\[ \text{4. Proof of T6407.} \]

In this and the next paragraph, the properties of the partial ordering \( \mathcal{P} \) of D6303, considered now as a notion of forcing, will be investigated. The convention on variables of \( \mathcal{P} \) is maintained.

D6400 Let \( \mathcal{B} \) be the algebra over \( \mathcal{P} \).

D6401 Let \( \dot{x} \) be the element of \( \mathcal{V}^\mathcal{B} \) defined by

\[
\text{dom}(\dot{x}) = \{ n \in \omega \} \backslash \{ \dot{\omega} \}, \\
\dot{x}(\dot{n}) = \sum \mathcal{B} \{ O^\mathcal{B} \mid n \epsilon s \land \langle s, s \rangle \leq \langle \mathcal{P} \rangle \}.
\]

As before, for \( \langle s, s \rangle \leq \langle \mathcal{P} \rangle \), \( \omega^s \in \mathcal{L}^\mathcal{B} \),

D6402 \( \langle s, s \rangle \models \omega^s \iff O^\mathcal{B} \leq [\omega^s]^\mathcal{B} \).

The next group of theorems, till T6413, establish elementary properties of \( \mathcal{B} \) and \( \dot{x} \).

T6403 \( \models [\dot{x} \leq \omega] = 1 \).

Proof For \( \dot{y} \in \text{dom}(\dot{x}) \), \( \models [\dot{y} \in \omega] = 1 \), so

\[
\models [\dot{y} \in \dot{x}] = \sum \mathcal{B} \{ \dot{y} \in \dot{x} \} \cdot \dot{x}(\dot{y})
\]

\[
= \sum \mathcal{B} \{ \dot{y} \in \dot{x} \} \cdot \dot{x}(\dot{y})
\]

\[
\leq \sum \mathcal{B} \{ \dot{y} \in \dot{x} \} \cdot \dot{y} \cdot \dot{x}(\dot{y}) \text{ by the identity axiom.}
\]

\[
\leq [\dot{y} \in \dot{\omega}]^\mathcal{B}.
\]
\[ [y \in x \rightarrow y \in \omega]^B = 1 \text{ for } y \in y, \]
\[ [x \in \omega]^B = 1. \]

Q.E.D.

\[ T6404 \quad \vdash F \vdash [\forall \alpha \exists \beta] = \alpha(\beta). \]

\[ \text{Proof.} \quad [\forall \alpha \exists \beta] = \sum_{\alpha < \alpha} [\forall \alpha' \exists \beta']^{\alpha} \cdot \alpha(\beta') = \alpha(\beta) \]
\[ \alpha \vdash [\forall \alpha' \exists \beta']^{\alpha} = 0 \quad \text{ when } \alpha = n, \quad \exists \beta. \]

\[ T6405 \quad \vdash F \vdash \forall \langle s, S \rangle \forall n < \omega \left[ \right. \]
\[ \left. \begin{array}{l}
\quad (n \in s \iff \forall \langle t, T \rangle \exists \langle s, S \rangle (t < T) \text{ and } n \in \langle s, S \rangle) \\
\quad \text{and } (n \notin S) \iff \neg \exists \langle t, T \rangle \iff \langle s, S \rangle (n \in T) \right].
\]

\[ \text{Proof.} \]
\[ 1) \quad n \notin s, \text{ let } V T \subseteq S \text{ and } n \notin T.
\]
\[ \langle s, T \rangle < \langle s, S \rangle, \text{ and there is no }
\]
\[ \langle t', T' \rangle < \langle s, T \rangle \text{ s.t. } n \in T', \text{ so } n \notin S \cup T.
\]
\[ \text{If } n \in S \cup T, \text{ then set } T = \sup \{ n \},
\]
\[ T = \frac{S}{\{ n \}}; \text{ then } \langle t, T \rangle < \langle s, S \rangle. \quad \text{Q.E.D.} \]

\[ T6406 \quad \vdash F \vdash \forall n < \omega \forall \langle s, S \rangle: \]
\[ (i) \quad \alpha(n) = \bigcup \{ O^{\langle s, S \rangle} \mid n \in s \}
\]
\[ (ii) \quad \langle s, S \rangle \vdash \forall n \exists \alpha \iff n \in s
\]
\[ (iii) \quad \langle s, S \rangle \vdash \forall n \notin \alpha \iff n \notin S \cup S
\]
\[ (iv) \quad \langle s, S \rangle \vdash \forall \gamma \in \alpha \subseteq S \cup S.
\]

\[ \text{Proof.} \quad \text{Immediate from the definition and } T \vdash T6404, S. \]
Let $M$ be a transitive model of $\mathsf{ZF}$. I shall also consider these concepts continued in $M$. Write $D_{6407}$ \[ P^M = \langle |P^M|, \lesssim^M \rangle \] for the set of $M$ satisfying $\subseteq M$. \[ D_{6302} \] \[ D_{6302} \]

Thus, $|P^M| = |P| \cap M$; $\lesssim^M$ is the restriction of $\lesssim$ to $|P^M|$, and it is the case in $M$ that $P^M$ is a partial ordering without minimal elements and the maximum element $\langle 0, w \rangle$.

$D_{6408}$ Write $B^M$ for the notion of $M$ which in $M$ satisfies $D_{6400}$. That is, in $M$ $B^M$ is the algebra on $P^M$. Write $M^B$ for the Boolean-valued universe constructed in $M$ w.r.t. the algebra $B^M$.

$D_{6409}$ Write $x^M$ for the element of $M^B$ satisfying $\subseteq M$. \[ D_{6401} \]

$D_{6410}$ \[ F \stackrel{M}{\rightarrow} x \iff y \] \[ F \in M \] \[ \text{is a } M \text{-generic filter} \]
on $P^M$, $\widehat{F}$ the $M$-complete collapsing on $B^M$ it generates, \[ \phi^F : M^B \rightarrow M[F] \] the "collapsing" mapping of $D_{6207}$, and \[ x = \phi^F (x^M). \]

$D_{6411}$ \[ x \text{ is } M \text{-generic over } M \iff V F (F \stackrel{M}{\rightarrow} x). \]
Theorem 6412. \( \text{ZF} \)\textsuperscript{+} Let \( F \) be an \( M \)-generic filter, and \( x \in w. \) The following are equivalent:

(a) \( F \xrightarrow{M} x \)
(b) \( x = \{ \bar{y} \mid \exists y \in x^M \in F \} \)
(c) \( x = \bigcup \{ s \mid \forall s \in M \land \langle s, s \rangle \in F \} \)
(d) \( F = \{ \langle s, s \rangle \in \mathcal{P}(M) \mid s \subseteq x \subseteq s \cup s \} \).

Proof. The equivalence of (a), (b) and (c) is immediate from the definition. Note that \( s \subseteq x \subseteq s \cup s \) \( \implies s \in x. \)

Suppose (c) (and therefore (a)) holds:

Then by (a), \( \langle s, s \rangle \in F \implies s \subseteq x \subseteq s \cup s. \)

and (T 6406(iv))

Conversely, if \( \langle s, s \rangle \in M \) and \( s \subseteq x \subseteq s \cup s, \)

then \( \langle s, s \rangle \) is compatible with every element of \( F: \) for let \( \langle t, t \rangle \in F; \) then \( t \subseteq x \subseteq t \cup t, \)
so \( t \subseteq x \) and \( s \subseteq x, \) and therefore \( s \cup t = t \lor s, \)
\( t \subseteq s \subseteq s \cup t \subseteq t. \)

Then \( \langle s \cup t, S \cup t \rangle \leq^M \langle s, s \rangle \)
and \( \langle s \cup t, S \cup t \rangle \leq^M \langle t, t \rangle. \)

Now the set \( \{ \langle t, t \rangle \mid \langle t, t \rangle \leq^M \langle s, s \rangle \lor \langle t, t \rangle \} \)

is incompatible with \( \langle s, s \rangle (\leq^M) \)

in \( M, \) and is dense and \( \leq \)-closed.

\( \therefore \forall \langle t, t \rangle \in F \) s.t. \( \langle t, t \rangle \leq^M \langle s, s \rangle \) (or it has been already shown that \( \neg \langle t, t \rangle \in F \) is incompatible with \( \langle s, s \rangle \); hence \( \langle s, s \rangle \in F \) by D 6217 (ii); so (d) holds.

If (d) holds, let \( F \xrightarrow{M} y \); then for every \( z \subseteq y, \) \( t \subseteq z \) \( \implies s \subseteq z \) \( \implies s \subseteq x. \) \( \Box \).
T 6412 (c) and (d) together show that

\[ T_{6413} \quad \text{If } F \xrightarrow{M} x \text{ then } M[F] = M[x]. \]

T 6017 is now restated:

\[ T_{6017} \quad \text{ZF} \vdash \text{Let } M \text{ be a transitive model of } ZF + DC, \]
\[ x \text{ IP-generic over } M \text{ and } y \subseteq x. \text{ Then } y \]
\[ \text{is also IP-generic over } M. \]

**REMARKS**

1. It is not asserted that for a general
   such M any such x exist.

2. It is not assumed that \( y \in M[x] \).

The proof is in three steps: I shall give the
first two as theorems in ZF + DC about TP. They will
then hold in M about TP.

\[ D_{6414} \quad D = \{ \Delta \in \text{TP} | \Delta \text{ is dense and } \leq \text{-closed} \}. \]

\[ D_{6415} \quad \text{Let } \Delta \in D, \langle s, S \rangle \in \text{TP}. \]
\[ \langle s, S \rangle \text{ captures } \Delta \iff \forall T \subseteq S \forall t \in T \text{ such that } \langle s \cup t, S \rangle \in \Delta. \]

\[ T_{6416} \quad \text{ZF + DC} \vdash \bigwedge \Delta \in D \forall \langle s, S \rangle \forall S' \subseteq S \langle s, S \rangle \text{ captures } \Delta. \]

**Proof**

Let \( \Delta \) be given.

1. Define \( P_\Delta = \{ X \subseteq \omega | \forall S \in X \langle s, X \rangle \in \Delta \}. \)

Then \( P_\Delta \) is CR+.
[67]

For by T 6316, either \( P \) is CR or for some \( \langle s, S \rangle \), \( P \) is CSF on \( 2^\langle s, S \rangle \). Let \( \langle s, S \rangle \in |P| \). As \( \Delta \) is dense, there is an \( \langle s', S' \rangle \leq \langle s, S \rangle \) and that \( \langle s', S' \rangle \in \Delta \).

But then \( 2^\langle s', S' \rangle \leq P_\Delta \), i.e., \( X \in 2^\langle s', S' \rangle \). Then \( s' \in X \) and \( \langle s', X \rangle \leq \langle s', S' \rangle \) and so \( \langle s', X \rangle \in \Delta \) as \( \Delta \) is \( \leq \)-closed; and hence \( X \in P_\Delta \).

It follows that \( P \) is CSF on no \( 2^\langle s, S \rangle \), and is therefore CR. So given \( \langle s, S \rangle \in |P| \), there is an \( S' \subseteq S \) such that either \( 2^\langle s, S' \rangle \leq P_\Delta \) or \( 2^\langle s, S' \rangle \cap P_\Delta \): but by the argument above, the second alternative is impossible, so there is an \( \langle s'', S'' \rangle \leq \langle s, S' \rangle \), with \( \langle s'', S'' \rangle \in \Delta \); and then \( s'' \in S'' \in P_\Delta \).

(1) is proved. Let now \( \langle s, S \rangle \) be given.

By (1) there is an \( S_0 \subseteq S \) such that \( 2^\langle s, S_0 \rangle \leq P_\Delta \). Let \( \langle s_n \rangle_{n < \omega} \) be the wonderful enumeration (cf. D 6313). Define a sequence of sets \( \langle S_i \rangle_{i < \omega} \) as follows:

\[ S_0 \text{ is chosen s.t. } s < S_0 \text{ and } 2^\langle s, S_0 \rangle \leq P_\Delta. \]

At the \( n \)-th stage,

if \( s_n \neq S_n \), set \( S_{n+1} = S_n \); if \( s_n \subseteq S_n \) and there is an \( X \subseteq S_n \) such that \( \langle s_n, X \rangle \in \Delta \), then pick such an \( X \) and set \( S_{n+1} = \frac{s_n \cup X}{S_n} \);

if there is no such \( X \), set \( S_{n+1} = S_n \).
Then $S_0 \supset S_1 \supset S_2 \supset \ldots$, and for all $n$, $S < \subseteq S_n$.

Let $S' = \bigcap_{n<\omega} S_n$. Then $S < \subseteq S'$

Exactly as in the proof of T 6314, it is seen that $S'$ is infinite.

I assert that $S'$ has the required properties. For let $T \subseteq S'$: then $s < \subseteq T$ and $suT \in Z<s, S_0>$ and no $suT \in P_{\Delta}$. There is therefore a $t' \in suT$ such that $\langle t', \frac{T}{t'} \rangle \in \Delta$. Now set $t'' = s u t'$: (note that either $s \not\in t'$ or $t' \in s$; then $s \not\in t''$, and $\langle t'', \frac{T}{t''} \rangle \preceq \langle t', \frac{T}{t'} \rangle \in \Delta$, and so $\langle t'', \frac{T}{t''} \rangle \in \Delta$, as $\Delta$ is $\leq$-closed. Write $t'' = s u t$ where $s < \subseteq t$. Then $t = s_m$ some $m$, and so $\langle su t, \frac{T}{s u t} \rangle \in \Delta$, $S_{s_m + 1}$, was of the form $S_m \cup X$, where $\langle su t, X \rangle \in \Delta$; but then for $S' \subseteq X$, $\langle su t, \frac{S'}{s u t} \rangle \in \Delta$, as required.

( $\frac{S'}{s u t} = \frac{S'}{t} \supset s < \subseteq S'$, so $\langle su t, \frac{S'}{s u t} \rangle = \langle su t, \frac{S'}{t} \rangle$.)

As in T 6314 it is seen that DC is enough for the whole proof. Q.E.D.
\[69\]

Let \( \mathcal{A} = 2^\omega \).

\[6417\]

\[2F + DC \vdash \forall \Delta \in \mathcal{D} \forall S \exists \dot{\mathcal{A}}(x \in S \text{ and } i \in \dot{\mathcal{V}}, \\
\langle 0, S \rangle \text{ captures } \Delta \rangle^B = 1. \]

\( T6417 \) may be rephrased as

\[6418\]

\[2F + DC \vdash \forall \Delta \in \mathcal{D} \forall S \exists \dot{\mathcal{A}}(x \in S \land \\
\forall T \in \mathcal{A} \forall \beta < \omega \exists T (\langle \epsilon, S \rangle \in D)^B = 1, \]

and hence the contrapositive.

\[6419\]

\[2F \vdash \text{Let } M \text{ be a transitive model of } 2F + DC, \]

and \( x \) \( P \)-generic over \( M \). Then for any \( \Delta \in \mathcal{D}^M \),

there is an \( S \in M \) such that \( x \in S \) and in \( M \),

\( \langle 0, S \rangle \text{ captures } \Delta. \)

(Here \( D^M \) is in \( M \) and is the set that in \( M \)

satisfies \( D6414 \)).

Derivation of \( T6419 \) from \( T6417 \)

Let \( F \rightarrow x \). \( T6417 \) holds in \( M \) (replacing \( x \) by \( \dot{x}^M \)) and therefore as \( \phi_F (^\mathcal{V}) = M \) and \( \phi_F (\dot{x}^M) = x \), the following is true in \( M[x] \): for any \( \Delta \in \mathcal{D}^M \) there is an \( S \in M \) such that

\( x \subseteq S \) and in \( M \), \( \langle 0, S \rangle \text{ captures } \Delta. \)

As \( x \subseteq S \leftrightarrow x \subseteq S \text{ in } M[x] \), and the other

quantifiers are relativised to \( M \), the conclusion of \( T6419 \) follows.

Q.E.D.
Proof of T 6417.

It suffices to show that 

6420 if \( \Delta \in \mathcal{D} \) and \( \langle s, S \rangle \in |P| \) then there is an \( \langle s', S' \rangle \subseteq \langle s, S \rangle \) and an \( X \subseteq W \) such that 

\[
\langle s', S' \rangle \models \exists \bar{x} \land \forall T \subseteq \bar{x} (T \in \mathcal{A} \rightarrow \forall \bar{t} \in T (\langle \bar{t}, \bar{x} \rangle \in \Delta)),
\]

as for such an \( \langle s', S' \rangle \),

\[
\langle s', S' \rangle \models \forall \bar{x} \in \mathcal{A} (\exists \bar{x} \subseteq X \land \langle \bar{x}, \bar{y} \rangle \in \mathcal{A} \cap \Gamma (X \text{ captures } \Delta))
\]

and so \( \{ \langle s', S' \rangle \models \langle s, S \rangle \models \forall \bar{x} \in \mathcal{A} (\exists \bar{x} \subseteq X \land \langle \bar{x}, \bar{y} \rangle \in \mathcal{A} \cap \Gamma (X \text{ captures } \Delta)) \} \)

is dense in \( |P| \), which shows that for each \( \Delta \), that sentence has Boolean value \( 1 \).

Let then \( \Delta \in \mathcal{D} \), \( \langle s, S \rangle \in |P| \).

Enumerate the finite subseq \( s \) as \( t_0, \ldots, t_k \).

Define a sequence \( S_0, \ldots, S_k \) as follows:

pick \( S_0 \subseteq S \) such that \( \langle t_0, S_0 \rangle \text{ captures } \Delta \);

such an \( S_0 \) exist by T 6416.

Pick \( S_1 \subseteq S_0 \) such that \( \langle t_1, S_1 \rangle \text{ captures } \Delta \);

Pick \( S_k \subseteq S_{k-1} \) such that \( \langle t_k, S_k \rangle \text{ captures } \Delta \).

Remark that if \( \langle t, T \rangle \text{ captures } \Delta \) and \( T' \subseteq T \), then \( \langle t, T' \rangle \text{ captures } \Delta \). As \( S_k \subseteq S_i \) each \( i = 0, \ldots, k \),
\( \langle t_i, S_k \rangle \) captures \( \Delta \) for \( i = 0, 1, \ldots k \).

I assert that \( \langle 0, su S_k \rangle \) captures \( \Delta \). For let \( T \subseteq su S_k \), and set \( t = T \cap s \). Then \( t = t_i \) for some \( i \leq k \). Then \( T \subseteq 2^{\langle t_i, S_k \rangle} \), and so \( \langle t_i, S_k \rangle \) captures \( \Delta \), there is a \( t' \in T \setminus \frac{t_i}{t_i} \) such that \( \langle t_i \cup t', \frac{S_k}{t'} \rangle \in \Delta \).

But \( t_i \cup t' \in T \) and \( \frac{S_k}{t_i} = \frac{S_k}{t'} \), so \( \forall t'' \in T \langle t'', \frac{S_k}{t''} \rangle \in \Delta \), as required.

Set \( X = su S_k \), and \( \langle s', S_k \rangle = \langle s, S_k \rangle \).

Then \( \langle s', S_k \rangle \leq \langle s, S_k \rangle \)

and \( \langle s', S_k \rangle \vdash x \in \check{X} \) by T6406(N).

As \( \langle 0, X \rangle \) captures \( \Delta \),

\( \langle s', S_k \rangle \vdash \check{V}, \langle 0, X \rangle \) captures \( \Delta \).

QED

In the final step of the proof of T6017 I shall use a standard result, which I give as T6423.

D 6421 (ZF) A partial ordering \( \langle Q, R \rangle \) is a tree iff

(i) \( Q \) is countable and \( \forall q \in Q (\neg q Rq) \),

(ii) \( \forall q \in Q \{ q' \mid q Rq' \} \) is finite and

linearly ordered by \( R \).

REMARK None of the conventions about partial orderings assumed in
1 and 2 are intended to apply in D 64-21.

D 64-22 (ZF) A tree \( \langle Q, R \rangle \) is well-founded if
there is no function \( g : \omega \to Q \) such that
\[
\forall i < \omega \quad g(i+1) R g(i),
\]
that is, no infinite \( R \)-descending paths.

T 64-23 (ZF) A tree \( \langle Q, R \rangle \) is well-founded if
there is a function \( f : Q \to \text{On} \) such that
\[
\forall q, q' \in Q \quad q R q' \to f(q) < f(q').
\]

Proof. If there is an \( f : Q \to \text{On} \) such that
\[
\forall q, q' \in Q \quad q R q' \to f(q) < f(q'),
\]
then \( \langle Q, R \rangle \) is well-founded: for every \( g : \omega \to Q \)
to be such that \( \forall i < \omega \quad g(i+1) R g(i) \), then
\[
\forall i < \omega \quad f(g(i+1)) < f(g(i)),
\]
contradicting the well-ordering of the ordinals,
and \( Q \) not empty.

Let \( \Xi \) be the set of all functions \( \xi \) such
that
(a) \( \text{dom}(\xi) \subseteq Q \)
(b) \( \text{range}(\xi) \subseteq \text{On} \),
(c) \( q \in \text{dom}(\xi) \land q R q' \to q' \in \text{dom}(\xi) \).
(d) if \( \neg V q' q' R q \) and \( q \in \text{dom}(\xi) \),
then \( \xi(q) = 0 \).
(e) if \( V q' q' R q \) and \( q \in \text{dom}(\xi) \),
then \( \xi(q) = \sup \{ \xi(q') + 1 \mid q' R q \} \).

Then
[73]

(1) \( \forall q \in Q \rightarrow \forall q' \in Q \; q \land R \land q' \land q' \in Q \).

Suppose not. Let \( \psi : a \rightarrow Q \) be a well-ordering of \( Q \), which (such exist as \( Q \) is countable).

Define \( \gamma(0) = \text{the first element of } Q \text{ in the well-ordering } \psi \)

(\"the } \psi \text{-first\")

\( \gamma(i+1) = \text{the } \psi \text{-first } q \in Q \text{ such that } q \land R \land \gamma(i) \).

Then \( \forall i < \omega \; \gamma(i+1) \land R \land \gamma(i) \), contradicting

the well-foundedness of \( Q \).

(2) \( \exists i \in \omega \).

For let \( q \in Q \) s.t. \( \neg \forall q' \in Q \; q \land R \land q' \).

Define \( \delta \) by \( \delta(q) = 0 \),

\( \delta \) undefined otherwise.

Then \( \delta \in \omega \).

(3) If \( \delta, \delta' \in \omega \) and \( q \in \text{dom}(\delta) \cap \text{dom}(\delta') \)

then \( \delta(q) = \delta'(q) \).

For if \( \neg \forall q' \; q \land R \land q', \) then \( \delta(q) = \delta'(q) = 0; \)

\( \therefore \); \( \delta(q) \neq \delta'(q) \).

\( \forall q'(q' \land R \land q', \) and \( \delta(q') \neq \delta'(q') \),

by (2), (3).

But then pick the \( \psi \)-first such \( q' \); there is

a \( q'' \land R \land q' \land \delta(q'') \neq \delta'(q'') \) ---

clearly there is then a map \( \gamma : \omega \rightarrow Q \) s.t.

\( \forall i \; \gamma(i+1) \land R \land \gamma(i) \), again contradicting well-foundedness.
Define $f = \bigcup \{ S | S \in \Xi \}$. Then by (3), $f$ is a function,
$\text{dom}(f) \subseteq Q$, $\text{range}(f) \subseteq A$.
$f(q)$ is defined iff $\forall \xi \in \Xi$ s.t.
$\xi(q)$ is defined, and then $f(q) = \xi(q)$.

(4) $f$ is everywhere defined.
For if $q \in Q$ and $\{ q' | q'Rq \} \subseteq \text{dom}(f)$,
then define $\xi(q) = f(q')$ if $q'Rq$,
$\xi(q) = \sup \{ f(q') + 1 | q'Rq \}$.
Then $\xi \in \Xi$, and so $q \in \text{dom}(f)$.
So if $\text{dom}(f) \neq Q$,
define $g(0) = \text{the } f\text{-first } q \text{ not in } \text{dom}(f)$;
then $\forall q' q'Rq$ and $q' \notin \text{dom}(f)$;
let $g(1) = \text{the } f\text{-first such } q'$.

$g(2+1) = \text{the } f\text{-first } q \in \text{dom}(f)$
\[ s.t. \quad q \not\in \text{Rg}\{i\} \]
\[ \text{let } \xi : g(2+1) \rightarrow \text{Rg}\{i\} \]
\[ \text{So } \quad f : Q \rightarrow \text{dom } g \text{ and } qRq' \rightarrow f(q) < f(q') \text{. qed } \]

I shall now prove T6017.
Let $M$ be a transitive model of $\text{ZF+DC}$, $\mathcal{M}$ generic
over $M$, and $y \in M$.
Define $E_y = \{ (s, S) \in M \mid s \in y \subseteq y \subseteq S \}$. 
I assert that $F_y$ is an M-generic filter on $PM$. 

(1) Let $\langle s, S \rangle \in F_y$ and $\langle s', S' \rangle \in F_y$.

Then $s \in y \subseteq s \in s' \subseteq S$,

and $s \in y, s' \in y$ so that $s \in s'$ or $s' \in s$.

Suppose the first, without loss of generality. Then $s' \in y \subseteq s' \subseteq s'$.

So $y \subseteq s' \subseteq (S \cap S')$. Set $T = S \cap S'$.

Then $\langle s', T \rangle \leq \langle s', S' \rangle$, $\langle s', T \rangle \leq \langle s', S' \rangle \leq \langle s, S \rangle$,

and $s' \in y \subseteq s' \subseteq T$, so $\langle s', T \rangle \in F_y$.

(2) Let $\langle s, S \rangle \in F_y, \langle s, S \rangle \leq \langle s', S' \rangle$.

Then $s' \in s \subseteq s \in s' \subseteq s' \subseteq S'$, so $\langle s', S' \rangle \in F_y$.

(3) Let $\Delta \in DM$: that $\Delta$ is a dense and $\subseteq$-closed subset of $|PM|$. By T 6419 there is an $S^2 \in x$ such that $S \in M$ and in $M$, $S^2$ captures $\Delta$.

Let $Q^\Delta = \{ t \in S^2 | \langle t, S^2 \rangle \in \Delta \}$. $Q^\Delta \in M$.

If $Q^\Delta$ is empty, then $\langle 0, S^2 \rangle \in \Delta$; but $0 \in y \subseteq x \subseteq S^2 = 0 \cup S^2$, so then $\langle 0, S^2 \rangle \in F_y$, as required.

Suppose that $Q^\Delta$ is not empty. Define a relation
$[76]$ 

$R^A$ on $Q^A$ by

$t_1 R^A t_2 \iff t_2 \in t_1$ and $t_1 \not= t_2$. 

That is, $t_1 R^A t_2$ if $t_2$ is a proper end extension of $t_2$. $R^A \in M$. Set $T^A = \langle Q^A, R^A \rangle$.

$[6424]$ 

Prop. (A) $R^A$ is transitive, irreflexive, and for all $s \subseteq Q$,

$s R^A t_1 \& s R^A t_2 \rightarrow t_1 \in s$ and $t_2 \in s$

$\rightarrow t_2 \in t_1$ or $t_1 \in t_2$

$\rightarrow t, R^A t_1, t_1 = t_2$ or $t_2 R^A t_1$.

So as $s$ has only finitely many initial segments,

$\{ t \mid s R^A t \}$ is finite, and linearly ordered by $R^A$.

(B) $Q^A$ is countable, being a subset of the set of finite subsets of $w$.

(C) Suppose $\langle t_i \mid i < \omega \rangle \in M$ is a sequence such that $\forall i \exists t_{i+1} R^A t_i$ so that in particular $t_i \in Q^A$ each $i$, and $i < j \rightarrow t_i \not= t_j$.

Set $T = \bigcup_{i < \omega} t_i$.

Then $T \subseteq S$, and as $S$ captures $A$, there is a $t \in T$ such that $\langle t, \frac{S}{t} \rangle \in A$. But then for some $i$, $t \in t_i$; and $\langle t_i, \frac{S}{t_i} \rangle \neq \langle t, \frac{S}{t} \rangle$.

$\Rightarrow A$ is $\leq$-closed, $\langle t_i, \frac{S}{t_i} \rangle \in A$, and so $t_i \in Q^A$. 

$QED$
Now $M$ is a model of $2F$, and therefore by $T6423$, there is an $f: Q^A \rightarrow \text{On}^M$, $(\gamma \in M)$ such that
\[ \lambda q, q' \in Q^A \quad q'^A q' \rightarrow f(q) < f(q'). \]
Now $M$ is transitive, and so the ordinals of $M$ are an initial segment of the ordinals in $V$, so in $V$, $f$ is a function $f: Q^A \rightarrow \text{On}$, and
\[ \lambda q, q' \in Q^A \quad q'^A q' \rightarrow f(q) < f(q'). \]
Applying $T6423$ ($\in V$),

$T6425$ $T^A$ is a well founded tree.

$y \subseteq x \subseteq S^A$; so by $T6425$, there is an $s \subseteq y$ such that $\langle s, S^A \rangle \in \Delta$ (otherwise $\{s \cap s \subseteq y\}$ would form an infinite $RA$-descending path in $Q^A$, contradicting $T6425$.) But then $s \subseteq y \subseteq s$ $\langle s, S^A \rangle \in F_y \cap \Delta.$

I shall show therefore that $\forall \Delta \in DM^M \quad F_y \cap \Delta \neq 0.$

That, and point (1), (2) on page 75, show that $F_y$ is indeed an $M$-generic filter a $PM^M$. By $T6412$, $F_y \rightarrow y$, and so $y$ is $1P$-generic over $M$.

The proof of $T6017$ is complete.
D 6426 Let $2F_n$ denote the conjunction of the first
$n$ axioms of $ZF$ in some fixed recursive enumeration
such that the axiom of extensionality is first, the
axiom of foundations the second, and the
axiom of infinity is third.

In defining $⟨M, ≤, B, T_\alpha^M⟩$, the functions
$x ∈ y$, $x ⊆ y$ (which are not sets), and $D$, the
set of dense and $≤$-closed subsets of $1\|M$, certain axioms
of $ZF$ have been used to ensure that these are sets satisfying
the various definitions: only finitely many of these axioms
have been used, say a subset of the first $m_0$.

For $M$ a transitive model of $ZF_{m_0} + DC$,
"$x$ is $\Pi_1$-generic over $M$" may be defined as before:
that is, that there is an $F ≤ 1\|M$ such that
for all $Δ ∈ DM$, $F∩Δ ≠ 0$, and $F \rightarrow x$.

If $n ≥ m_0$ and $M$ is a transitive model of $ZF_n + DC$,
and $x$ is $\Pi_1$-generic over $M$, then $M[x]$ will
satisfy at least those axioms of $ZF$ for
which $[\mathcal{D}]$ can be defined in $M$ and forced to
equal $1$.

T 6427 $ZF ⊢$ There is an $n ≥ m_0$ such that if
$M$ is a transitive model of $ZF_n + DC$, $x$
$\Pi_1$-generic over $M$ and $y ≤ x$, then $y$ is also
$\Pi_1$-generic over $M$. 
\[7.9\]

Proof. By examining the proof of T6017, which used the finite \( n \in M \) of only a finite number of axioms of \( ZF \). That is, pick \( n \) so large that the definition of the following statement is meaningful, and such that the statement is a theorem of \( ZF+DC \), when written with all variables bound and defined terms replaced by their defining clauses written out in full:

\[ T6313 \text{ and its constellations, } T6315,16; T6321, T6412, T6415, T6418 \text{ of which } T6419 \text{ will then be a constancy; } T6423; \text{ and } T6424, (\text{with } \Delta \text{ a bound variable}). \]

\[ \text{QED} \]

\[64.28\]

Let \( \mathbb{N}_0 \) be the least such \( n \).

T6427 will not be used in the proof of T6001, but in applying the notion of a \( P \)-generic real to establish Silver's theorem T6013.

I conclude the paragraph with three further observations about \( P \): the first is due to Jensen, and will simplify some later proofs.

\[ T6429 \text{ (Jensen) } ZF+DC \vdash \text{ Let } \delta_1 \in \mathcal{L}_B \text{ and } \langle s, S \rangle \in \mathcal{P}(\mathcal{P}). \text{ (} B \text{ is the algebra on } P\text{). Then there is an } S' \subseteq S \text{ such that } \langle s, S' \rangle \models \delta_1 \text{ or } \langle s, S' \rangle \models \neg \delta_1. \]

Proof: (not Jensen's) The set \( \Delta = \{ \langle \xi, T \rangle | \forall \delta \cdot \langle \xi, T \rangle \models \delta \text{ or } \langle \xi, T \rangle \models \neg \delta \} \)
is dense and \( \leq \)-closed, by the basic property of
joining \( F 1 \), page 21. By \( T \ 6416 \) there is an \( S'' \in S \)
such that \( \langle S, S'' \rangle \) captures \( \Delta \). Consider the family
\[
P = \{ X \in \mathcal{P}_\omega \mid \forall t \in X \langle t, S'' \rangle \models \Box \Delta \}.
\]

Let \( \langle t, T \rangle \leq \langle S, S'' \rangle \); there is a \( t' \in T \) such
that \( \langle t \cup t', S'' \rangle \in \Delta \), so \( \langle S, S'' \rangle \) captures \( \Delta \).

Then \( \mathcal{P}_\omega \langle t \cup t', S'' \rangle \subseteq \mathcal{P} \) or \( \subseteq S(\omega) \setminus \mathcal{P} \); so
\( \mathcal{P} \) is not CSF on any \( \langle t, T \rangle \) with \( \langle t, T \rangle \leq \langle S, S'' \rangle \),
and therefore \( \mathcal{P} \) is CR on \( \mathcal{P}_\omega \langle S, S'' \rangle \), by \( T \ 6316 \) applied
to \( \mathcal{P}_\omega \langle S, S'' \rangle \) rather than \( \mathcal{P}_\omega \langle 0, \omega \rangle \). So there is an \( S' \subseteq S'' \)
such that either \( \mathcal{P}_\omega \langle S, S' \rangle \subseteq \mathcal{P} \) or \( \mathcal{P}_\omega \langle S, S' \rangle \cap \mathcal{P} = \emptyset \).

Suppose the first. Then for no \( \langle t, T \rangle \leq \langle S, S' \rangle \) does
\( \langle t, T \rangle \models \neg \Box \Delta \), as then
\( t \cup T \subseteq \mathcal{P} \cup S' \in S'' \). So therefore \( \langle S, S' \rangle \models \Box \Delta \).
If the second holds, \( \langle S, S' \rangle \models \neg \Box \Delta \), by a similar argument.

\( \Box \text{QED} \)

\( T \ 6430 \ \text{ZF} \) Let \( B \) be the algebra over \( \mathcal{P} \). Then
\( B \) is homogenous.

\textit{Proof:} By \( T \ 6126 \), it is enough to show that for
all \( p \in \mathcal{P}_l \), \( B \upharpoonright p \cong B \).
$\{O_q \mid q \leq p \}$ is dense in $B|0_p$, and so $B|0_p \cong$ the algebra over $\{\{q \mid q \leq p\}, \leq\}$.

but let $p = \langle s, S \rangle$, by the homeomorphism (Def. 6.30)

$$h: 2^{\langle s, S \rangle} \rightarrow 2^\omega.$$  

Then $h$ is an isomorphism with $\leq$, and so $B|0_p \cong B$.

Q.E.D.

Theorem 6.431. $ZF \vdash$ Let $M$ be a transitive model of $ZF + DC$. Suppose there is a real $\mathbb{P}$-generic over $M$. Then for all $\langle s, S \rangle \in |\mathbb{P}^M|$ there is an $M$-complete filter on $\mathbb{P}^M$ containing $\langle s, S \rangle$.

Proof: Let $x$ be $\mathbb{P}$-generic on $M$, and let $F_x \rightarrow x$,

and $\hat{F}_x$ the $M$-complete ultralower on $B$ generated by $F_x$.

Let $\langle s, S \rangle \in |\mathbb{P}^M|$; pick $b \in \hat{F}_x$, $b \neq 1$.

Set $b' = 0 \langle s, S \rangle$. By T.6430, there is an automorphism $\phi \in M$ with $\phi(b) = b'$. Let $\hat{F} = \{ \phi(c) \mid c \in \hat{F}_x \}$.

I assert that $\hat{F}$ is an $M$-complete ultralower on $B$:

(i) $\phi$ is an automorphism, $\hat{F}$ is an ultralower.

(ii) Let $A \in \hat{F}$; then $\{ \langle s, S \rangle \mid \phi(c) \in A \} \subseteq \hat{F}_x$ and is in $M$ as $\phi$ and $A$ are both in $M$, and so $\bigwedge \{ \langle s, S \rangle \mid \phi(c) \in A \} \in \hat{F}_x$; so $\bigwedge \{ \langle s, S \rangle \mid \phi(c) \in A \} \in \hat{F}$.

Therefore $\langle \langle s, S \rangle \mid O_{\langle s, S \rangle} \in \hat{F} \rangle$ is an $M$-generic filter on $\mathbb{P}^M$ containing $\langle s, S \rangle$.

Q.E.D.