]

(2) If $b \in \dot{F}_x$, then $\dot{\varphi}_x(b \cdot c) = \dot{\varphi}_x(c)$ for every c ;

$$\text{for } \dot{\varphi}_x(c) = \dot{\varphi}_x(b \cdot c + (c - b))$$

$$= \dot{\varphi}_x(b \cdot c) + \dot{\varphi}_x(c - b),$$

$$\text{and } \dot{\varphi}_x(c - b) \leq \dot{\varphi}_x(-b) \quad (\text{as } c - b \leq -b) \\ = \emptyset!$$

(3) Let $c, c' \in |\dot{\mathbb{C}}^x| \setminus \{\emptyset\}$: then

$$\dot{\varphi}_x(c) = \dot{\varphi}_x(c') \iff c = c'.$$

For $\dot{\varphi}_x(c) = \dot{\varphi}_x(c')$ $\rightarrow \forall b (b \notin \dot{F}_x \text{ and}$

$$(c - c') + (c' - c) \leq b)$$

\rightarrow for such a b , $((c - c') + (c' - c)) \circ -b = \emptyset$;

now $-b \in \dot{F}_x$ (as \dot{F}_x is an ultrafilter), and

$\Rightarrow -b \neq \emptyset$; \Rightarrow by T6744 ($\emptyset \notin \dot{H}$),

$$(c - c') + (c' - c) = \emptyset, \text{ and so } c' = c.$$

So the map $\dot{\varphi}_x$ restricted to $|\dot{\mathbb{C}}^x|$ is 1-1 and a homomorphism.

(4) The image of $|\dot{\mathbb{C}}^x| \setminus \{\emptyset\}$ under $\dot{\varphi}_x$ is dense in $\dot{\mathbb{C}}_x / \dot{F}_x^+$: for any non-zero element of

$|\dot{\mathbb{C}}_x / \dot{F}_x^+|$ is of the form $\dot{\varphi}(d)$, where $-d \notin \dot{F}_x^+$;

further $d \in |\dot{\mathbb{C}}_x|$, and so there is an $\mathfrak{X} \in \dot{X}$ such

[144] $d = \sum_{\mathbb{C}_x}^{\vee} \{b_x \cdot c_x \mid x \in \mathbb{X}\}$, by [6744].

$\dot{\psi}_{\dot{x}}$ preserves the supremum of each $\mathbb{X} \subseteq \dot{X}$, by T6220 (for a complete filter is readily characterised as one dual to the kernel of a complete homomorphism) and so

$$\dot{\psi}_{\dot{x}}(d) = \sum_{x \in \mathbb{X}} \dot{\psi}_{\dot{x}}(b_x \cdot c_x);$$

there is therefore an $x \in \mathbb{X}$ such that

$$\dot{\psi}_{\dot{x}}(b_x \cdot c_x) \neq 0.$$

By (1) $b_x \in F_x$, and so by (2),

$\dot{\psi}_{\dot{x}}(b_x \cdot c_x) = \dot{\psi}_{\dot{x}}(c_x)$, which is in the image of $\dot{\mathbb{C}}^{\dot{x}}$ by $\dot{\psi}_{\dot{x}}$.

But $0 \neq \dot{\psi}(c_x) \leq \dot{\psi}(d)$, and so $\dot{\psi}_{\dot{x}}((\dot{\mathbb{C}}^{\dot{x}}))$ is indeed dense in $\dot{\mathbb{C}}_{\dot{x}}^{\dot{x}} / F_{\dot{x}}^+$.

But $\dot{\mathbb{C}}^{\dot{x}} \cong \dot{\mathbb{C}}_x$ (indeed such an isomorphism exists in $\dot{\mathcal{V}}$ (6732)) and so there is an embedding of $\dot{\mathbb{P}}_{\dot{x}}^{\dot{x}}$ as a dense subset of $\dot{\mathbb{C}}^{\dot{x}}$; further $\dot{\mathbb{C}}^{\dot{x}}$ is embedded as a dense subset of $\dot{\mathbb{D}}$, its regular minimal completion; and so these embeddings compose to yield a dense embedding of $\dot{\mathbb{P}}_{\dot{x}}^{\dot{x}}$ in $\dot{\mathbb{D}}$

QED

If M is a transitive model of ZF, then write

$$K^M =_{df} \overline{S(S^M(S^M(\omega)))^M}.$$

D 6745

[145]

[6746]

By a $\text{ZF}^{\check{V}}$ -sentence I mean a sentence of L^{C_δ} in which the predicate letter \check{V} may occur but in which no (names of) elements of V^{C_δ} occur. By

$$T6746 \quad [\forall x(x \subseteq \dot{\omega} \rightarrow \check{V}^{[x]} < \dot{\omega}_1)]^{C_\delta} = \mathbb{I}.$$

Proof. The formula is a $\text{ZF}^{\check{V}}$ -sentence, and so by T6726 and T6120, has truth value $\mathbb{0}$ or \mathbb{I} . (For the predicate letter \check{V} , see the remark following T6120).

Suppose $\mathbb{0}$. Then

$$[\forall x(x \subseteq \dot{\omega} \wedge \check{V}^{[x]} \geq \dot{\omega}_1)]^{C_\delta} = \mathbb{I}.$$

By the Maximum Principle, there is an $\dot{x} \in V^{C_\delta}$ such that

$$[\dot{x} \subseteq \dot{\omega}]^{C_\delta} = [\check{V}^{[\dot{x}]} \geq \dot{\omega}_1]^{C_\delta} = \mathbb{I}.$$

But by T6715, $[\dot{x} = \dot{\omega}_1]^{C_\delta} = \mathbb{I}$ and by T6736, $[\dot{x} \text{ is strongly inaccessible in } \check{V}^{[\dot{x}]}]^{C_\delta} = \mathbb{I}$ — a manifest contradiction. Q.E.D.

Remark that no names of elements of V^{C_δ} occur in the formula $\dot{\Omega}_2$ of D6738.

T6747 $[\forall x \subseteq \dot{\omega} \vee B \vee F(B \in \check{V}^{[x]}) \text{ and in}$

$\check{V}^{[x]}, \dot{\Omega}_2(B)$; and F is a $\check{V}^{[x]}$ -complete ultrafilter on B , and $\check{V} = \check{V}^{[x][F]}]^{C_\delta} = \mathbb{I}$.

[146]

[6748]

Before the proof, a remark.

6748 Let M be a transitive model of ZF , B, B' CBA's in M , and F an M -complete ultrafilter on B . Suppose that in M , B and B' are isomorphic, and let $\pi \in M$ be an isomorphism:

$$\pi : B \xrightarrow{\sim} B'.$$

Define $F' = \{\pi(b) \mid b \in F\}$. Then

$$(i) \quad F = \{\pi^{-1}(b') \mid b' \in F'\}$$

$$(ii) \quad M[F] = M[F']$$

and (iii) F' is an M -complete ultrafilter on B' .

Proof. (i) is obvious; (ii) from (i), the definition of F' and the fact that π and π^{-1} are in M ; (iii) as $\pi \in M$.

Proof of 6747.

The formula in question is again a ZFV -sentence.

Suppose T6747 false: by the same reasoning as in the proof of T6746, there is an $\dot{x} \in V^{C_{\dot{x}}}$ such that

$$[\dot{x} \subseteq \dot{\omega}] = \mathbb{1} \text{ and } [V B V F (B \in \dot{V}[\dot{x}] \wedge \dot{\Omega}_2^{\dot{V}[\dot{x}]}(B) \wedge \dots)]^{C_{\dot{x}}} = 0.$$

By 6740, 6741, 6742 and 6734, there are $\dot{D}_{\dot{x}}, \dot{E}_{\dot{x}}$ and $\dot{F}_{\dot{x}}^+$ $\in V^{C_{\dot{x}}}$ such that

$$[\dot{D}_{\dot{x}} \in \dot{V}[\dot{x}] \wedge \dot{E}_{\dot{x}} \in \dot{V}[\dot{x}] \wedge \dot{F}_{\dot{x}}^+ \in \dot{V}[\dot{x}],$$

$$(\dot{\Omega}_2(\dot{E}_{\dot{x}}), \dot{D}_{\dot{x}} \approx \dot{E}_{\dot{x}} \text{ and } \dot{D}_{\dot{x}} \text{ is r.m.c. of } \dot{C}_{\dot{x}}/\dot{F}_{\dot{x}}^+)]^{C_{\dot{x}}} = \mathbb{1}.$$

By 6218, 6227, 6228 and the third line of page 43, there is an $\dot{F}^{\dot{x}} \in V^{C_{\dot{x}}}$ such that $[\dot{V} \equiv \dot{V}[\dot{x}] [\dot{F}^{\dot{x}}]]^{C_{\dot{x}}} = \mathbb{1}$ and $[\dot{F}^{\dot{x}} \text{ is a } \dot{V}[\dot{x}]\text{-complete ultrafilter on } \dot{D}_{\dot{x}}, \text{ with basis } \dot{F}/\dot{F}_{\dot{x}}^+]^{C_{\dot{x}}} = \mathbb{1}$. That and 6748 (applied to $\dot{D}_{\dot{x}}, \dot{E}_{\dot{x}}$ and $\dot{V}[\dot{x}]$) give a contradiction. QED