¶ 4. Proof of T6017.

In this and the next paragraph the properties of the partial ordering $\mathbb{P}$ of D6303, considered now as a notion of forcing, will be investigated. The convention on variables of ¶ 3 is maintained.

D6400 Let $\mathbb{B}$ be the algebra over $\mathbb{P}$.

D6401 Let $\dot{x}$ be the element of $V^{\mathbb{B}}$ defined by

$$\mathrm{dom}(\dot{x}) = \{\check{n} \mid n < \omega\}$$

$$\dot{x}(\check{n}) = \sum^{\mathbb{B}} \{0^{\mathbb{B}}_{\langle s,S\rangle} \mid n \in s \wedge \langle s,S\rangle \in |\mathbb{P}|\}.$$

As before, for $\langle s,S\rangle \in |\mathbb{P}|$, $\dot{\Omega} \in \mathcal{L}^{\mathbb{B}}$,

D6402 $$\langle s,S\rangle \Vdash \dot{\Omega} \longleftrightarrow 0^{\mathbb{B}}_{\langle s,S\rangle} \leq [\![\dot{\Omega}]\!]^{\mathbb{B}}.$$

The next group of theorems, till T6413, establish elementary properties of $\mathbb{B}$ and $\dot{x}$.

T6403 $ZF \vdash [\![\dot{x} \subseteq \dot{\omega}]\!]^{\mathbb{B}} = \mathbb{1}$.

Proof For $\dot{\eta} \in \mathrm{dom}(\dot{x})$, $[\![\dot{\eta} \in \dot{\omega}]\!]^{\mathbb{B}} = \mathbb{1}$, so

$$[\![\dot{y} \in \dot{x}]\!]^{\mathbb{B}} = \sum_{\dot{\eta} \in \mathrm{dom}(\dot{x})}^{\mathbb{B}} [\![\dot{\eta} \equiv \dot{y}]\!]^{\mathbb{B}} \cdot \dot{x}(\dot{\eta})$$

$$= \sum_{\dot{\eta} \in \mathrm{dom}(\dot{x})}^{\mathbb{B}} [\![\dot{\eta} \in \dot{\omega}]\!]^{\mathbb{B}} \cdot [\![\dot{\eta} \equiv \dot{y}]\!]^{\mathbb{B}} \cdot x(\dot{\eta})$$

$$\leq \sum_{\dot{\eta} \in \mathrm{dom}(\dot{x})}^{\mathbb{B}} [\![\dot{\eta} \in \dot{\omega}]\!]^{\mathbb{B}} \cdot [\![\dot{\eta} \equiv \dot{y}]\!]^{\mathbb{B}} \cdot \dot{x}(\dot{\eta}) \quad \text{by the identity axioms.}$$

$$\leq [\![\dot{y} \in \dot{\omega}]\!]^{\mathbb{B}};$$

hence $[\![\dot{y} \in \dot{x} \longrightarrow \dot{y} \in \dot{\omega}]\!]^{\mathbb{B}} = \mathbb{1}$ for all $\dot{y}$,

so $[\![\dot{x} \subseteq \dot{\omega}]\!]^{\mathbb{B}} = \mathbb{1}$. QED.

T6404 $ZF \vdash [\![\check{n} \in \dot{x}]\!]^{\mathbb{B}} = \dot{x}(\check{n})$.

Proof $[\![\check{n} \in \dot{x}]\!]^{\mathbb{B}} = \sum_{m<\omega} [\![\check{n} \equiv \check{m}]\!]^{\mathbb{B}} \cdot \dot{x}(\check{m}) = \dot{x}(\check{n})$

as $[\![\check{n} \equiv \check{m}]\!]^{\mathbb{B}} = 0$ unless $n = m$.

QED.

T6405 $ZF \vdash \bigwedge \langle s, S \rangle \bigwedge n < \omega \Big[$

$(n \in s \longleftrightarrow \bigwedge \langle t, T \rangle \preccurlyeq \langle s, S \rangle \bigvee \langle t', T' \rangle \preccurlyeq \langle t, T \rangle \; n \in t')$

and $(n \notin (s \cup S) \longleftrightarrow \neg \bigvee \langle t, T \rangle \preccurlyeq \langle s, S \rangle \; (n \in t))\Big]$.

Proof If $n \notin s$, then $\bigvee T \subseteq S \; n \notin T$;

$\langle s, T \rangle \preccurlyeq \langle s, S \rangle$, and there is no

$\langle t', T' \rangle \preccurlyeq \langle s, T \rangle$ s.t. $n \in t'$, as $n \notin s \cup T$.

If $n \in s \cup S$, $n \notin s$, then set $t = s \cup \{n\}$,

$T = \frac{S}{\{n\}}$; then $\langle t, T \rangle \preccurlyeq \langle s, S \rangle$. QED

T6406 $ZF \vdash \bigwedge n < \omega \bigwedge \langle s, S \rangle$:

(i) $\dot{x}(\check{n}) = \bigcup \{ 0^{\mathbb{B}}_{\langle s, S \rangle} \mid n \in s \}$

(ii) $\langle s, S \rangle \Vdash \check{n} \in \dot{x} \longleftrightarrow n \in s$

(iii) $\langle s, S \rangle \Vdash \check{n} \notin \dot{x} \longleftrightarrow n \notin s \cup S$

(iv) $\langle s, S \rangle \Vdash \check{s} \subseteq \dot{x} \subseteq \check{s \cup S}$.

Proof. Immediate from the definitions and TT 6404, 5.

Let M be a transitive model of ZF. I shall also consider these concepts construed in M. Write

D 6407 $\mathbb{P}^M = \langle |\mathbb{P}^M|, \precsim^M \rangle$ for the set of M satisfying in M DD 6302, 3.

Thus $|\mathbb{P}^M| = |\mathbb{P}| \cap M$; $\precsim^M$ is the restriction of $\precsim$ to $|\mathbb{P}^M|$, and it is true in M that $\mathbb{P}^M$ is a partial ordering without minimal elements and the maximum element $\langle 0, \omega \rangle$.

D6408 Write $\mathbb{B}^M$ for the member of M which in M satisfies D 6400: that is, in M $\mathbb{B}^M$ is the algebra over $\mathbb{P}^M$. Write $M^{\mathbb{B}}$ for the Boolean-valued universe constructed in M w.r.t. the algebra $\mathbb{B}^M$.

D 6409 Write $\dot{x}^M$ for the element of $M^{\mathbb{B}}$ satisfying in M D 6401.

D 6410 $F \underset{M}{\longrightarrow} x \longleftrightarrow_{df}$ F is an M-generic filter on $\mathbb{P}^M$, $\tilde{F}$ the M-complete ultrafilter on $\mathbb{B}^M$ it generates, $\phi_{\tilde{F}} : M^{\mathbb{B}} \longrightarrow M[F]$ the "collapsing" mapping of D 6207, and $x = \phi_{\tilde{F}}(\dot{x}^M)$.

D 6411 x <u>is $\mathbb{P}$-generic over M</u> $\longleftrightarrow$ $\mathrm{VF}(F \underset{M}{\longrightarrow} x)$.

T6412 $ZF \vdash$ Let F be an M-generic filter, and $x \subseteq \omega$. The following are equivalent:

(a) $F \xrightarrow[M]{} x$

(b) $x = \{n \mid [\![\check{n} \in \dot{x}^M]\!] \in \tilde{F}\}$

(c) $x = \bigcup\{s \mid \bigvee S \in M\,(S \subseteq \omega \wedge \langle s, S\rangle \in F)\}$

(d) $F = \{\langle s, S\rangle \in |\mathbb{P}^M| \mid s \subseteq x \subseteq s \cup S\}$.

<u>Proof:</u> The equivalence of (a), (b) and (c) is immediate from the definitions. Note that $s \subseteq x \subseteq s \cup S \rightarrow s \underline{\text{in}}\, x$.

Suppose (c) (and therefore (a)) holds:

then by (a), (and T6406(iv)) $\langle s, S\rangle \in F \rightarrow s \subseteq x \subseteq s \cup S$.

Conversely, if $\langle s, S\rangle \in M$ and $s \subseteq x \subseteq s \cup S$, then $\langle s, S\rangle$ is compatible with every element of F: for let $\langle t, T\rangle \in F$; then $t \subseteq x \subseteq t \cup T$, so $t \underline{\text{in}}\, x$ and $s \underline{\text{in}}\, x$, and therefore $s \cup t = t$ or s, $t \setminus s \subseteq S$ and $s \setminus t \subseteq T$.

Then $\langle s \cup t, S \cap T\rangle \preccurlyeq^M \langle s, S\rangle$

and $\langle s \cup t, S \cap T\rangle \preccurlyeq^M \langle t, T\rangle$.

Now the set $\{\langle t, T\rangle \mid \langle t, T\rangle \preccurlyeq^M \langle s, S\rangle$ or $\langle t, T\rangle$ is incompatible with $\langle s, S\rangle$ (in M)$\}$

is in M, and is dense and $\leqslant$-closed.

$\therefore \bigvee \langle t, T\rangle \in F$ s.t. $\langle t, T\rangle \leqslant \langle s, S\rangle$ (as it has been already shown that no $\langle t, T\rangle \in F$ is incompatible with $\langle s, S\rangle$); hence $\langle s, S\rangle \in F$ by D6217 (ii): so (d) holds.

If (d) holds, let $F \xrightarrow[M]{} y$; then for every $t \underline{\text{in}}\, y$, $t \underline{\text{in}}\, x$; so $y = x$. QED

T 6412 (c) and (d) together show that

T6413 If $F \underset{M}{\longrightarrow} x$ then $M[F] = M[x]$.

T 6017 is now restated:

T6017 $ZF \vdash$ Let M be a transitive model of $ZF+DC$, x $\mathbb{P}$-generic over M and $y \subseteq x$. Then y is also $\mathbb{P}$-generic over M.

<u>REMARKS</u> (1) It is not asserted that for a general such M any such x exist.

(2) It is not assumed that $y \in M[x]$.

The proof is in three steps: I shall give the first two as theorems in $ZF+DC$ about $\mathbb{P}$. They will then hold in M about $\mathbb{P}^M$.

D 6414 $D =_{df} \{\Delta \subseteq |\mathbb{P}| \mid \Delta$ is dense and $\preccurlyeq$-closed$\}$.

D 6415 Let $\Delta \in D$, $\langle s, S\rangle \in |\mathbb{P}|$.

$\langle s, S\rangle$ <u>captures</u> $\Delta \longleftrightarrow_{df} \bigwedge T \subseteq S \bigvee t \underline{\text{in}} T$ such that $\langle s \cup t, \frac{S}{t}\rangle \in \Delta$.

T 6416 $ZF+DC \vdash \bigwedge \Delta \in D \bigwedge \langle s, S\rangle \bigvee S' \subseteq S$ $\langle s, S'\rangle$ captures Δ.

<u>Proof</u> Let Δ be given.

(1) Define $P_\Delta = \{X \subseteq \omega \mid \bigvee s \underline{\text{in}} X \ \langle s, \frac{X}{s}\rangle \in \Delta\}$.

Then P_Δ is CR^+.

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For by T 6316, either P_Δ is CR or for some $\langle s, S\rangle$, P is CSF on $2^{\langle s,S\rangle}$. Let $\langle s, S\rangle \in |\mathbb{P}|$. As Δ is dense, there is an $\langle s', S'\rangle \preceq \langle s, S\rangle$ such that $\langle s', S'\rangle \in \Delta$. But then $2_\infty^{\langle s', S'\rangle} \subseteq P_\Delta$, for if $X \in 2_\infty^{\langle s', S'\rangle}$, then s' in X and $\langle s', \frac{X}{s'}\rangle \preceq \langle s', S'\rangle$ and so $\langle s', \frac{X}{s'}\rangle \in \Delta$ as Δ is $\preceq$-closed; and $\therefore X \in P_\Delta$. It follows that P is CSF on no $2^{\langle s,S\rangle}$, and is therefore CR. So given $\langle s, S\rangle \in |\mathbb{P}|$ there is an $S' \subseteq S$ such that either $2_\infty^{\langle s, S'\rangle} \subseteq P_\Delta$ or $2_\infty^{\langle s, S'\rangle} \cap P_\Delta$: but by the argument above, the second alternative is impossible, as there is an $\langle s'', S''\rangle \preceq \langle s, S'\rangle$, with $\langle s'', S''\rangle \in \Delta$; and then $s'' \cup S'' \in P_\Delta$.

(1) is proved. Let now $\langle s, S\rangle$ be given.

By (1) there is an $S_0 \subseteq S$ such that $2_\infty^{\langle s, S_0\rangle} \subseteq P_\Delta$. Let $\langle s_n \mid n < \omega\rangle$ be the wonderful enumeration (of D 6313). Define a sequence of sets $\langle S_i \mid i < \omega\rangle$ as follows:

S_0 is chosen s.t. $s <_0 S_0$ and $2_\infty^{\langle s, S_0\rangle} \subseteq P_\Delta$.

at the nth stage,

if $s_n \not\subseteq S_n$, set $S_{n+1} = S_n$;

if $s_n \subseteq S_n$ and there is an $X \subseteq \frac{S_n}{s_n}$ such that $\langle s \cup s_n, X\rangle \in \Delta$, then pick such an X and set $S_{n+1} = \frac{s \cup s_n}{S_n} \cup X$;

if there is no such X, set $S_{n+1} = S_n$.

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Then $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$, and for all n, $s <_0 S_n$.

Let $S' = \bigcap_{n<\omega} S_n$. Then $s <_0 S'$

Exactly as in the proof of T 6314 it is seen that S' is infinite.

I assert that S' has the required properties. For let $T \subseteq S'$: then $s <_0 T$ and $s \cup T \in 2_\infty^{\langle s, S_0 \rangle}$ and so $s \cup T \in P_\Delta$. There is therefore a t' in $s \cup T$ such that $\langle t', \frac{T}{t'} \rangle \in \Delta$. Now set $t'' = s \cup t'$: (note that either s in t' or t' in s;) then s in t'', and $\langle t'', \frac{T}{t''} \rangle \preccurlyeq \langle t', \frac{T}{t'} \rangle \in \Delta$, and so $\langle t'', \frac{T}{t''} \rangle \in \Delta$, as Δ is $\preccurlyeq$-closed. Write $t'' = s \cup t$ where $s <_0 t$. Then $t = s_m$ some m, and so as $\langle s \cup t, \frac{T}{s \cup t} \rangle \in \Delta$, S_{m+1} was of the form $\frac{s \cup t}{S_m} \cup X$, where $\langle s \cup t, X \rangle \in \Delta$; but then as $\frac{S'}{s \cup t} \subseteq X$, $\langle s \cup t, \frac{S'}{s \cup t} \rangle \in \Delta$, as required.

$$\left(\frac{S'}{s \cup t} = \frac{S'}{t} \text{ as } s <_0 S', \text{ so } \langle s \cup t, \frac{S'}{s \cup t} \rangle = \langle s \cup t, \frac{S'}{t} \rangle . \right)$$

As in T 6314 it is seen that DC is enough for the whole proof. Q.E.D.

Let $A = 2^{\omega}_{\infty}$.

T 6417 $ZF + DC \vdash [\![\bigwedge \Delta \in \check{D} \bigvee S \in \check{A} (\dot{x} \subseteq S \text{ and in } \check{V}, \langle \dot{0}, S \rangle \text{ captures } \Delta)]\!]^{\mathbb{B}} = \mathbb{1}$.

T 6417 may be rephrased as

T 6418 $ZF + DC \vdash [\![\bigwedge \Delta \in \check{D} \bigvee S \in \check{A} (\dot{x} \subseteq S \wedge \bigwedge T \in \check{A} (T \subseteq S \rightarrow \bigvee t \underline{\text{in}}\, T (\langle t, \tfrac{S}{t} \rangle \in \Delta)))]\!]^{\mathbb{B}} = \mathbb{1}$,

and has the corollary

T6419 $ZF \vdash$ Let M be a transitive model of $ZF + DC$, and x $\mathbb{P}$-generic over M. Then for any $\Delta \in D^M$, there is an $S \in M$ such that $x \subseteq S$ and in M, $\langle 0, S \rangle$ captures Δ.

(Here D^M is in M and is the set that in M satisfies D 6414).

<u>Derivation of T 6419 from T 6417</u>

Let $F \xrightarrow[M]{} x$. T 6417 holds in M (replacing $\dot{x}$ by $\dot{x}^M$) and therefore as "$\phi_{\tilde{F}}(\check{V}) = M$" and $\phi_{\tilde{F}}(\dot{x}^M) = x$, the following is true in $M[x]$:

for any $\Delta \in D^M$ there is an $S \in M$ such that $x \subseteq S$ and in M, $\langle 0, S \rangle$ captures Δ.

As $x \subseteq S \longleftrightarrow x \subseteq S$ in $M[x]$, and the other quantifiers are relativised to M, the conclusion of T 6419 follows.

QED

Proof of T 6417.

It suffices to show that

6420 if $\Delta \in D$ and $\langle s, S\rangle \in |\mathbb{P}|$ then there is an $\langle s', S'\rangle \precsim \langle s, S\rangle$ and an $X \subseteq \omega$ such that

$$\langle s', S'\rangle \Vdash \check{X} \supseteq \dot{x} \wedge \bigwedge T \subseteq \check{X} (T \in \check{A} \to \bigvee t \underline{\text{in}}\, T(\langle t, \underset{t}{\check{X}}\rangle \in \check{\Delta})),$$

as for such an $\langle s', S'\rangle$,

$$\langle s', S'\rangle \Vdash \bigvee X \in \check{A} (\dot{x} \subseteq X \wedge \text{in } \check{V}, \langle 0, X\rangle \text{ captures } \check{\Delta})$$

and so $\{\langle s, S\rangle \mid \langle s, S\rangle \Vdash \bigvee X \in \check{A} (\dot{x} \subseteq X \text{ and in } \check{V}, \langle 0, X\rangle \text{ captures } \check{\Delta})\}$

is dense in $\mathbb{P}$, which shows that for each Δ, that sentence has Boolean value $\mathbb{1}$.

Let then $\Delta \in D$, $\langle s, S\rangle \in |\mathbb{P}|$.

Enumerate the finite subsets of s as $t_0, \ldots, t_k$.

Define a sequence $S_0 \supseteq \ldots \supseteq S_k$ as follows:

pick $S_0 \subseteq S$ such that $\langle t_0, S_0\rangle$ captures Δ: such an S_0 exists by T 6416.

Pick $S_1 \subseteq S_0$ such that $\langle t_1, S_1\rangle$ captures Δ;

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Pick $S_k \subseteq S_{k-1}$ such that $\langle t_k, S_k\rangle$ captures Δ.

Remark that if $\langle t, T\rangle$ captures Δ and $T' \subseteq T$, then $\langle t, T'\rangle$ captures Δ. As $S_k \subseteq S_i$ each $i = 0, 1, \ldots k$,

$\langle t_i, S_k \rangle$ captures Δ for $i = 0, 1, \ldots k$.

I assert that $\langle 0, s \cup S_k \rangle$ captures Δ. For let $T \subseteq s \cup S_k$, and set $t = T \cap s$. Then $t = t_i$ some $i \leq k$. Then $T \in 2^{\langle t_i, S_k \rangle}_{\infty}$, and as $\langle t_i, S_k \rangle$ captures Δ, there is a t' in $\frac{T}{t_i}$ such that $\langle t_i \cup t', \frac{S_k}{t'} \rangle \in \Delta$.

But $t_i \cup t'$ in T and $\frac{S_k}{t'} = \frac{S_k}{t_i \cup t'}$;

so $\bigvee t''$ in T $\langle t'', \frac{S_k}{t''} \rangle \in \Delta$, as required.

Set $X = s \cup S_k$, and $\langle s', S' \rangle = \langle s, S_k \rangle$.

Then $\langle s', S' \rangle \preceq \langle s, S \rangle$

and $\langle s', S' \rangle \Vdash \dot{x} \subseteq \check{X}$ by T6406(iv).

As $\langle 0, X \rangle$ captures Δ,

$\langle s', S' \rangle \Vdash$ in $\check{V}$, $\langle \dot{0}, \check{X} \rangle$ captures $\check{\Delta}$.

QED

In the final step of the proof of T6017 I shall use a standard result which I give as T6423.

D 6421 (ZF) A partial ordering $\langle Q, R \rangle$ is a <u>tree</u> iff
(i) Q is countable and $\bigwedge q \in Q (\neg q R q)$.
(ii) $\bigwedge q \in Q$ $\{q' \mid q R q'\}$ is finite and linearly ordered by R.

<u>REMARK</u> None of the conventions about partial orderings assumed in

¶¶ 1 and 2 are intended to apply in D 6421.

D6422 (ZF) A tree $\langle Q, R\rangle$ is <u>well founded</u> iff there is no function $g : \omega \longrightarrow Q$ such that $\bigwedge i < \omega \quad g(i+1) R g(i)$; that is, "no infinite R-descending paths."

T6423 ZF$\vdash$ A tree $\langle Q, R\rangle$ is well-founded iff there is a function $f : Q \longrightarrow On$ such that $\bigwedge q, q' \in Q \quad qRq' \longrightarrow f(q) < f(q')$.

<u>Proof</u> If there is an $f : Q \longrightarrow On$ such that $\bigwedge q, q' \in Q \quad qRq' \longrightarrow f(q) < f(q')$, then $\langle Q, R\rangle$ is well founded; for were $g : \omega \longrightarrow Q$ to be such that $\bigwedge i < \omega \; g(i+1) R g(i)$, then $\bigwedge i < \omega \quad f(g(i+1)) < f(g(i))$, contradicting the well-ordering of the ordinals.

, Suppose $\langle Q, R\rangle$ is well-founded, and Q not empty.

Let Ξ be the set of all functions ξ such that

(a) $\mathrm{dom}(\xi) \subseteq Q$

(b) $\mathrm{range}(\xi) \subseteq On$.

(c) $q \in \mathrm{dom}(\xi) \wedge q'Rq \longrightarrow q' \in \mathrm{dom}(\xi)$.

(d) if $\neg \bigvee q' \; q'Rq$ and $q \in \mathrm{dom}(\xi)$, then $\xi(q) = 0$.

(e) if $\bigvee q' \; q'Rq$ and $q \in \mathrm{dom}(\xi)$, then $\xi(q) = \sup\{\xi(q')+1 \mid q'Rq\}$.

Then

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(1) $\forall q \in Q \neg \forall q' \in Q \; q' R q$.

Suppose not. Let $\psi : \alpha \longleftrightarrow Q$ be a well ordering of Q, which (such exist as Q is countable).

Define $g(0)$ = the first element of Q in the well ordering ψ.

("the ψ-first").

$g(i+1)$ = the ψ-first $q \in Q$ such that $q \, R \, g(i)$.

Then $\bigwedge i < \omega \; g(i+1) \, R \, g(i)$, contradicting the well foundedness of Q.

(2) Ξ is not empty.

For let $q \in Q$ s.t. $\neg \forall q' \in Q \; q' R q$.

Define ξ by $\xi(q) = 0$,

ξ undefined otherwise.

Then $\xi \in \Xi$.

(3) if $\xi, \xi' \in \Xi$ and $q \in \mathrm{dom}(\xi) \cap \mathrm{dom}(\xi')$ then $\xi(q) = \xi'(q)$:

for if $\neg \forall q' \; q' R q$, then $\xi(q) = \xi'(q) = 0$;

so if $\xi(q) \neq \xi'(q)$,

$\forall q' (q' R q$ and $\xi(q') \neq \xi'(q'))$,

by (2), (c).

But then pick the ψ-first such q': there is a $q'' R q'$ $\xi(q'') \neq \xi'(q'')$ ....,

clearly there is then a map $g : \omega \to Q$ s.t. $\bigwedge i \; g(i+1) \, R \, g(i)$, again contradicting well foundedness.

Define $f = \bigcup\{\xi \mid \xi \in \Xi\}$.

Then by (3), f is a function,

$\mathrm{dom}(f) \subseteq Q$, $\mathrm{range}(f) \subseteq On$.

$f(q)$ is defined iff $\bigvee \xi \in \Xi$ s.t.

$\xi(q)$ is defined, and then $f(q) = \xi(q)$.

(4) f is everywhere defined.

For if $q \in Q$ and $\{q' \mid q' R q\} \subseteq \mathrm{dom}(f)$,

then define $\xi(q') = f(q')$ $q' R q$,

$\xi(q) = \sup\{f(q') + 1 \mid q' R q\}$.

Then $\xi \in \Xi$, and so $q \in \mathrm{dom}(f)$.

So if $\mathrm{dom}(f) \neq Q$,

define $g(0) =$ the ψ-first q not in $\mathrm{dom}(f)$;

then $\bigvee q'$ $q' R q$ and $q' \notin \mathrm{dom}(f)$;

let $g(1) =$ the ψ-first such q'.

$g(i+1) =$ the ψ-first $q \notin \mathrm{dom}(f)$

s.t. $q \; R \; g(i)$;

then $\bigwedge i$ $g(i+1) \; R \; g(i)$ #.

So $f : Q \to On$ and $q R q' \to f(q) < f(q')$. QED

I shall now prove T 6017.

Let M be a transitive model of ZF+DC, x $\mathbb{P}$-generic over M, and $y \subseteq x$.

Define $F_y = \{\langle s, S\rangle \in M \mid s \subseteq y \subseteq s \cup S\}$.

[75].

I assert that F_y is an M-generic filter on $\mathbb{P}^M$.

(1) Let $\langle s, S\rangle \in F_y$ and $\langle s', S'\rangle \in F_y$.

Then $s \subseteq y \subseteq s \cup S$, $s' \subseteq y \subseteq s' \cup S'$,

and $s \underline{\text{in}}\ y$, $s' \underline{\text{in}}\ y$ so that

$s \underline{\text{in}}\ s'$ or $s' \underline{\text{in}}\ s$.

Suppose the first, without loss of generality. Then

$$s' \subseteq y \subseteq s' \cup \frac{S}{s'},$$

so $y \subseteq s' \cup \left(S' \cap \frac{S}{s'}\right)$. Set $T = S' \cap \frac{S}{s'}$.

Then $\langle s', T\rangle \preccurlyeq \langle s', S'\rangle$, $\langle s', T\rangle \preccurlyeq \langle s', \frac{S}{s'}\rangle \preccurlyeq \langle s, S\rangle$,

and $s' \subseteq y \subseteq s' \cup T$, so $\langle s', T\rangle \in F_y$.

(2) Let $\langle s, S\rangle \in F_y$, $\langle s, S\rangle \preccurlyeq \langle s', S'\rangle$.

Then $s' \subseteq s \subseteq y \subseteq s \cup S \subseteq s' \cup S'$, so $\langle s', S'\rangle \in F_y$.

(3) Let $\Delta \in D^M$: that is Δ is in M a dense and $\preccurlyeq$-closed subset of $|\mathbb{P}^M|$. By T 6419 there is an $S^\Delta \supseteq x$ such that $S \in M$ and in M,

S^Δ captures Δ.

Let $Q^\Delta = \{t \subseteq S^\Delta \mid \langle t, \frac{S^\Delta}{t}\rangle \notin \Delta\}$.

$Q^\Delta \in M$.

If Q^Δ is empty, then $\langle 0, S^\Delta\rangle \in \Delta$; but $0 \subseteq y \subseteq x \subseteq S^\Delta = 0 \cup S^\Delta$, so then $\langle 0, S^\Delta\rangle \in F_y$, as required.

Suppose that Q^Δ is not empty. Define a relation

R^Δ on Q^Δ by

$$t_1 R^\Delta t_2 \longleftrightarrow_{df} t_2 \underline{\text{in}}\, t_1 \text{ and } t_1 \neq t_2:$$

that is, $t_1 R^\Delta t_2$ iff t_1 is a proper end extension of t_2. $R^\Delta \in M$. Set $T^\Delta = \langle Q^\Delta, R^\Delta \rangle$.

T 6424 In M, T^Δ is a well-founded tree.

<u>Proof</u> (A) R^Δ is transitive, irreflexive, and for all $s \in Q$,

$$s R^\Delta t_1 \;\&\; s R^\Delta t_2 \longrightarrow t_1 \underline{\text{in}}\, s \text{ and } t_2 \underline{\text{in}}\, s$$

$$\longrightarrow t_1 \underline{\text{in}}\, t_2 \text{ or } t_2 \underline{\text{in}}\, t_1$$

$$\longrightarrow t_1 R^\Delta t_2,\ t_1 = t_2 \text{ or } t_2 R^\Delta t_1.$$

So ~~as~~ s has only finitely many initial segments, $\{t \mid s R^\Delta t\}$ is finite, and linearly ordered by R^Δ.

(B) Q^Δ is countable, being a subset of the set of finite subsets of ω.

(C) Suppose $\langle t_i \mid i < \omega \rangle \in M$ is a sequence such that $\bigwedge_i\ t_{i+1} R^\Delta t_i$: so that in particular $t_i \in Q^\Delta$ each i, and $i < j \rightarrow t_i \underline{\text{in}}\, t_j$.

Set $T = \bigcup_{i<\omega} t_i$.

Then $T \subseteq S$, and as S captures Δ, there is a t $\underline{\text{in}}$ T such that $\langle t, \frac{S}{t} \rangle \in \Delta$. But then for some i, $t \underline{\text{in}}\, t_i$; and $\langle t_i, \frac{S}{t_i} \rangle \preccurlyeq \langle t, \frac{S}{t} \rangle$; as Δ is $\preccurlyeq$-closed, $\langle t_i, \frac{S}{t_i} \rangle \in \Delta$, and so $t_i \notin Q^\Delta$. ※.

QED

Now M is a model of ZF, and therefore by T6423, there is an $f: Q^\Delta \to On^M$, $(f \in M)$ such that

$$\bigwedge q, q' \in Q^\Delta \quad q R^\Delta q' \to f(q) < f(q').$$

Now M is transitive, and so the ordinals of M are an initial segment of the ordinals in V, so in V, f is a function $f: Q^\Delta \to On$, and

$$\bigwedge q, q' \in Q^\Delta \quad q R^\Delta q' \to f(q) < f(q').$$

Applying T6423 (in V),

T6425 $\quad T^\Delta$ is a well founded tree.

$y \subseteq x \subseteq S^\Delta$; so by T6425, there is an s $\underline{\text{in}}$ y such that $\langle s, \frac{S}{s} \rangle \in \Delta$ (otherwise $\{s \mid s \underline{\text{in}} y\}$ would form an infinite R^Δ-descending path in Q^Δ, contradicting T 6425.) But then

$$s \subseteq y \subseteq s \cup \frac{S}{s}, \text{ so } \langle s, \frac{S}{s} \rangle \in F_y \cap \Delta.$$

I have shown therefore that

$$\bigwedge \Delta \in D^M \quad F_y \cap \Delta \neq 0.$$

That, and points (1), (2) on page 75 show that F_y is indeed an M-generic filter on $\mathbb{P}^M$.

By T6412, $F_y \xrightarrow[M]{} y$, and so y is $\mathbb{P}$-generic over M.

The proof of T 6017 is complete.

D 6426 Let ZF_n denote the conjunction of the first n axioms of ZF in some fixed recursive enumeration such that the axiom of extensionality is first, the axiom of foundations the second, and the axiom of infinity is third.

In defining $\langle |\mathbb{P}|, \preccurlyeq \rangle$, $\mathbb{B}$, $V_\alpha^{\mathbb{B}}$, the functions $[\![x \in y]\!]$, $[\![x \equiv y]\!]$ (which are not sets), and D, the set of dense and $\preccurlyeq$-closed subsets of $|\mathbb{P}|$, certain axioms of ZF have been used to ensure that there are sets satisfying the various definitions: only finitely many of the axioms have been used, say a subset of the first m_0.

For M a transitive model of $ZF_{m_0} + DC$, "x is $\mathbb{P}$-generic over M" may be defined as before: that is, that there is an $F \subseteq |\mathbb{P}^M|$ such that for all $\Delta \in D^M$, $F \cap \Delta \neq 0$, and $F \xrightarrow[M]{} x$.

If $n \geq m_0$ and M is a transitive model of $ZF_n + DC$, and x is $\mathbb{P}$-generic over M, then $M[x]$ will satisfy at least those axioms Ω of ZF for which $[\![\mathring{\Omega}]\!]$ can be defined in M and proved to equal $\mathbb{1}$.

T 6427 $ZF \vdash$ There is an $n \geq m_0$ such that if M is a transitive model of $ZF_n + DC$, x $\mathbb{P}$-generic over M and $y \in x$, then y is also $\mathbb{P}$-generic over M.

<u>Proof.</u> By examining the proof of T 6017, which used the truth in M of only a finite number of axioms of ZF. That is, pick n so large that the definitions of the following statement are meaningful, and such that the statements are theorems of $ZF_n + DC$, when written with all variables bound and defined terms replaced by their defining clauses written out in full:

T 6313 and its corollaries, TT 6315, 16; T 6321, T 6412, T 6415, T 6418 of which T 6419 will then be a corollary; T 6423; and T 6424, (with Δ a bound variable). QED

D 6428 Let n_0 be the least such n.

T 6427 will not be used in the proof of T 6001, but in applying the notion of a $\mathbb{P}$-generic real to establish Silver's theorem T 6013.

I conclude the paragraph with three further observations about $\mathbb{P}$: the first is due to Jensen, and will simplify some later proofs.

T 6429 (Jensen) $ZF + DC \vdash$ Let $\mathring{\Omega} \in \mathcal{L}^{\mathbb{B}}$, and $\langle s, S\rangle \in |\mathbb{P}|$. ($\mathbb{B}$ is the algebra over $\mathbb{P}$). Then there is an $S' \subseteq S$ such that

$$\langle s, S'\rangle \Vdash \mathring{\Omega} \text{ or } \langle s, S'\rangle \Vdash \neg\mathring{\Omega}.$$

<u>Proof:</u> (not Jensen's) The set $\Delta =_{df} \{\langle t,T\rangle \mid \langle t,T\rangle \Vdash \mathring{\Omega} \text{ or } \langle t,T\rangle \Vdash \neg\mathring{\Omega}\}$

is dense and $\leq$-closed, by the basic property of forcing F 1, page 21. By T 6416 there is an $S'' \subseteq S$ such that $\langle s, S'' \rangle$ captures Δ. Consider the family

$$P =_{df} \{ X \in 2_\infty^{\langle s, S'' \rangle} \mid \forall t \text{ in } X \ \langle t, \tfrac{S''}{t} \rangle \Vdash \Omega \}.$$

Let $\langle t, T \rangle \lesssim \langle s, S'' \rangle$; there is a t' in T such that $\langle t \cup t', \frac{S''}{t'} \rangle \in \Delta$, as $\langle s, S'' \rangle$ captures Δ.

Then $2_\infty^{\langle t \cup t', \frac{S''}{t'} \rangle} \subseteq P$ or $\subseteq S(\omega) \setminus P$; so P is not CSF on any $\langle t, T \rangle$ with $\langle t, T \rangle \lesssim \langle s, S'' \rangle$, and therefore P is CR on $2^{\langle s, S'' \rangle}$, by T 6316 applied to $2^{\langle s, S'' \rangle}$ rather than $2^{\langle 0, \omega \rangle}$. So there is an $S' \subseteq S''$ such that <u>either</u> $2_\infty^{\langle s, S' \rangle} \subseteq P$ or $2_\infty^{\langle s, S' \rangle} \cap P = 0$.

Suppose the first. Then for no $\langle t, T \rangle \lesssim \langle s, S' \rangle$ does

$\langle t, T \rangle \Vdash \neg \Omega$, as then

$t \cup T \notin P$, (as $S' \subseteq S''$). So therefore $\langle s, S' \rangle \Vdash \Omega$.

If the second holds, $\langle s, S' \rangle \Vdash \neg \Omega$, by a similar argument.

QED.

T 6430 ZF $\vdash$ Let $\mathbb{B}$ be the algebra over $\mathbb{P}$. Then $\mathbb{B}$ is homogeneous.

Proof: By T 6126, it is enough to show that for all $p \in |\mathbb{P}|$, $\mathbb{B} | 0_p \cong \mathbb{B}$.

$\{O_q \mid q \preccurlyeq p\}$ is dense in $\mathbb{B} \mid O_p$, and so

$\mathbb{B} \mid O_p \cong$ the algebra over $\langle \{q \mid q \preccurlyeq p\}, \preccurlyeq \rangle$:

but let $p = \langle s, S \rangle$, h the homeomorphism (D 6305)

$$h : 2^{\langle s, S \rangle} \xleftrightarrow{\sim} 2^{\omega}.$$

Then h is an isomorphism w.r.t. $\preccurlyeq$,

and so $\mathbb{B} \mid O_p \cong \mathbb{B}$.

QED.

T6431 ZF $\vdash$ Let M be a transitive model of ZF+DC. Suppose there is a real $\mathbb{P}$-generic over M. Then for all $\langle s, S \rangle \in |\mathbb{P}^M|$ there is an M-complete filter on $\mathbb{P}^M$ containing $\langle s, S \rangle$.

<u>Proof:</u> Let x be $\mathbb{P}$-generic over M, and let $F_x \underset{M}{\longrightarrow} x$, and $\widetilde{F}_x$ the M-complete ultrafilter on $\mathbb{B}$ generated by F_x.

Let $\langle s, S \rangle \in |\mathbb{P}^M|$; pick $b \in \widetilde{F}_x$, $b \neq \mathbb{1}$.

Set $b' = O_{\langle s, S \rangle}$. By T6430, there is an automorphism ϕ of $\mathbb{B}$, $\phi \in M$, with $\phi(b) = b'$. Let $\widetilde{F} = \{\phi(c) \mid c \in \widetilde{F}_x\}$.

I assert that $\widetilde{F}$ is an M-complete ultrafilter on $\mathbb{B}$:

(i) as ϕ is an automorphism, $\widetilde{F}$ is an ultrafilter.

(ii) let $\mathcal{X} \subseteq \widetilde{F}$ $(\mathcal{X} \in M)$; then $\{c \mid \phi(c) \in \mathcal{X}\} \subseteq \widetilde{F}_x$ and is in M as ϕ and $\mathcal{X}$ are both in M, and so $\prod \{c \mid \phi(c) \in \mathcal{X}\} \in \widetilde{F}_x$; so $\prod \mathcal{X} = \phi(\prod\{c \mid \phi(c) \in \mathcal{X}\}) \in \widetilde{F}$.

Therefore $\{\langle t, T \rangle \mid O_{\langle t, T \rangle} \in \widetilde{F}\}$ is an M-generic filter on $\mathbb{P}^M$ containing $\langle s, S \rangle$. QED