

¶ 5. Further properties and uses of P-generic reals.

The convention on variables of ¶¶ 3, 4 is maintained.

Before stating the main results of this paragraph, I recall some definitions from the Survey, §1, ¶1.

$$\text{D 1105 } X \leq_L Y \leftrightarrow_{\eta} X \in L[Y].$$

$$X =_L Y \leftrightarrow_{\eta} X \leq_L Y \wedge Y \leq_L X.$$

$$X <_L Y \leftrightarrow_{\eta} X \leq_L Y \text{ and } Y \notin L[X].$$

$$(\text{So } X =_L 0 \leftrightarrow X \in L).$$

$$\text{D 6500 } X \text{ is of minimal } L\text{-degree} \leftrightarrow \forall Y (Y <_L X \leftrightarrow Y \in L).$$

In the opening pages of the Survey, four types of reals obtained by forcing are discussed, viz-

generic, random, Sacks and Silver.

(See DD 1101, 1102, 1103, 1104, 1107, 1109, 1110, 1111. The Survey discusses countable transitive models M of $ZF + V = L$; I shall instead discuss L itself; the definitions are the same).

I draw the reader's attention to T 1118 (which I now restate in a slightly different form) and invite him to end the discussion in the Survey of pages 1 - 5.

T 1118 The properties of a real number being

[83]

[6501]

generic, random, Sacks or Silver over L are mutually exclusive; moreover if x is a real having any one of these properties, $L[x]$ contains no real having any of the other three.

I now state the main result of this paragraph.

T 6501 $ZF \vdash$ If x is \mathbb{P} -generic over L , $y \subseteq x$,
 $z \subseteq x$ and $y \cap z$ is infinite, then $y \not\subseteq_L z$;
in particular x is not of minimal L -degree.

REMARK y, z need not be in $L[x]$; y need not be $\subseteq_L y$.

I shall show in § 6 that the property of x stated in T 6501 is definitely not enjoyed by reals of any of the four other types mentioned. I am though (as yet) unable to answer many questions suggested by T 1118; for example, whether there are reals of minimal L -degree in $L[x]$. All I can say about that is that (by T 6501) no ^{infinite} subset of x is of minimal L -degree, which may be proved too, directly from T 6017, as every $y \subseteq x$ is \mathbb{P} -generic over L .

In establishing T 6501 I shall use Silver's theorem T 6013, an improvement of T 6011, and Jensen's observation that \mathbb{P} does not collapse cardinals. I shall prove all these in the course of the paragraph; my proof of Silver's result uses T 6017 and an absoluteness argument which I shall soon discuss. Before that, I prove Silver's result T 6014, again using T 6017, as an illustration of the method.

[84]

I begin with a review of known results given in the Survey, which I slightly rephrase.

T 3100 (Mostowski) $ZF \vdash$ Let M be a transitive model of ZF ; a_1, \dots, a_k reals in M ; $\Omega(e_1, \dots, e_k)$ a Σ_1^1 formula; then

$$\Omega^M(a_1, \dots, a_k) \longleftrightarrow \Omega(a_1, \dots, a_k).$$

That is, $\Omega(a_1, \dots, a_k)$ is true in M iff it is true in V .

T 3101 (Sheenfield) $ZF \vdash$ If M is a transitive model of ZF such that $\omega_1 \subseteq M$, a_1, \dots, a_k reals in M and $\Omega(e_1, \dots, e_k)$ a Σ_2^1 predicate, then

$$\Omega^M(a_1, \dots, a_k) \longleftrightarrow \Omega(a_1, \dots, a_k);$$

i.e., $\Omega(a_1, \dots, a_k)$ is true in M iff it is true in V .

CAUTION In the Survey I use "transitive model" to mean a set; here I include inner models, for which T 3100 and 3101 are also true.

Notice that T 3100, 3101 are for given M theorems and not schemata of ZF : for by the remark 6007 in the introduction, T 3100, e.g., may be cast in the form: for any Σ_1^1 formula Ω (of the appropriate object language), and reals $a_1, \dots, a_k \in M$,

$$\langle (2^\omega)^M, \in, \epsilon, +, \cdot, \leq, f_1, f_2, \dot{\Omega} \rangle \models \dot{\Omega}[a_1, \dots, a_k]$$

$$\longleftrightarrow \langle 2^\omega, \omega, \in, \epsilon, +, \cdot, \leq, f_1, f_2, \dot{\Omega} \rangle \models \dot{\Omega}[a_1, \dots, a_k].$$

T 3101 is proved in [13]; the proof of T 3100 is discussed presently.

[85]

[6502]

In proving T 6501, I shall use the following facts about " \leq_L ", proofs of which may be found in Addison [1].

D 6502 Let $a \in \omega$, $\alpha \in \text{On}$. a codes $\alpha \iff \{\langle n, m \rangle \mid 2^n \cdot 3^m \in a\}$ is a well-ordering of ω of type α .

T 6503 "a codes an ordinal" is a Π_1^0 predicate of a .

For a codes an ordinal $\iff \{\langle n, m \rangle \mid 2^n \cdot 3^m \in a\}$ is a linear ordering (an arithmetical notion) which has no sequence $\langle n_i \mid i < \omega \rangle$ such that $\forall i (2^{n_{i+1}} \cdot 3^{n_i} \in a)$, and this last clause is easily written in Π_1^0 form.

If $y \leq_L z$ then $y \in L[z]$, and there is an ordinal α in $F(L[z])$ such that $y = F_z(\alpha)$, where F_z is Gödel's constructibility function for constructing in $L[z]$: it is Gödel's fundamental lemma in his proof that $L[z]$ satisfies GCH that the least such α is countable in $L[z]$. There is therefore a real $a \in L[z]$ that codes the ordinal $\alpha + 1$; then for such an a ,

T 6504 [1] there are Σ_1^0 and Π_1^0 predicates Q and R such that

$$\lambda n \lambda y (y \leftrightarrow Q(n, a, z))$$

and

$$\lambda n \lambda y (y \leftrightarrow R(n, a, z)).$$

Q and R may be paraphrased as saying respectively

[86]

[3114]

that there is a construction on the ordinal coded by α which yields γ at the last step and $n \in \gamma$, and that for any construction on the ordinal coded by α , $n \in \gamma$ is the set constructed at the last step.

T 3114 (Addison) $ZF \vdash X \leq_L Y$ is a Σ_2^1 predicate of X and Y .

For $X \leq_L Y \leftrightarrow \forall a(a \text{ codes an ordinal and } \Lambda n(n \in X \rightarrow R(n, a, Y) \text{ and } \Lambda n(n \notin X \rightarrow \neg Q(n, a, Y)))$ which by Shoenfield's rules is seen to be Σ_2^1 . But here the difficulty mentioned in 6008 occurs: the following argument shows that use of AC can be avoided in this and similar cases.

Let R_o be arithmetical, and $\mathbb{Z} \subseteq \omega$.

$ZF \vdash \Lambda n \forall X \forall Y R_o(\mathbb{Z}, X, Y, n) \leftrightarrow \forall X \Lambda n \forall Y R'_o(\mathbb{Z}, X, Y, n)$ for an R'_o that is also arithmetical.

For suppose $\Lambda n \forall X \forall Y R_o$: then for every n ,

$$\forall X \forall Y R_o(\mathbb{Z}, X, Y, n);$$

and so by Shoenfield's theorem, T3101,

is $L[\mathbb{Z}]$, $\forall X \forall Y R_o(\mathbb{Z}, X, Y, n)$;

as that holds for every n ,

is $L[\mathbb{Z}]$, $\Lambda n \forall X \forall Y R_o(\mathbb{Z}, X, Y, n)$; conversely, if false in V , it will be false in $L[\mathbb{Z}]$.

now AC is true in $L[\mathbb{Z}]$, and so to each n the first X in $L[\mathbb{Z}]$ the canonical $L[\mathbb{Z}]$ -definable well ordering

[87)

[6505]

of $S^{L(\omega)}(\omega)$ (call it X^n) can be chosen:

set $X = \{2^m \cdot 3^n \mid m \in X^n\}$:

then $X \in L(\mathbb{Z})$, and now R'_0 can be written down so that $\in L(\mathbb{Z})$,

$$\lambda n \forall x \lambda y R_0(z, x, y, n) \leftrightarrow \forall x \lambda n \lambda y R'_0(z, x, y, n);$$

but (by coalescing the quantifiers $\lambda n, \lambda y$ into 1) the right hand side is a \sum_2' predicate of \mathbb{Z} ; and so holds $\in L(\mathbb{Z})$ iff $\in V$, whereas it has already been shown that the l.l.s. holds $\in L(\mathbb{Z})$ iff $\in V$.

QED

A similar argument shows that DC may be avoided in proving that no SF is Borel.

Let me now prove

T 6014 (Silver) $ZF + MC \vdash$ no SF $\in \sum_2'$ (hence by T 6324 it follows that every \sum_2' set is CR).

In fact I shall show that

T 6505 $ZF \vdash \lambda x \vee y (y \text{ is } P\text{-generic over } L[x])$
implies that no SF is \sum_2' ,

from which Silver's result follows by the theorem of Rasiowa and Sikorski and the theorem of Gaifman and Robinson [10] that

T 6506 $ZF + MC \vdash \lambda x (\omega_1 \text{ is inaccessible in } L[x]).$

[88]

Proof of T6505.

Let P be \sum_2^1 , so that for some $a \in \omega$,

$$P = \{X \subseteq \omega \mid Q(a, X)\}$$

where $Q \in \Sigma_2^1$. Now reason in $L[a]$: consider the sentence in the language L^B (for making P -generic extensions of $L[a]$)

$$\dot{Q}(\dot{a}, \dot{x}).$$

By T6429, which is true in $L[a]$ as AC is true there, there is an $S \in L[a]$ s.t.

$$\text{either } \langle 0, S \rangle \Vdash \dot{Q}(\dot{a}, \dot{x})$$

$$\text{or } \langle 0, S \rangle \Vdash \neg \dot{Q}(\dot{a}, \dot{x}).$$

By hypothesis there is an x P -generic over $L[a]$:

Therefore by T6431

there is an $L[a]$ -generic filter F containing $\langle 0, S \rangle$.

Let this $F \xrightarrow[M]{} x$.

Then I assert that $\forall y \subseteq x (\forall y \in P \rightarrow x \in P)$:
for $x \in P \rightarrow Q(a, x)$

\rightarrow in $L[a][x]$ $Q(a, x)$, by

Shoenfield's theorem T3107, as $Q \in \Sigma_2^1$

$\rightarrow \bigvee \langle s', s' \rangle \in F \quad \langle s', s' \rangle \Vdash Q(a, x)$

$\rightarrow \langle 0, S \rangle \Vdash Q(a, x)$; but by T6017

for every $y \subseteq x$, y is P -generic, and $\langle 0, S \rangle \in F_y$,

(89)

(6507)

where $F_y \xrightarrow{L[a]} y$, and so in

$$L[a][y], Q(a, y)$$

and so in V , $Q(a, y)$ ($\text{as } Q \in \Sigma_2^1$)

and then $y \in P$.

Conversely, if $x \notin P$, then $(0, 3) \Vdash \neg Q(a, x)$,
and so no $y \in x$ is in P .

P is therefore not an SF.

Q.E.D.

Before proving T 6013, I discuss the proof of Mostowski's theorem T 3100, to show that it can be refined to

T 6507 There is an $n < \omega$ such that $ZF \vdash$ if
 M is a transitive model of ZF_n , a_1, \dots, a_k reals in
 M and $\Omega(x_1, \dots, x_k)$ a Π_1^1 predicate, then
 $\Omega^M(a_1, \dots, a_k) \longleftrightarrow \Omega(a_1, \dots, a_k)$.

The steps in the proof of Mostowski's theorem may be given as follows:

D 6508 Let a be a real: define $f_a \in \omega^\omega$ by

$$f_a(n) = 0 \quad \text{if } n \notin a$$

$$f_a(n) = 1 \quad \text{if } n \in a.$$

(1) $ZF \vdash$ the function $\varphi: a \mapsto f_a$ exists.

(2) $ZF \vdash$ For any $\Pi_1^1(2^\omega)$ predicate
 $Q(a_1, \dots, a_k)$ there is a $\Pi_1^1(\omega^\omega)$ predicate $R_0(f_{a_1}, \dots, f_{a_k})$
such that for all a_1, \dots, a_k ,
 $Q(a_1, \dots, a_k) \longleftrightarrow R_0(f_{a_1}, \dots, f_{a_k})$

[90]

(Proof): replace ' $n \in a$ ' $\in Q$ by ' $f_a(n) = 1$ ';
replace quantifiers

$\forall x$ by $\forall f (\lambda_n(f(n) = 0 \text{ or } 1) \rightarrow \dots)$

$\exists x$ by $\exists f (\lambda_n(f(n) = 0 \text{ or } 1) \text{ and } \dots)$

etc.)

(3) (Kleene [5]) $ZF \vdash$ There is a recursive predicate T such that every $\Pi_1^1(\omega^\omega)$ set $A \subseteq \omega^\omega$ has the form

$$6509 \quad A = \{f \mid \lambda g \forall n T(\bar{f}(n), \bar{g}(n), \bar{h}(n))\}$$

for some $h \in \omega^\omega$.

Here $\bar{f}(n) = \prod_{i < n} p_i^{f(i)+1}$ where
 p_i is the i^{th} odd prime.

(4) For a set A in the form 6509, define for $f \in \omega^\omega$:

$$6510 \quad Q_{A,f} = \{\bar{g}(n) \mid n \in \omega \text{ and all } m \leq n, \neg T(\bar{f}(m), \bar{g}(m), \bar{h}(m))\}$$

define a relation $R_{A,f}$ on $Q_{A,f}$ by

$$6511 \quad m_1 R_{A,f} m_2 \longleftrightarrow \forall g \forall n_1, n_2 \text{ s.t.}$$

$$n_1 > n_2, m_1 = \bar{g}(n_1) \text{ and } m_2 = \bar{g}(n_2), \text{ and} \\ n_1 > n_2.$$

Then

$$(5) ZF \vdash \forall A \forall f (f \in A \longleftrightarrow \langle Q_{A,f}, R_{A,f} \rangle \text{ is a well-founded tree.})$$

[91]

[6512]

(Note that $Q_{A,f}$, $R_{A,f}$ are uniformly definable from f).

Choose n so large that $(n - s)$ and T 6423 are provable in ZF_n .

D 6512 Let $\circled{m_1}$ be the least such n .

As remarked during the proof of T 6017, if M satisfies ZF_{m_1} and $\langle Q, R \rangle \subseteq M$ well founded, then by T 6423, there is a function $f: Q \rightarrow \text{On}^M$ such that

$$\forall q, q' \in Q \quad qRq' \rightarrow f(q)Rf(q').$$

The ordinals in M are actual ordinals, and so $\langle Q, R \rangle$ is well founded in V . Conversely, if $\langle Q, R \rangle$ is well founded, there are no R-descending infinite paths; a fortiori there are no such paths in M , and therefore $\langle Q, R \rangle$ is well founded in M . That is, the notion of a well-founded tree is absolute w.r.t. transitive models.

As M is a model of enough of ZF to express any Π_1 -predicate in the form (5), T 6507 follows.

QED

I shall also, here and in §9, use the reflection principle established by Lévy [7]:

T 6513 (Lévy) Let $\Omega(u_1, \dots, u_k)$ be any ZF -formula.
Then

$$ZF \vdash \Lambda u_1, \dots, u_k \Lambda \beta \in \text{On} \vee \beta \in \text{On}$$

$\beta > \max \{d, p(u_1), \dots, p(u_k)\}$ and

$$\Omega(u_1, \dots, u_k) \longleftrightarrow \Omega^{V_\beta}(u_1, \dots, u_k).$$

(In particular, $u_i \in V_\beta$ for $i = 1, \dots, k$).

[92]

[6514]

ρ is the rank function (cf. D 6202).

I shall now prove

T 6013 (Silver) $ZF \vdash$ every Σ_1^1 set is CR.

By T 6324 it suffices to show that no Σ_1^1 set is an SF.

D 6514 Let n_2 be such that $n_2 \geq n_1, n_2 \geq n_0$,
(D 6428), $ZF_{n_2} + AC \vdash$ T 6429, and

$ZF_{n_2} + AC \vdash$ for every axiom α of
 $ZF_{n_2} + AC$, $[\alpha]^B = 1$, where B is
the algebra over P .

D 6515 Let b_r be the conjunction of the axioms of
 $ZF_{n_2} + AC$.

Let P be Σ_1^1 ; then for some $a \in \omega$,

$P = \{X \mid R(a, X)\}$ where $R \in \Sigma_1^1$.

Work in $L[a]$.

By the reflection principle, T 6513, there is an $\alpha \in On$
such that $\in L[a]$,

$$a \in V_\alpha \wedge b_r^{V_\alpha} \wedge [V = L[a]]^{V_\alpha}.$$

Apply the downward Löwenheim-Skolem theorem

[93]

To obtain a countable elementary submodel $\langle M, \in \cap M^2 \rangle$ of V_α such that $a \in M$. M need not be transitive, but the interpretation M is well-founded, and M satisfies the axiom of extensionality, so by Mostowski's Isomorphism Theorem, there is an isomorphism

$$\psi : \langle M, \in_M \rangle \xrightarrow{\sim} \langle N, \in \rangle$$

where N is transitive; ψ is given by

$$\psi(u) = \{ \psi(v) \mid v \in u \} \quad \text{for } u \in M.$$

(The existence of ψ is proved in a way similar to T6423.)

As $M \prec V_\alpha$, and each $n \in \omega$ is definable, say by $\phi_n(x)$, $\omega \subseteq M$ and is in M , $\psi_n(n)$.

By recursion on n , $\psi(n) = n$ for all $n \in \omega$.

$$\text{So } \psi(a) = \{ n \mid n \in a \text{ in } M \}$$

$$= \{ n \mid n \in a \} \quad (\text{as } \omega \subseteq M)$$

$$= a.$$

6516 Therefore N is a countable transitive model of $\text{AC} + \text{ZF}_{\aleph_2}$ and $a \in N$.

(The above argument is well-known; cf. Levy [7a].)

Now consider P -generic extension of N : consider in particular the L^{B^N} -sentence

$$\dot{R}(\dot{a}, \dot{x}^N)$$

There is by T6329, which holds in N by choice of n_2 ,

[94]

an $\langle o, s \rangle \in |P^N|$ s.t.

$$\langle o, s \rangle \Vdash \dot{R}(\dot{a}, \dot{x}^N) \text{ or } \langle o, s \rangle \Vdash \neg \dot{R}(\dot{a}, \dot{x}^N).$$

Suppose the first: let F be an N -generic filter such that $\langle o, s \rangle \in F$. Such a F exists as N is countable. Let $F \xrightarrow[N]{} x$. I assert that

$$2_\infty^\chi \in P.$$

For let $y \leq x$: then by T 6427, y is P -generic over N , and as $o \in y \in x \in s$,

$$\langle o, s \rangle \in F_y \text{ where } F_y \xrightarrow[N]{} y.$$

Therefore in $N(y)$, $R(o, y)$.

But R is Σ_1^1 , so $\neg R(o, y)$ is a Π_1^1 predicate of a and y . By choice of n_2 , $N(y)$ is a model of $ZF_{(n_2)}$, and so as

$\neg R(o, y)$ is false in $N(y)$,

$\neg R(o, y)$ is false in the universe, by T 6507 that is, $R(o, y)$ holds, and so $y \in P$.

I) the second, then a similar argument shows that

$$2_\infty^\chi \cap P = \emptyset.$$

QED.

The next results improve T 6011. (The proof of T 6517 uses Cohen's method for T 6011).

[95]

[6517]

Let $a \in \omega$, and let $R(n, a, y) \in \Sigma_1^1$.

Let $\mathcal{X} = \{x \mid \forall y \leq x \quad xy \text{ infinite} \rightarrow \neg \exists n (n \in x \leftrightarrow R(n, a, y))\}$.

T6517 ZF + DC $\vdash \mathcal{X}$ is CR⁺.

Proof Let $\langle s, S \rangle$ be given. I shall show that there is an $S' \subseteq S$ s.t. $2_{\omega}^{(s, S')} \subseteq \mathcal{X}$. First note that for all $n \in \omega$, the family

[**] $P_n = \{y \mid R(n, a, y)\}$ is Σ_1^1 , and so CR.

Let $t_0 = \bar{s}$.

Let $n_0^{(0)}, \dots, n_0^{(2^{k_0})}$ be the first $2^{2^{k_0}} + 1$ elements of S . By repeated application of [**] it may be shown that there is a $T_0 \subseteq S$ such that

(i) for all $t \in s$, and all $i \leq 2^{2^{k_0}}$, P_i is trivial on 2^t , and (ii) $n_0^{(2^{k_0})} < \min T_0$.

There are 2^k $t \in s$, so there are $n, n' \in \{n_0^{(0)}, \dots, n_0^{(2^{k_0})}\}$ ($n < n'$) such that

$\forall t \in s \quad R(n, a, t \cup T_0) \leftrightarrow R(n', a, t \cup T_0)$.

Define $t_0 = \sup\{n\}$; it is also convenient to define a partial function $\phi : \omega \rightarrow \omega$ during the construction: set now $\phi(n) = n'$.

Repeat the above process, starting from $\langle t_0, T_0 \rangle$.

A sequence $\langle s, S \rangle \geq \langle t_0, T_0 \rangle \geq \langle t_1, T_1 \rangle \geq \dots$

is obtained in which

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t_{k+1} has precisely one more element than t_k .
 If $\{n\} = t_{k+1} \setminus t_k$, then for each $t \in t_k$,
 and all $y \in 2^{< t, T_{k+1}}$,

$$R(n, a, y) \longleftrightarrow R(\phi(n), a, y)$$

$$\text{Let } S' = \bigcup \{t_i \setminus s \mid i < \omega\}.$$

The $S' \subseteq S$; I assert that S' has the required property.

Let $z \in 2^{< s, S'}$, and $y \in z$: suppose that

$$\lambda n (n \in z \longleftrightarrow R(n, a, y)).$$

Then $\lambda n (n \in z \setminus s \rightarrow n \notin y)$: for let $n \in z \setminus s$; then for some k , $n = t_{k+1} \setminus t_k$: if $n \notin y$, then $y \in 2^{< t, T_{k+1}}$, where $t = n \cap t_k$, and so by choice of T_{k+1} ,

$$R(n, a, y) \longleftrightarrow R(\phi(n), a, y);$$

but as $n \in z$, $R(n, a, y)$; and so $R(\phi(n), a, y)$, and therefore $\phi(n) \in z$, contradicting the fact that $\phi(n) \notin S'$, and a portion not in z . So $n \in y$.

So $z \setminus y$ is finite.

Q.E.D.

REMARK. T6517 is true too for R $\Pi^1_{1, \alpha}$ every Π^1_α set is CR, (the complement of a Π^1_α set being Σ^1_α).

[97].

[6518]

As the set of Σ'_1 and Π'_1 predicates is countable, by T6517 and T6322

T6518 $ZF + DC \vdash$ Let $a \in \omega$. Then the set $\{x \mid \text{for no } y \leq x \text{ with } x \sim y \text{ infinite and for no } \Sigma'_1 \text{ or } \Pi'_1 R \text{ is it true that } \lambda n(n < x \leftrightarrow R(n, y))\}$ is CR.

REMARKS

6519 The use of DC in T6517 and T6518 may be eliminated: given a and $\langle s, S \rangle$, work first in $L[a][S]$, where AC holds, and obtain on S' of the required sort; then by Shoenfield's theorem, $\langle s, S' \rangle$ will have the same properties in V .

These arguments show more generally that

T6520 $ZF + DC \vdash$ Let $n < \omega$. If there are no \sum'_{n+1} SFs then

$\{x \mid \text{for no } y \leq x \text{ with } x \sim y \text{ infinite is } x \in \sum'_n \text{ or } \Pi'_n \text{ in } y\}$ is CR.

For Δ'_{n+1} only a weaker result holds: the problem being that if $x \in \Delta'_{n+1} \cap y$, then, by definition, there are Σ'_n and Π'_n predicates, Q and R, such that

$\lambda n(n < x \leftrightarrow Q(n, y))$ and $\lambda n(n < x \leftrightarrow R(n, y))$;

[98]

in particular,

$$\lambda n(Q(n,y) \leftrightarrow R(n,y)).$$

But it does not follow that for another $y' \neq y$,

$$\lambda n(Q(n,y') \leftrightarrow R(n,y')).$$

Call therefore x guaranteed Δ'_n in y if there are Q, R such that

$$\lambda n \forall y' (Q(n,y') \leftrightarrow R(n,y'))$$

$$\text{and } \lambda n (n \in x \leftrightarrow Q(n,y)).$$

Then the result for Δ'_n reads

T6521 ZF + DCT Let $n < \omega$. If there are no Δ'_n SFs, then

$\{x \mid \text{for no } y \subseteq x \text{ with } x \setminus y \text{ infinite is } x \text{ guaranteed } \Delta'_n \text{ in } y\}$ is c.t.

In TT 6520 and 1, as in T6517, a fixed parameter a may be admitted.

T6522 ZF + DCT $\boxed{\forall a \in S(\omega) \lambda y \subseteq x \lambda z \subseteq y (\text{if there is an } \Sigma_1^0 \text{ or } \Pi_1^0 \text{ predicate } R(\cdot, \cdot, \cdot) \text{ such that } \lambda n (n \in y \leftrightarrow R(n, a, z)) \text{ then } y \setminus z \text{ is finite})} = \mathbb{B}$,
 (where \mathbb{B} is the algebra over \mathbb{P}).

T6522 has a model-theoretic corollary:

[6521]

[99]

[6523]

T6523 $ZF \vdash$ Let M be a transitive model of $ZF + DC$, and let x be \mathbb{P} -generic over M . Then in $M[x]$ the following is true :

$\forall a \in M \forall y \subseteq x \forall z \in y (\text{if } a \in M \text{ and there is a } \Sigma_1^1$
 $\text{or } \Pi_1^1 \text{ relation } R(\cdot, \cdot, \cdot) \text{ such that } \forall n (n \in y \leftrightarrow R(n, a, z))$
 $\text{then } y - z \text{ is finite}).$

REMARK. If in addition, M contains all the countable ordinals, then the above is true in V : for it is equivalent to the set of sentences of the form

$\forall y \subseteq x \forall z \in y (\forall n (n \in y \leftrightarrow R(n, a, z)) \rightarrow y - z \text{ is finite})$

(for $a \in M$, $R \Sigma_1^1$ or Π_1^1) and each of these is a Π_2^1 predicate of the a that occurs in it.

Proof of T6522

Let $a \in \omega$, $\langle s, S \rangle \in |\mathbb{P}|$, and $R \Sigma_1^1$ or Π_1^1 . Then

6524 $\mathcal{X} = \{w \subseteq \omega \mid \forall r \subseteq w (w \setminus r \text{ infinite} \rightarrow \neg \forall n (n \in w \leftrightarrow R(n, a, r)))\}$

is c.t., and therefore there is an $S' \subseteq S$ such that $\bigcup_{s \in S'} S' \subseteq \mathcal{X}$.

That says that

$\forall w \subseteq \bigcup_{s \in S'} S' \forall r \subseteq w (w \setminus r \text{ infinite} \rightarrow \neg \forall n (n \in w \leftrightarrow R(n, a, r)))$

which is seen by Stoerfeld's idea, and the remark on page 86

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[6525]

to be a Π_2^1 predicate of $s \in S'$ and a . Applying Shoenfield's theorem in \dot{V}^B , as $\langle s, S' \rangle \Vdash \dot{x} \leq_s \dot{S}'$,

$$\begin{aligned}\langle s, S' \rangle \Vdash \lambda y \in \dot{x} \lambda z \in y (y \text{ is infinite} \rightarrow \\ \rightarrow \lambda n (n \in y \leftrightarrow R(n, \dot{a}, z)))\end{aligned}$$

$\langle s, S' \rangle \preceq \langle s, S \rangle$, and $\langle s, S \rangle$ was arbitrary: accordingly the above sentence has B -value \mathbb{I} for every a, R , and the theorem is proved.

I shall now prove T 6416 to prove

T 6525 (Jensen) $\text{ZF} + \text{AC} + 2^{\aleph_0} = \aleph_1 \vdash$ Let α be a cardinal. Then $[\dot{\alpha} \text{ is a cardinal}]^B = \mathbb{I}$.

Proof (note Jensen's)

(1) $|\dot{P}| = 2^{\aleph_0} = \aleph_1$; so as \dot{P} is dense in B ,

B satisfies the \aleph_2 chain condition. Therefore by T 6129, if $\alpha = \text{cf}(\alpha) \geq \aleph_2$, $[\dot{\alpha} \text{ is a cardinal}]^B = \mathbb{I}$. As all regular cardinals $\geq \aleph_2$ are preserved, so are all singular ones. It remains therefore only to show that

$[\dot{\aleph}_2 \text{ is a cardinal}]^B = \mathbb{I}$.

Now $\dot{\aleph}_2$ is a cardinal is an \dot{L}^B sentence whose sole constant is in \dot{V} ; by the homogeneity of B , proved in T 6430, and by T 6120, its B -value is 0 or \mathbb{I} . Suppose 0 . Then

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$$[\dot{\Sigma}_1^V \text{ is countable}]^B = 1,$$

so by the Maximum principle, there is an $\dot{f} \in V^B$
s.t. $[\dot{f} : \omega \leftrightarrow \dot{\Sigma}_1^V]^B = 1.$

For each $n < \omega$, the set

$$\Delta_n = \{ \langle s, S \rangle \mid \forall \beta < \dot{\Sigma}_1^V \langle s, S \rangle \Vdash \dot{f}(n) = \dot{\beta} \}$$

is dense and \leq -closed. By T 6416,

$\lambda n < \omega \lambda \langle s, S \rangle \in [P] \forall S' \subseteq S \langle s, S' \rangle \text{ captures } \Delta_n$,
and so for each n , the set

$$Q_n = \{ X \subseteq \omega \mid \langle 0, X \rangle \text{ captures } \Delta_n \}$$

is CR⁺: for given $\langle s, S \rangle$ w.h.t. $S' \subseteq S$ s.t. for
each $s' \leq s$, $\langle s', S' \rangle$ captures Δ_n .

Then $Q = \bigcap_{n < \omega} Q_n$ is also CR⁺, by T 6322,
and so there is an $S \subseteq \omega$ such that

$$\lambda n \langle 0, S \rangle \text{ captures } \Delta_n.$$

Now let $\mathcal{X} = \{ f < \dot{\Sigma}_1^V \mid \forall s \subseteq S \forall n \langle s, \dot{\Sigma}_s^V \rangle \Vdash \dot{f}(n) = \dot{\beta} \}.$

For each $s \subseteq S$,

$$\mathcal{X}_s = \{ \beta \mid \forall n \langle s, \dot{\Sigma}_s^V \rangle \Vdash \dot{f}(n) = \dot{\beta} \}$$

is countable and so $\mathcal{X} = \bigcup \mathcal{X}_s$ is countable, being
the union of countably many countable
sets, which by AC is countable.

I now assert that $\langle 0, S \rangle \Vdash \text{Range } (\dot{f}) \subseteq \dot{\Sigma}_s^V$.

[6526]

[6526]

For if $\langle s', S' \rangle \leq \langle 0, S \rangle$ and $\langle s', S' \rangle \Vdash f(\dot{\alpha}) = \dot{\beta}$,
 then there is a t in $s' \cup S'$ such that

for some β' , $\langle t, \frac{S}{\dot{\alpha}} \rangle \Vdash f(\dot{\alpha}) = \dot{\beta}'$

(as $\langle 0, S \rangle$ captures Δ_n); but as $\langle s', S' \rangle \leq \langle t, S \rangle$,
 and $0 \Vdash f$ is 1-1,

$\beta = \beta'$, and so $\beta \in \mathcal{X}$.

As \mathcal{X} is countable, ^{by AC} there is an $\alpha < \aleph_1$ such
 that

$\langle 0, S \rangle \Vdash \text{range of } (\dot{f}) \subseteq \dot{\omega}$

contradicting $0 \Vdash \dot{f} : \dot{\omega} \rightarrow \dot{\aleph}_1$. Q.E.D.

I turn now to the proof of T 6501.

I first prove

T 6526 $ZF + V = L \vdash$

$$[\forall y \in \dot{x} \forall z \in y (y \in_L z \rightarrow y - z \text{ finite})]^\mathbb{B} = 1.$$

Proof. Suppose

$$\langle s, S \rangle \Vdash \dot{y} \in \dot{x} \wedge \dot{z} \in \dot{y} \wedge \dot{y} \in_L \dot{z}.$$

There is an $\langle s', S' \rangle \leq \langle s, S \rangle$ and an ordinal α
 such that

$$\langle s', S' \rangle \Vdash \dot{y} = F_{\dot{\alpha}}(\dot{\lambda}) \wedge \dot{\lambda} \text{ is countable in } L[\dot{z}];$$

but by T 6525 $0 \Vdash \dot{s}_1^L[\dot{z}] = \dot{s}_2^L$,

so α is countable in L . ($= V$)

[103]

[6527]

Pick $\alpha, \beta (\in L)$ so that codes $\alpha+1$; let Q be the Σ_1^0 predicate in T 6504; there is then an $\langle s'', S'' \rangle \leq \langle s', S' \rangle$ such that

6527 $\langle s'', S'' \rangle \Vdash \text{An}(n \dot{e} i \dot{\in} Q(n, \dot{\alpha}, \dot{\beta}))$

But by T 6522,

$\langle s'', S'' \rangle \Vdash \dot{y} \dot{\sim} \dot{z}$ is finite.

I have proved that every condition $\langle s, S \rangle$ which forces

$$y \leq x \quad n \dot{e} y \dot{\in} \dot{z} \quad n \dot{e} y \dot{\leq} \dot{z}$$

can be refined to one which forces

$$y \dot{\sim} \dot{z} \text{ is finite};$$

and therefore for every $y, z \in V^B$

$$\text{Off}(y \leq x \wedge z \dot{\in} y \wedge y \dot{\leq} z) \rightarrow (y \dot{\sim} z \text{ is finite});$$

T 6526 is therefore proved.

Let now x be \mathbb{P} -generic over L . Then by T 6526 the following holds in $L[x]$:

$$[\ast\ast] \quad \begin{aligned} \lambda y \lambda z (y \leq x \wedge z \dot{\in} y \wedge y \dot{\sim} z \text{ infinite}) \\ \rightarrow y \dot{\sim} z \leq_L z. \end{aligned}$$

Now ' $y \leq_L z$ ' is Σ_2^0 , by T 3114 (page 86) and so ' y is not $\leq_L z$ ' is Π_2^0 . Thus the sentence $[\ast\ast]$ is a Π_2^0 predicate of x , and therefore by Shoenfield's theorem, holds in V .

[04]

[6528]

Suppose now that $y \subseteq x$ and $z \subseteq x$ and $y \leq_L z$:

then $y \cup z \subseteq x$ and $y \cup z \leq_L z$, so

$(y \cup z) \setminus z$ is finite; and therefore $y \setminus z$ is finite.

To see that x is not of minimal L-degree, pick

$x_1 \subseteq x$, $x_2 \subseteq x$ s.t. $x_1 \cap x_2 = \emptyset$ and $x = x_1 \cup x_2$;

then if x is of minimal L-degree,

$x_1 \in L$ ($\because x_1 <_L x$)

$x_2 \in L$ ($\because x_2 <_L x$);

and so $x_1 \cup x_2 \in L$; $\therefore x \in L$. \star

Theorem 6501 is now proved.

These arguments indeed establish the following:

T 6528 ZF+ Let M be a transitive model of ZF+
 $V=L$ and let x be \mathbb{P} -generic over M . Then in
 $M[x]$, x is not of minimal L-degree; further in
 $M[x]$, $\forall y \subseteq x \forall z \subseteq y (y \leq_L z \rightarrow y \setminus z \text{ finite})$. If
 M contains all countable ordinals, then Silver's
theorem shows that in V ,

$\forall y \subseteq x \forall z \subseteq y (y \leq_L z \rightarrow y \setminus z \text{ finite})$,

and so $\forall y \subseteq x \forall z \subseteq x (y \leq_L z \rightarrow y \setminus z \text{ finite})$.