

[49]

[6300]

¶ 3. A combinatorial lemma.

Before stating the lemma I introduce some notation.

Lower case s, t (possibly with sub- or super-script) are used as variables for finite subsets of ω .

S, T, x, y are used for infinite subsets of ω .
 n, m, \dots for elements of ω ; X, Y for reals (subsets of ω).

D 6300 $X <_o Y \iff \bigwedge_m \bigwedge_n (m \in X \wedge n \in Y \rightarrow m < n)$.

D 6301 $s \text{ in } s' \iff s \subseteq s' \wedge s <_o (s' \setminus s)$.

$s \text{ in } s'$ says that s is an initial segment of s' .

D 6302 A condition is a pair $\langle s, S \rangle$ s.t. $s <_o S$.

D 6303 The set $|P|$ of conditions is given the following partial ordering: $\langle s, S \rangle \leq \langle s', S' \rangle \iff \langle s, s' \rangle, \langle S, S' \rangle$ are conditions, $s' \text{ in } s, S \subseteq S'$ and $s \setminus s' \subseteq S'$.

Let $P = \langle |P|, \leq \rangle$.

Symbols like $\langle s, S \rangle$, $\langle s, T \rangle$ are always intended to denote members of $|P|$.

D 6304 $2^S = \{Y \subseteq \omega \mid Y \subseteq S\}$; $2_\infty^S = \{x \subseteq \omega \mid x \subseteq S\}$;

$2^{\langle s, S \rangle} = \{X \subseteq \omega \mid s \subseteq X \subseteq s \cup S\}$;

$2^{\langle s, S \rangle}_\infty = \{x \subseteq \omega \mid s \subseteq x \subseteq s \cup S\}$.

2^X is of course the power set of X , $S(X)$; and $2^X = 2^{<0, X>}$ (0 is the empty set).

As S is infinite, $2^{\langle s, S \rangle}$ is canonically homeomorphic to 2^ω by the mapping

[50]

$$D6305 \quad h(X) = \{s_i \mid s_i \in X\} \quad \text{for } X \in 2^{(s, S)}$$

where $\{s_0, s_1, s_2, \dots\}$ is the monotonic enumeration of S .

That is, if 2^ω is given the topology of page 4 and $2^{(s, S)}$ the subspace topology, then h is a homeomorphism. The only property of h (which I shall call the canonical homeomorphism between $2^{(s, S)}$ and 2^ω) I shall use is

$$X \subseteq Y \iff h(X) \subseteq h(Y)$$

for $X, Y \in 2^{(s, S)}$.

D6306 Let P be a set of reals, $\langle s, S \rangle \in |P|$. P is an SF on $2^{(s, S)}$ \iff_{df}

$$\forall X \in 2_\infty^{(s, S)} \vee Y \in 2_\infty^{(s, S)} (Y \subseteq X, X \in P \iff Y \notin P).$$

D6307 P is completely Scott (P is a CSF) \iff_{df}
for every $\langle s, S \rangle \in |P|$, P is an SF on $2^{(s, S)}$.

Example The SF constructed in T6005 is a CSF.

D6308 P is a CSF on $2^{(s, S)}$ \iff_{df} for every $\langle s', S' \rangle \in |P|$,
 $\langle s', S' \rangle \leq \langle s, S \rangle \rightarrow P$ is an SF on $2^{(s', S')}$

D6309 P is trivial on $2^{(s, S)}$ \iff_{df} $2_\infty^{(s, S)} \subseteq P$ or
 $2_\infty^{(s, S)} \cap P = \emptyset$.

D6310 P is completely Ramsey (CR) on $2^{(s, S)}$ \iff_{df}
 $\forall \langle s', S' \rangle \leq \langle s, S \rangle \vee S'' \subseteq S' (P$ is trivial on $2^{(s', S'')}$).

D6311 P is CR $\iff P$ is CR on $2^{(0, \omega)}$.

[51]

$$D6312 \quad \frac{X}{s} =_y \{n \mid n \in X \wedge s <_o \{n\}\};$$

$$\frac{s}{X} =_y X \setminus \frac{X}{s}.$$

I shall often use an enumeration of the finite subsets of ω , and shall want one which lies in every transitive model of ZF.

D 6313 Let $\langle s_n \mid n < \omega \rangle$ be the following enumeration
(hereinafter called the wonderful enumeration):

$$\begin{array}{ccccccccc} 0 & \{0\} & \{1\} & \{0,1\} & \{2\} & \{0,2\} & \{1,2\} & \dots \\ s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & \dots \end{array}$$

given by the rules

$$s_0 = 0, \quad s_{2^n+k} = s_k \cup \{n\} \quad \text{for } 0 \leq k < 2^n, \quad n < \omega,$$

and for which

$$n < m \rightarrow \max(s_n) \leq \max(s_m);$$

$\Gamma_m \in s_n$ " is a recursive predicate of m and n .

T 6314 ZF + DC \vdash Let P be a set of reals. Then there is an infinite $X \subseteq \omega$ such that either P is trivial on 2^X or P is a CSF on 2^X .

Proof. I shall first use AC to construct a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ and shall later show that the construction can be made using only DC. I shall define a function ϕ on the set of finite sequences while constructing the sequence $\langle X_i \mid i < \omega \rangle$.

[6312]

[52]

(a) Let $X_0 = \omega$.

(b) If X_n has been defined and $s_n \notin X_n$, then set $X_{n+1} = X_n$, and $\psi(s_n) = 0$.

(c) If $s_n \in X_n$ and there is a $Y \subseteq \frac{X_n}{s_n}$ such that P is a CSF on $2^{\langle s_n, Y \rangle}$, then set $\psi(s_n) = 1$, pick such a Y and set

$$X_{n+1} = \frac{s_n}{X_n} \cup Y.$$

(d) If $s_n \in X_n$ and there is no such Y but there is a $Z \subseteq \frac{X_n}{s_n}$ such that P is trivial on $2^{\langle s_n, Z \rangle}$, then set $\psi(s_n) = 2$, pick one such Z and set

$$X_{n+1} = \frac{s_n}{X_n} \cup Z.$$

(e) If $s_n \in X_n$ and there is no such Y or Z , set $\psi(s_n) = 3$ and $X_{n+1} = X_n$.

These cases exhaust all possibilities.

Set $X = \bigcap_{n \in \omega} X_n$.

(1) X is infinite: for suppose not, and that $X = \{n_0, \dots, n_k\}$, where $n_0 < \dots < n_k$. Note that in all four cases ($\psi(s_n) = 0, 1, 2, 3$),

$$X_{n+1} \supseteq \frac{s_n}{X_n}.$$

Thus if $0 \in X_1$, then $0 \in$ every X_n and so $0 \in X$; more generally, if $m \in X_n$ and $m \leq \max(s_n)$, then $m \in$ every X_{n+1} and so $m \in X$. Now if $X = \{n_0, \dots, n_k\}$, then for $n \geq 2^{n_k+1}$, $\max(s_n) \geq n_k$; so let m be the least number in $X_{2^{n_k+1}}$ bigger than n_k .

[53]

Then for every $s_j \in X_{2^{n_k+1}}$, where $j \geq 2^{n_k+1}$,
 $m \leq \max(s_j)$ and so $m \in X_{j+1}$, and therefore $m \in X$,
contradicting the belief that $X = \{n_0, \dots, n_k\}$. Thus X
is indeed infinite.

(2) X is of the sort required.

To see that, remark that

(3) if $\langle s, S \rangle \preceq \langle s', S' \rangle$ and $P \subseteq \text{CSF on } 2^{\langle s, S \rangle}$ then
 $P \subseteq \text{CSF on } 2^{\langle s', S' \rangle}$: that is true as \preceq
is a transitive relation on \mathbb{P} ;

(4) if $\langle s, S \rangle \preceq \langle s', S' \rangle$ and P is trivial on $2^{\langle s, S \rangle}$ then
 P is trivial on $2^{\langle s', S' \rangle}$;

(5) if $s \in X$, then $\psi(s) \neq 0$;

(6) if $s \in X$ and $\psi(s) = 1$, say $s = s_n$, then by (3) and
the fact that $\langle s, \frac{X}{s} \rangle \preceq \langle s, \frac{X_{n+1}}{s} \rangle$,

$P \subseteq \text{CSF on } 2^{\langle s, \frac{X}{s} \rangle}$;

(7) if $s \in X$ and $\psi(s) = 2$, say $s = s_n$, then by (4) and
the fact that $\langle s, \frac{X}{s} \rangle \preceq \langle s, \frac{X_{n+1}}{s} \rangle$,

P is trivial on $2^{\langle s, \frac{X}{s} \rangle}$;

(8) $\psi(0) \neq 0$; if $\psi(0) = 1$, by (6) $P \subseteq \text{CSF on } 2^X$;

if $\psi(0) = 2$, by (7) P is trivial on 2^X ; so that it
remains only to show that $\psi(0) \neq 3$.

(54)

(9) If $s \subseteq X$ and $\psi(s) = 3$, then there are only finitely many $n \in \frac{X}{s}$ such that $\psi(s \cup \{n\}) \neq 3$.

(9) requires some argument. Suppose $s \subseteq X$ and $\psi(s) = 3$.

$$W = \{n \in \frac{X}{s} \mid \psi(s \cup \{n\}) \neq 3\};$$

$$Y = \{n \in \frac{X}{s} \mid \psi(s \cup \{n\}) = 1\};$$

$$Z = \{n \in \frac{X}{s} \mid \psi(s \cup \{n\}) = 2\}.$$

$$\text{By (5)} \quad W = Y \cup Z.$$

If Y is infinite, then P is a CSF on $2^{\langle s, Y \rangle}$: for let $\langle s', Y' \rangle \leq \langle s, Y \rangle$, $x \in 2_{\infty}^{\langle s', Y' \rangle}$.

If $s' \neq s$ set $n = \min s' - s$.

$$\text{Then } \langle s', \frac{x}{s'} \rangle \leq \langle s \cup \{n\}, \frac{Y}{\{n\}} \rangle \leq \langle s \cup \{n\}, \frac{X}{\{n\}} \rangle;$$

but by (6), P is a CSF on $\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle$, and so there is $y \in 2_{\infty}^{\langle s', \frac{x}{s'} \rangle}$ with $y \leq x$ and $y \in P \iff x \notin P$. But this $y \in 2_{\infty}^{\langle s', Y' \rangle}$ as required, as $s' \subseteq y \leq x \subseteq s \cup Y'$.

If $s' = s$, set $n = \min \frac{X}{s}$.

$$\text{Then } x \in 2_{\infty}^{\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle}, \text{ and}$$

$$\langle s \cup \{n\}, \frac{x}{\{n\}} \rangle \leq \langle s \cup \{n\}, \frac{X}{\{n\}} \rangle; \text{ by (6) again,}$$

(and the fact that $\psi(s \cup \{n\}) = 1$), there is $y \leq x$ s.t.

$$y \in P \iff x \notin P \quad \text{and } y \in 2_{\infty}^{\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle};$$

but then $y \in 2_{\infty}^{\langle s', Y' \rangle}$, again since $s = s' \leq y \leq x \leq s \cup Y'$.

[55]

But $Y \subseteq X$; say $s = s_n$: then $\frac{Y}{s} \subseteq \frac{X}{s_n}$, contradicting $\psi(s) = 3$. Thus Y is finite.

Suppose that Z is infinite.

$$\text{Let } Z_0 = \{n \in \mathbb{N} \mid 2_{\infty}^{\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle} \subseteq P\}$$

$$Z_1 = \{n \in \mathbb{N} \mid 2_{\infty}^{\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle} \cap P = \emptyset\}.$$

$$\text{Then } Z = Z_0 \cup Z_1.$$

If Z_0 is infinite, then P is trivial on $2^{\langle s, Z_0 \rangle}$,

$$\text{for } 2_{\infty}^{\langle s, Z_0 \rangle} = \bigcup_{n \in Z_0} 2_{\infty}^{\langle s \cup \{n\}, \frac{Z_0}{\{n\}} \rangle} \subseteq \bigcup_{n \in Z_0} 2_{\infty}^{\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle} \subseteq P,$$

but $Z_0 \subseteq X$, contradicting $\psi(s) = 3$; (for if $s = s_n$, X_n could then have been refined to $\frac{s_n}{X} \cup Z_0$).

If Z_1 is infinite, then P is trivial on $2^{\langle s, Z_1 \rangle}_{\infty}$,

$$2_{\infty}^{\langle s, Z_1 \rangle} = \bigcup_{n \in Z_1} 2_{\infty}^{\langle s \cup \{n\}, \frac{Z_1}{\{n\}} \rangle} \subseteq \bigcup_{n \in Z_1} 2_{\infty}^{\langle s \cup \{n\}, \frac{X}{\{n\}} \rangle} \subseteq S(\omega) \setminus P,$$

again contradicting $\psi(s) = 3$.

So Z is finite, and so W is now known to be finite which proves point (q).

(r) Suppose therefore that $\psi(o) = 3$. By (q) there are only finitely many n s.t. $\psi(\{n\}) \neq 3$: let n_0 = the first integer greater than each of these.

Then $\psi(o) = \psi(\{n_0\}) = 3$.

Applying (q) to $\{n_0\}$ I conclude that

[56]

There is an $n' > n_0$ s.t. for all $n \geq n'$,

$\psi(\{n_0, n\}) = 3$; further by choice

of n_0 , $\psi(\{n'\}) = 3$. Let n_1 be the least such no....

Let n_{k+1} be the least $m > n_k$ such that for all $n \geq m$ and for all $s \subseteq \{n_0, \dots, n_k\}$, $\psi(s \cup \{n\}) = 3$: by (9) such an m exists. Let $X' = \{n_0, n_1, n_2, \dots\}$.

By construction, for every $s \subseteq X'$, $\psi(s) = 3$.

As $\psi(\emptyset) = 3$, P is not CSF on $2^{X'}$, and so there is an $\langle s, s' \rangle \in \langle \emptyset, X' \rangle$ such that P is trivial on $2^{\langle s, s' \rangle}$,

so $\psi(s) \neq 3$ for that s , which contradicts $s \subseteq X'$.

So $\psi(\emptyset) \neq 3$, and X fulfills the assertions of the lemma. It only remains to be seen that DC is an adequate choice axiom for the proof.

Let \mathcal{X} be the set of finite sequences of infinite subsets of ω , of length at least 1 satisfying the rules (a) - (e) of the construction.

Define $\langle X_0, \dots, X_n \rangle R \langle X'_0, \dots, X'_m \rangle$ iff

$m = n + 1$ and $X_i = X'_i$ for each $i = 0, \dots, n$.

By DC, as $\bigcup u \in \mathcal{X} \vee v \in \mathcal{X} \wedge u R v$, there is a sequence $\langle u_i : i \in \omega \rangle$ s.t. $\bigcup u_i \in u_i R u_{i+1}$.

Then $u_0 = \langle X_0, \dots, X_n \rangle$ for some X_0, \dots, X_n .

$u_1 = \langle X_0, \dots, X_n, X_{n+1} \rangle$ some X_{n+1}

$u_2 = \langle X_0, \dots, X_n, X_{n+1}, X_{n+2} \rangle, \dots$

[57]

[6315]

evidently from the sequence $\langle u_i | i < \omega \rangle$ a sequence $\langle X_i | i < \omega \rangle$ can be constructed satisfying the requirements of the proof of the lemma; the function f is definable from the sequence $\langle X_i | i < \omega \rangle$. The axiom of choice was not used elsewhere.

The proof of the lemma is complete.

T 6315 ZF + DC \vdash Let P be a family of sets, and $\langle s, S \rangle \in |\mathcal{P}|$. Then there is an $S' \subseteq S$ such that P is either trivial on $2^{\langle s, S' \rangle}$ or a CSF on $2^{\langle s, S' \rangle}$.

Proof. Let h be the canonical homeomorphism (D6305)

$$h: 2^{\langle s, S \rangle} \xrightarrow{\sim} 2^\omega.$$

Let $P^* = \{x \mid \forall y \in 2^{\langle s, S \rangle} x = h(y)\}$.

Apply T6314 to P^* to obtain an X s.t. on 2^X , P^* is either trivial or a CSF.

$$h^{-1}(X) = s \cup S' \text{ for some } S' \supseteq s, S' \subseteq S.$$

Then on $2^{\langle s, S' \rangle}$ P is either trivial or a CSF.

QED.

The next result is used continually in §§ 4-6.

T 6316 ZF + DC \vdash Let P be a family of sets. Then either there is an $\langle s, S \rangle$ such that P is a CSF on $2^{\langle s, S \rangle}$ or P is CR.

Proof. If there is no $\langle s, S \rangle$ s.t. P is a CSF on $2^{\langle s, S \rangle}$, then by T6315 for any $\langle s, S \rangle$ there is an $S' \subseteq S$ s.t. P is trivial on $\langle s, S' \rangle$; that is, P is CR.

QED.

[58]

[6317]

T 6317 (Nash-Williams et alii)

 $ZF + DC \vdash$ Let P be open. Then P is CR.Proof. Let $\langle s, S \rangle \in |P|$.

If $P \cap 2_{\infty}^{(s, S)}$ is empty, then P is trivial on $2^{(s, S)}$. If not, let $x \in 2_{\infty}^{(s, S)} \cap P$. As P is open, there is a t such that $x = t \cup \frac{x}{t}$ and

$$\lambda y (y \in 2^{(t, \frac{w}{t})} \rightarrow y \in P).$$

Therefore $\langle s \cup t, \frac{S}{s \cup t} \rangle \not\leq \langle s, S \rangle$ and $2_{\infty}^{\langle s \cup t, \frac{S}{s \cup t} \rangle} \subseteq P$: thus

P is not CSF on $2^{(s, S)}$; and so by T6316, as $\langle s, S \rangle$ was arbitrary, P is CR. QED

D 6318 P is CR^+ on $2^{(s, S)}$ \iff

$$\lambda \langle t, T \rangle \leq \langle s, S \rangle \forall T' \subseteq T \quad 2_{\infty}^{(t, T')} \subseteq P$$

D 6319 P is $CR^+ \iff P$ is CR^+ on $2^{(0, \omega)}$.T 6320 $ZF \vdash$ Let P be CR^+ . Then

$$\lambda \langle s, S \rangle \forall S' \subseteq S \quad 2_{\infty}^{s \cup S'} \subseteq P.$$

Proof Enumerate the subsets t_0, \dots, t_k of s .Pick successively $S \supseteq S_0 \supseteq S_1 \supseteq \dots \supseteq S_k$ s.t.

$$2_{\infty}^{\langle t_i, S_i \rangle} \subseteq P.$$

$$\text{Then } 2_{\infty}^{s \cup S_k} = \bigcup_{i \leq k} 2^{\langle t_i, S_k \rangle} \subseteq P.$$

[59]

[6321]

Similarly by relativizing to $2^{(t, T)}$:

T 6321 ZF \vdash Let P be CR+ on $2^{(t, T)}$. Then
 $\langle s, S \rangle \leq \langle t, T \rangle \vee S' \subseteq S \quad 2_{\infty}^{\langle t, \frac{s \cup S'}{t} \rangle} \subseteq P$.

The next result is useful.

T 6322 ZF + DC \vdash Let $\langle P_i | i < \omega \rangle$ be a sequence of CR+ families. Then $\bigcap_{i < \omega} P_i$ is also CR+.

Proof. Let $\langle s, S \rangle \in P_0$. Set $t_0 = s$.

Pick $T_0 \subseteq S$ s.t. $2^{t_0 \cup T_0} \subseteq P_0$ (by T 6320).

Define $t_1 = t_0 \cup \min T_0$.

Pick $T_1 \subseteq \frac{T_0}{t_1}$ s.t. $2^{t_1 \cup T_1} \subseteq P_1$.

Define $t_2 = t_1 \cup \min T_1$.

$t_{n+1} = t_n \cup \min T_n$

Pick $T_{n+1} \subseteq T_n$ such that $2_{\infty}^{t_{n+1} \cup T_{n+1}} \subseteq P_{n+1}$.

A sequence is obtained:

$$\langle s, S \rangle \geq \langle t_1, T_1 \rangle \geq \langle t_2, T_2 \rangle \geq \dots$$

Set $S' = \bigcup_i \{t_i | i < \omega\}$.

Then S' is infinite and $2_{\infty}^{S'} \subseteq \bigcap_{i < \omega} P_i$,

a fortiori, $2_{\infty}^{\langle s, \frac{S'}{s} \rangle} \subseteq \bigcap_{i < \omega} P_i$ Q.E.D.

I close the paragraph with three remarks.

[60]

[6323]

Príkry's basic observation in his proof of T6012 may be formulated as

T6323 (Príkry) ZF + DC \vdash If $\langle P_i \mid i < \omega \rangle$ is a sequence of CR families, then $\bigcap_{i < \omega} P_i$ is also CR,

which may be proved using the method of T6322 and applying T6316, and is not used by me.

It will be seen in §T5 that DC can be avoided in the proof of T6317, and in the application of T6323 to Borel families.

T6324 ZF \vdash Let $n < \omega$. If no SF $\subseteq \Sigma_n^1$, then every Σ_n^1 set is CR.

Proof. Let $\mathcal{X} = \{X \mid R(X, Y)\}$ where R is Σ_n^1 .

Let $\langle s, S \rangle \in \text{IMP}$, h the mapping (D6305)

$$h: 2^{(s, S)} \xrightarrow{\sim} 2^\omega.$$

Then the predicate $h(Z) = w$ is arithmetical with the parameter S (w as it may be written

$$\lambda n(n \in \omega \leftrightarrow \text{the } n^{\text{th}} \text{ element of } S \text{ is in } Z))$$

and so

$$\mathcal{X}_1 = \{X \mid \forall Z(s \subseteq Z \subseteq s \cup S \wedge R(Z, Y) \wedge X = h(Z))\}$$

may also be expressed in Σ_n^1 form (with parameters S, Y) without using DC. For let

$$R(X, Y) \longleftrightarrow \bigvee W_1 \wedge W_2 \wedge \dots \wedge W_m Q(W_1, \dots, W_m, X, Y)$$

where Q is arithmetical.

[61]

$$\text{Then } X \in \mathcal{X}_1 \longleftrightarrow V_2 V_{W_1} \wedge_{W_2} \dots \wedge_{W_n} [$$

$$Q(W_1 \dots W_n, Z, Y) \wedge S \subseteq Z \subseteq s \cup S \wedge X = h(Z);$$

The first two quantifiers may be contracted to one by Shoenfield's rules, and so \mathcal{X}_1 is indeed $\sum_1^1 n$. Then \mathcal{X}_1 is not an SF, so there is an x s.t.

\mathcal{X}_1 is trivial on 2^x , and \mathcal{X} is then

trivial on $2^{(S, \frac{f^{-1}(x)}{S})}$,

and $S \subseteq h^{-1}(x) \subseteq s \cup S$.

Q.E.D.

E The same method will prove T 6324 when " $\sum_1^1 n$ " is replaced by "projective" or "ordinal definable from a real" (D 6802).