

### ¶ 6. Related notions of forcing.

Recall D 6003.

D 6600 Set  $\tilde{Q} = \{x/f \mid x \text{ infinite and } x \leq \omega\}$ .

Give  $\tilde{Q}$  the partial ordering defined by

D 6601  $a \leq_{\tilde{Q}} b \iff \forall x \in a \forall y \in b \ x \leq y$   
(for  $a, b \in \tilde{Q}$ ).

Set  $Q = \langle \tilde{Q}, \leq_{\tilde{Q}} \rangle$ .

D 6602 Let  $\mathbb{C}$  be the canonical complete BA's  
Let  $\mathbb{B}$  be the algebra over  $\mathbb{P}$ .

T 6603 ZF  $\vdash$  There is a regular embedding of  $\mathbb{C}$  in  $\mathbb{B}$ .

A corollary to T 6603:

T 6604 ZF + AC +  $2^{\aleph_1} = \aleph_2$   $\vdash$   $\mathbb{C}$  does not collapse cardinals.  
That is,  $\alpha$  a cardinal  $\rightarrow [\alpha \text{ is a cardinal}]^{\mathbb{C}} = 1$ .

Proof. Let  $\mathbb{C} \cong \mathbb{D} \subseteq_{\text{reg}} \mathbb{B}$ .

Then  $V^{\mathbb{D}} \subseteq V^{\mathbb{B}}$ : by T 6525, under the continuum hypothesis  $\mathbb{B}$  does not collapse cardinals, and so a fortiori  $\mathbb{D}$  does not.

Before proving T 6603 I shall want a lemma:

T 6605 ZF  $\vdash$  Let  $M$  be a transitive model of ZF and let  $\mathbb{B}^M$  be in  $M$  the algebra over  $\mathbb{P}^M$ . Suppose there is an  $M$ -complete ultrafilter  $F$  for  $\mathbb{B}^M$ . Then there is an  $M$ -complete ultrafilter  $G$  for  $\mathbb{C}^M$  defined by

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taking as a basis the set

$$G_0 = \{ x/f \in M \mid \forall z \in x/f \ \forall s <_0 z \ O_{\langle s, z \rangle} \in F \}.$$

REMARK If  $x \in M$ , then  $x/f \in M$ , and is the  $\sim_f$  equivalence class of  $x$  in the sense of  $M$ , too.

Proof. Let  $\Delta \in M$  be dense and  $\leq_{\mathbb{Q}}$ -closed in  $\mathbb{Q}^M$ .

It must be shown that  $G_0 \cap \Delta \neq \emptyset$ .

$$\text{Define } \Delta' = \{ \langle s, S \rangle \mid \langle s, S \rangle \in \mathbb{P}^M \wedge S/f \in \Delta \}.$$

Then  $\Delta' \in M$ , and I assert that  $\Delta'$  is dense and  $\leq$ -closed in  $\mathbb{P}^M$ . For let  $\langle t, T \rangle \in \mathbb{P}^M$ ; there is an  $x/f \in \Delta$  s.t.  $x/f \leq_{\mathbb{Q}} T/f$ ; pick  $T' \in x/f$  such that  $T' \subseteq T$ ; then  $\langle t, T' \rangle \leq \langle t, T \rangle$  and  $\langle t, T' \rangle \in \Delta'$ . If  $\langle s, S \rangle \in \Delta'$  and  $\langle t, T \rangle \leq \langle s, S \rangle$ , then  $T \subseteq S$  and so  $T/f \leq_{\mathbb{Q}} S/f \in \Delta$ , which is  $\leq_{\mathbb{Q}}$ -closed, and so  $T/f \in \Delta$ , therefore  $\langle t, T \rangle \in \Delta'$ . So there is an  $\langle s, S \rangle \in \Delta'$  such that  $O_{\langle s, S \rangle} \in F$ ; then  $S/f \in G_0 \cap \Delta$ .

Now let  $x/f \in G_0$  and  $x/f \leq_{\mathbb{Q}} y/f$ ; pick  $x' \in x/f$ ,  $s$  st.  $s <_0 x'$  and  $O_{\langle s, x' \rangle} \in F$ .

Then pick  $y' \in y/f$  such that  $s <_0 y'$  and  $x' \in y'$ .

Then  $\langle s, x' \rangle \leq \langle s, y' \rangle$  and so  $O_{\langle s, y' \rangle} \in F$ , and

therefore  $y/f \in G_0$ .

Finally suppose  $x/f \in G_0$  and  $y/f \in G_0$ .

Pick  $s, S, s', S'$  st.  $s <_0 S \in x/f \wedge s' <_0 S' \in y/f$  and

$O_{\langle s, S \rangle} \in F$  and  $O_{\langle s', S' \rangle} \in F$ . Then  $s \underline{in} s'$  or  $s' \underline{in} s$ : say the first. Then  $\langle s', S \cap S' \rangle \leq \langle s, S \rangle$  and  $\langle s', S' \rangle$ , and  $O_{\langle s', S \cap S' \rangle} \in F$ ; and so  $(S \cap S')/f \in G_0$ : so any two

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elements of  $G_0$  have a common refinement in  $G_0$ .

$G_0$  is therefore an  $M$ -generic filter on  $\mathbb{Q}^M$ , and therefore is the basis of an  $M$ -complete ultrafilter on  $\mathbb{C}^M$ .

Q.E.D.

A superfluous lemma:

T 6606 ZF  $\vdash \mathbb{C}$  is homogeneous.

Proof By T 6126, it suffices to show that if

$0 \neq c \in |\mathbb{C}|$ , then there is a  $c' \in \mathbb{C}$  such that  $\mathbb{C}/c' \cong \mathbb{C}$ ,  $c' \leq c$  and  $c' \neq 0$ .

Let  $c \in \mathbb{C} \setminus \{0\}$ . Then there is an  $x/f \in \tilde{\mathbb{Q}}$  such that  $0_{x/f}^{\mathbb{C}} \leq c$ .

Set  $\tilde{\mathbb{Q}}/x/f = \{y/f \in \tilde{\mathbb{Q}} \mid y/f \leq_{\tilde{\mathbb{Q}}} x/f\}$ ,

and  $\mathbb{Q}/x/f = \langle \tilde{\mathbb{Q}}/x/f, \leq_{\tilde{\mathbb{Q}}} \upharpoonright (\tilde{\mathbb{Q}}/x/f) \rangle$ .

Then  $\mathbb{Q}/x/f \cong \mathbb{Q}$  as partial orderings: for pick  $x \in x/f$ , and let  $\psi: \omega \leftrightarrow x$ : that is,  $\psi$  is a 1-1 map of  $\omega$  onto  $x$ . Define  $\tilde{\psi}(y/f) = \psi''(y)/f$  for any  $y \in y/f$  ( $\tilde{\psi}$  is uniquely defined).

$\tilde{\psi}$  is the required isomorphism.

Then the algebra over  $\mathbb{Q}/x/f$ , which is  $\mathbb{C}/0_{x/f}^{\mathbb{C}}$ , is isomorphic to the algebra over  $\mathbb{Q}$ , which is  $\mathbb{C}$ .

Q.E.D.

Proof of T 6603.

Let  $\dot{F} \in \mathcal{V}^{\mathbb{B}}$  be the canonical  $\mathcal{V}$ -complete ultrafilter on  $|\mathbb{B}|$ .

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Define  $\dot{G} \in \mathcal{V}^B$  by

$$\text{dom}(\dot{G}) = \{x \mid u \in |C|\},$$

$$\dot{G}(u) = \sum^B \{0_{\langle s, s \rangle}^B \mid 0_{\langle s, f \rangle}^C \leq u\}.$$

Then, as in the proof of T 6211 it is seen that

$$[\check{u} \in \dot{G}]^B = \dot{G}(x), \text{ and}$$

$$[\check{u} \in \dot{G} \leftrightarrow \forall \langle s, s \rangle (0_{\langle s, s \rangle}^C \in \dot{F} \wedge 0_{\langle s, f \rangle}^C \leq u)]^B = 1;$$

$$\text{so that } [0_{\langle s, f \rangle}^C \in \dot{G} \leftrightarrow \forall \langle s, s \rangle (0_{\langle s, s \rangle}^C \in \dot{F} \wedge s_{\langle s, f \rangle} = x_{\langle s, f \rangle})]^B = 1$$

so that  $\dot{G}$  is the filter with basis  $\dot{G}_0$ , where  $\dot{G}_0$  is as defined in T 6605.

$$\text{Therefore } [\dot{G} \text{ is a } \check{V}\text{-complete ultrafilter on } \check{C}]^B = 1.$$

$$\text{Define } \pi(c) = [\check{c} \in \dot{G}] \text{ for } c \in |C|.$$

Then  $\pi$  is a complete homomorphism, for let  $X \subseteq |C|$ .

$$\text{Then } \pi(\sum X) = [\sum \check{x} \in \dot{G}]^B;$$

$$\text{now } [\sum \check{x} \equiv \sum^B \{b \mid b \in X\}]^B = 1$$

( $b$  is a bound variable here): so

$$[\sum \check{x} \in \dot{G} \leftrightarrow \forall b \in X \ b \in \dot{G}]^B = 1,$$

$$\text{and therefore } [\sum \check{x} \in \dot{G}]^B = \sum_{b \in X}^B [b \in \dot{G}]^B = \sum_{b \in X}^B \{\pi(b) \mid b \in X\}.$$

$$[\check{u} \in \dot{G} \leftrightarrow \neg \check{u} \notin \dot{G}]^B = 1 \text{ as } \dot{G} \text{ is an ultrafilter,}$$

$$\text{so } \pi(u) = \neg \pi(-u).$$

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$$\pi(0) = 0, \quad \pi(1) = 1.$$

Now if  $c \neq 0$ ,  $c \in |C|$ , then pick  $S \in c$ ;  
then  $\langle 0, S \rangle \in |P|$ ;

$$\text{and } 0 \neq 0_{\langle S, S \rangle}^B \leq \pi(c);$$

so  $\pi$  is 1-1.

Q.E.D.

I briefly consider a modification of the notion of forcing used in the main part of the paper.

Let  $\mathcal{D}$  be a non-principal ultrafilter on  $\omega$ .

6607. Define  $|P^{\mathcal{D}}| = \{ \langle s, S \rangle \in |P| \mid S \in \mathcal{D} \}$ .

$$P^{\mathcal{D}} = \langle |P^{\mathcal{D}}|, \leq \upharpoonright |P^{\mathcal{D}}| \rangle.$$

Then if  $\langle s, S \rangle, \langle t, T \rangle$  are in  $|P^{\mathcal{D}}|$ ,

then  $S \cap T \in \mathcal{D}$ , and so the two conditions are compatible. As there are only countably many  $S$ ,

6608  $B^{\mathcal{D}}$  satisfies the countable chain condition, where  $B^{\mathcal{D}}$  is the algebra over  $P^{\mathcal{D}}$ .

6609  $P$  does not satisfy C.C.C., for define for

$$\text{each } X \subseteq \omega, \quad S_X = \{ \bar{f}_X(n) \mid n < \omega \}$$

(D 6508, and 6509); then  $\{ S_X \mid X \subseteq \omega \}$  is a family of infinite subsets of  $\omega$ , any two of which have

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finite intersection, and so the set  $\{\langle 0, S_x \rangle \mid x \in \omega\}$  is of cardinality  $2^{\aleph_0}$ , and any two elements are incompatible conditions.

The idea of the next theorem was dictated to me by Jensen, and was used in the original proofs of TT 6013, 4 and 5, I understand.

T 6610 (Silver).  $\exists F + AC + 2^{\aleph_0} = \aleph_2 \mid$  There is an ultrafilter  $\mathcal{D}_0$  such that for every  $\sum_{\mathbb{N}}^1$  set  $\mathcal{X}$  and every  $\langle s, S \rangle \in |\mathbb{P}^{\mathcal{D}_0}|$  there is an  $S' \in \mathcal{D}_0$ ,  $S' \subseteq S$  such that  $\mathcal{X}$  is trivial on  $2_{\infty}^{\langle s, S' \rangle}$ .

Proof It is enough to show that there is a filter  $\mathcal{D}_0$  s.t.

$\wedge s \wedge \mathcal{X} (\mathcal{X} \sum_{\mathbb{N}}^1 \rightarrow \forall S' \in \mathcal{D}_0 (s \leq S' \wedge \mathcal{X} \text{ is trivial on } 2_{\infty}^{\langle s, S' \rangle}))$

as then given  $\langle s, S \rangle \in |\mathbb{P}^{\mathcal{D}_0}|$ , and  $\mathcal{X}$ ,

select such an  $S'$ , and then

$\mathcal{X}$  is trivial on  $2_{\infty}^{\langle s, S \cap S' \rangle}$  and

$\langle s, S \cap S' \rangle \in |\mathbb{P}^{\mathcal{D}_0}|$  and  $\langle s, S \cap S' \rangle \leq \langle s, S \rangle$ , and  $\mathcal{D}_0$  will necessarily be an ultrafilter, as  $\wedge x \in \omega \{y \mid y \subseteq x \vee y \cap x = \emptyset\}$  is  $\sum_{\mathbb{N}}^1$ .

Let  $\langle s_n \mid n < \omega \rangle$  be the wonderful enumeration.

Let  $\langle \mathcal{X}_\alpha \mid \alpha < \aleph_2 \rangle$  enumerate all  $\sum_{\mathbb{N}}^1$  sets.

Define a sequence of non-principal filters on  $\omega$ ,

$\langle F_\alpha^n \mid \alpha < \aleph_2, n < \omega \rangle$  such that

each has a countable basis: that is,

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There is a countable set  $Y_\alpha^n \subseteq S(\omega)$  such that

$$F_\alpha^n = \{Z \mid \forall Y \in Y_\alpha^n \ Y \subseteq Z\}.$$

Set  $F_0^0 =$  filter of cofinite sets.

If  $F_\alpha^n$  has been defined, enumerate its basis  $Y_\alpha^n$

$$\text{as } \{Z_i^{d,n} \mid i < \omega\}.$$

Construct a sequence  $n_0 < n_1 < n_2 < \dots$

by picking  $s_n < n_0 \in Z_0^{d,n}$ ,

$$n_0 < n_1 \in Z_0^{d,n} \cap Z_1^{d,n},$$

$$n_1 < n_2 \in Z_0^{d,n} \cap Z_1^{d,n} \cap Z_2^{d,n},$$

$$\text{Set } Y = \{n_i \mid i < \omega\}.$$

Then for any infinite  $Z \subseteq Y$ ,  $Z \cap Z_i^{d,n}$

is infinite for every  $i$ . As  $X_\alpha \in CR$ , there is a  $Y' \in Y$  such that  $X_\alpha$  is trivial on  $2^{\langle s_n, Y' \rangle}$ .

$$\text{Set } Y_\alpha^{n+1} = Y_\alpha^n \cup \{Y'\}.$$

If  $F_\alpha^m$  has been defined for all  $m$ , set

$$Y_{\alpha+1}^0 = \bigcup_{m < \omega} Y_\alpha^m \quad (\text{so } F_{\alpha+1}^0 = \bigcup_{m < \omega} F_\alpha^m).$$

If  $\lim \lambda (< \omega_1)$  and  $F_\alpha^0$  has been defined

for all  $\alpha < \lambda$ , set  $Y_\lambda^0 = \bigcup_{\alpha < \lambda} Y_\alpha^0$ .

Set  $\mathcal{D}_0 = \bigcup_{\alpha < \aleph_1} F_\alpha^0$ . That  $\mathcal{D}_0$  has the requisite properties is readily checked. Q.E.D.

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D 6611 Let  $E$  be the algebra over  $\mathbb{P}^{\mathcal{D}_0}$ .

T 6612 ZF  $\vdash$  If  $x$  is  $\mathbb{P}$ -generic over  $L$  and  $y \in x$  and  $z \in x$  and  $y \setminus z$  is infinite then  $y \neq_L z$ ; in particular  $x$  is not of minimal  $L$ -degree.

( " $\mathbb{P}^{\mathcal{D}_0}$ -generic over  $L$ " is defined analogously to " $\mathbb{P}$ -generic over  $L$ " ).

Proof It suffices to show that T 6526 holds with  $B$  replaced by  $E$ ;  $E$  preserves cardinals and it satisfies the C.C.C. (=  $\mathcal{S}_1$ -c.c.). The proof of T 6526 proceeds by the general theory of forcing until the line 6527, which using all of T 6504, can be strengthened to

$$\langle S'', S'' \rangle \Vdash \bigwedge n (n \in i_j \leftrightarrow Q(n, \check{\alpha}, j)) \wedge \bigwedge n (n \in i_j \leftrightarrow R(n, \check{\alpha}, j))$$

where  $Q$  and  $R$  are the  $\Sigma'_1$  and  $\Pi'_1$  predicates of T 6504; and now  $S'' \in \mathcal{D}_0$ .

$$\text{Set } P = \{x \mid \bigwedge y \in x (x \setminus y \text{ infinite} \rightarrow \neg \neg \bigwedge n (n \in x \leftrightarrow Q(n, \alpha, y))\}$$

$$\text{Then } P = \{x \mid \bigwedge y \in x (x \setminus y \text{ infinite} \rightarrow$$

$$[\bigvee n (n \notin x \wedge R(n, \alpha, y)) \text{ or } \bigvee n (n \in x \wedge \neg Q(n, \alpha, y))]\}$$

and by Shoenfield's rules is seen to be  $\Pi^1_1$ , and is CR<sup>+</sup>, by T 6517.  $\therefore$  as its complement is  $\Sigma^1_1$ , there is an  $S''' \in \mathcal{D}_0$  s.t.  $P$  is trivial on  $2_{\infty}^{\langle S''', S''' \rangle}$ ; set  $T = S''' \cap S''$ .

Then  $\langle S'', T \rangle \leq \langle S, S \rangle$  and  $\langle S'', T \rangle \Vdash y \setminus z$  is finite.

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The rest of the proof carries over.

Q.E.D.

Let  $x$  be  $\mathbb{P}$ -generic over  $L$ : then as  $x$  is not of minimal  $L$ -degree,  $x$  is not Silver or Sacks over  $L$ . Further, given any  $y \in L$ , either  $x \setminus y$  is finite or  $x \setminus (\omega \setminus y)$  is: as, given any  $\langle s, S \rangle \in |\mathbb{P}|$ : if  $y \cap S$  is finite, then

$$\langle s, S \rangle \Vdash x \setminus (\omega \setminus y) \text{ is finite}$$

but if  $y \cap S$  is infinite then  $\langle s, y \cap S \rangle \in |\mathbb{P}|$  and

$$\langle s, y \cap S \rangle \Vdash x \setminus y \text{ is finite.}$$

Now if  $\bar{y}$  is random or generic over  $L$ , then for any  $y \in L$ ,  $\overline{y \cap y} = \overline{y \cap \omega \setminus y} = \omega$ , if  $\overline{\bar{y}} = \overline{\omega \setminus y} = \omega$ , for

$$AC + ZF \vdash \Lambda y \subseteq \omega \left( \left\{ \omega \subseteq \omega \mid \overline{\omega \cap y} \text{ is finite or } \overline{\omega \cap \omega \setminus y} \text{ is finite} \right\} \right)$$

is of 1st. category and measure 0, provided that both  $y$  and  $\omega \setminus y$  are infinite);

hence  $x$  is not random or generic over  $L$ .

Now is  $x$   $\mathbb{P}^{\forall}$ -generic over  $L$  for any  $D$ : for let  $\langle s, S \rangle \in |\mathbb{P}|$ : then split  $S$  into two infinite disjoint subsets  $S_1, S_2$ . Then one of these, say  $S_1$  is not in  $D$ : and  $\langle s, S_1 \rangle \Vdash x \subseteq S_1 \cup s$

$$\text{so } \langle s, S_1 \rangle \Vdash x \text{ is not } \mathbb{P}^{\forall} \text{-generic.}$$

In fact, generic and random reals  $\bar{y}$  are always recursive in some  $\omega \subseteq \bar{y}$  with  $\bar{y} \setminus \omega$  infinite, a second property in which they disagree with  $\mathbb{P}$ -generic reals.