

## ¶ 2. Filters and models.

Suppose  $M$  a transitive model of ZF and  $Q$  a transitive set. Is there a transitive model  $N$  of ZF s.t.  $O_M \cap M = O_N \cap N$ ,  $M \subseteq N$ ,  $Q \in N$  and for any other transitive model  $N'$  such that  $O_M \cap M = O_{N'} \cap N'$ ,  $M \subseteq N'$  and  $Q \in N'$ ,  $N \subseteq N'$ ? If there is, then  $N$  is unique, and is called the constructible closure of  $M$  and  $Q$ . In the above exact sense it is the smallest transitive model of ZF containing  $M$  and  $Q$ , and with the same ordinals as  $M$ . If  $M \in V$ ,  $N$  such as  $N$  need not exist; but if  $M \notin V$ ,  $N$  does and its formation is now described.

D6200 (i) Let  $U$  be a transitive set. The language  $\mathcal{L}(U)$  is a first order language with an individual constant  $u$  for each  $u \in U$ , and <sup>the</sup> two two-place predicate symbols  $\varepsilon, \equiv$ .

(ii)  $\text{Def}(U) =$  the set of all  $x$  s.t. there is a formula  $\mathcal{O}(x)$  of  $\mathcal{L}(U)$  with one free variable and

$$x = \{y \in U \mid U \models \mathcal{O}(y)\},$$

(where  $\varepsilon, \equiv$  are interpreted as  $\in, =$ , and  $u$  as  $u$ ).

REMARK As  $U \in V$ ,  $\mathcal{L}(U)$  and the satisfaction

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relation  $\vDash$  may be defined in ZF uniformly from  $\cup$ . [6201]

D6201 (ZF) Let  $Q$  be a transitive set. Define

$$L_0[Q] = Q$$

$$L_{\alpha+1}[Q] = \text{Def } L_\alpha[Q]$$

$$\lim(\lambda) \rightarrow L_\lambda[Q] = \bigcup_{\gamma < \lambda} L_\gamma[Q]$$

$$L[Q] = \bigcup_{\alpha \in \text{On}} L_\alpha[Q].$$

Then  $L[Q]$  is an inner model of ZF and is definable uniformly from  $Q$ : that is, there is a formula  $\mathcal{Q}(x, y)$  of ZF with two free variables s.t. for  $Q$  transitive,

$$\mathcal{Q}(Q, y) \leftrightarrow y \in L[Q].$$

$L[0]$  is Gödel's constructible universe.

D6202. If  $Q$  is not transitive, write  $L[Q]$  to

mean  $L[C(Q)]$ , where  $C(Q) =_{df} \bigcap \{y \mid Q \subseteq y \subseteq S(y)\}$  is the transitive closure of  $Q$ .

$[C(Q) \in \mathcal{V}$ , as  $C(Q) \subseteq \mathcal{V}_{p(Q)}$  where  $p$  is the rank function defined by recursion on the well-founded relation  $\in$ :  $p(x) = \inf \{p(y) + 1 \mid y \in x\}$ , and

$$\mathcal{V}_\alpha = \{x \mid p(x) < \alpha\}. ]$$

It is well known that  $L$  is the smallest inner model and  $L[Q]$  the smallest inner model containing  $Q$ , is the

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[6203]

sense of "smallest" indicated above.

Suppose now that  $M$  is an inner model of ZF, and  $Q$  transitive. To construct  $M(Q)$  we have to ensure that all elements of  $M$  are in the final product; they cannot all be put in at once, as  $M \notin V$ ; so they must be inserted "one at a time". As  $M$  is transitive,  $p(x) = p^M(x)$  for  $x \in M$ ; write  $\mathcal{V}_\alpha^M = \{x \in M \mid p(x) < \alpha\}$ .

D6203

$$M_0[Q] = Q$$

$$M_{\alpha+1}[Q] = \mathcal{V}_\alpha^M \cup \text{Def}(M_\alpha[Q])$$

$$\lim(\lambda) \rightarrow M_\lambda[Q] = \bigcup_{\gamma < \lambda} M_\gamma[Q]$$

$$M[Q] = \bigcup_{\alpha \in \text{On}} M_\alpha[Q]$$

Then  $M[Q]$  is an inner model of ZF s.t.  $Q \in M[Q]$  and  $M \subseteq M[Q]$ ; and  $M[Q]$  is the smallest such.

D6204 If  $Q$  is not transitive, define

$$M[Q] = M[C(Q)].$$

Then  $M[Q]$  is the smallest transitive model of ZF containing  $Q$ , the elements of  $M$  and all ordinals.

Suppose now that  $M \in V$ ; and now define

$$M[Q] = \bigcup_{\alpha \in (\text{On} \cap M)} M_\alpha[Q].$$

$M[Q]$  need not be a model of ZF: the importance of forcing

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[6205]

is that if  $Q$  is "generic" with respect to some notion of forcing in  $M$ , then  $M[Q]$  will satisfy  $ZF$ .

D6205 Let  $M$  be a transitive model of  $ZF$ , and let  $\mathbb{B} \in M$  be s.t. in  $M$  it is a cBA.

A set  $F \subseteq |\mathbb{B}|$  is an  $M$ -complete filter on  $\mathbb{B}$

$$6206 \iff \bigwedge \mathcal{X} \in M (0 \notin \mathcal{X} \subseteq |\mathbb{B}| \rightarrow (\prod^{\mathbb{B}} \mathcal{X} \in F \leftrightarrow \mathcal{X} \subseteq F))$$

and  $F$  is a non-principal filter in the usual algebraic sense. (In particular,  $0^{\mathbb{B}} \notin F$ .)

" $\prod^{\mathbb{B}} \mathcal{X} \in F \leftrightarrow \mathcal{X} \subseteq F$ " is commonly read as " $F$  preserves the infimum of  $\mathcal{X}$ ". If  $F$  is also an ultrafilter, then  $\sum^{\mathbb{B}} \mathcal{X} \in F \leftrightarrow \mathcal{X} \cap F \neq \emptyset$ ;  $F$  then "preserves the supremum of  $\mathcal{X}$ ". As only atomfree Boolean algebras are considered, an ultrafilter will automatically be non-principal. Remark that for an ultrafilter  $F$ ,

$$F \text{ preserves the inf of } \mathcal{X} \iff F \text{ preserves the sup of } \{-b \mid b \in \mathcal{X}\}.$$

Working in  $M$ , form the associated Boolean-valued universe, written as  $M^{\mathbb{B}}$ .

Given an  $M$ -complete ultrafilter  $F$ , we can reduce  $M^{\mathbb{B}}$  to a (two-valued) transitive model: define the mapping  $\phi_F$  on  $M^{\mathbb{B}}$  by recursion on the well-founded relations  $i \in \text{dom}(i^-)$ :

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[6207]

D6207  $\phi_F(\dot{v}) = \{ \phi_F(\dot{u}) \mid \dot{u} \in \text{dom } \dot{v} \text{ and } [\dot{u} \in \dot{v}] \in F \}$ .

(If  $M \notin V$ , then  $\phi_F$  is given by a class term in the language of ZF with the extra predicate  $M(x)$ ).

Set  $N = \{ \phi_F(\dot{u}) \mid \dot{u} \in M^{\mathbb{B}} \}$ ;

T6208  $ZF \vdash N$  is a transitive model of ZF and all ZF-sentences  $\dot{\alpha}$  for which  $[\dot{\alpha}]^{\mathbb{B}} \in F$ .

T6208 is proved using the schema:

T6209  $ZF \vdash$  For each  $\dot{u}_1, \dots, \dot{u}_k \in M^{\mathbb{B}}$  and each ZF-formula  $\dot{\alpha}(x_1, \dots, x_k)$  with the  $k$  free variables shown,

$$[\dot{\alpha}(\dot{u}_1, \dots, \dot{u}_k)]^{\mathbb{B}} \in F \iff$$

$$\dot{\alpha}^N(\phi_F(\dot{u}_1), \dots, \phi_F(\dot{u}_k)),$$

which is proved by induction on the number of quantifiers in  $\dot{\alpha}$ , the essential step being that as

$$[\forall x \dot{\alpha}]^{\mathbb{B}} = \sum_{x \in M^{\mathbb{B}}} [\dot{\alpha}(x)]^{\mathbb{B}},$$

$$[\forall x \dot{\alpha}]^{\mathbb{B}} \in F \iff \forall x \in M^{\mathbb{B}} [\dot{\alpha}(x)]^{\mathbb{B}} \in F,$$

since  $F$  preserves all sups in  $M$ .

The next theorem makes precise the remark following D6204.

T6210  $ZF \vdash F \in N$  and  $N = M[F]$ .

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The proof will use the following important observation

T6211  $\exists F \vdash$  Let  $\mathbb{B}$  be a cBA. There is an  $\dot{F} \in \mathcal{V}^{\mathbb{B}}$  such  
 that  $[\dot{F} \text{ is a } \check{\mathcal{V}}\text{-complete ultrafilter on } \check{\mathbb{B}}]_{\mathbb{B}} = 1$   
 and for  $b \in |\mathbb{B}|$ ,  $[b^{\check{\vee}} \varepsilon \dot{F}]_{\mathbb{B}} = b$ .

Proof. Pick an  $\alpha$  s.t.  $\{b^{\check{\vee}} \mid b \in |\mathbb{B}|\} \subseteq \mathcal{V}_{\alpha}^{\mathbb{B}}$ . (D6100)

Define  $\dot{F}$  by setting

$$\text{dom}(\dot{F}) = \{b^{\check{\vee}} \mid b \in |\mathbb{B}|\};$$

$$\dot{F}(b^{\check{\vee}}) = b.$$

$$\text{Then } [b^{\check{\vee}} \varepsilon \dot{F}]_{\mathbb{B}} = \sum_{x \in \text{dom}(\dot{F})}^{\mathbb{B}} [x \equiv b^{\check{\vee}}]_{\mathbb{B}} \cdot \dot{F}(x);$$

$$[b'^{\check{\vee}} \equiv b^{\check{\vee}}]_{\mathbb{B}} = 0 \text{ unless } b' = b, \text{ so}$$

$$[b^{\check{\vee}} \varepsilon \dot{F}]_{\mathbb{B}} = b.$$

$$[0 \varepsilon \dot{F}]_{\mathbb{B}} = 0, \text{ so } [0 \not\varepsilon \dot{F}]_{\mathbb{B}} = 1.$$

Let  $0 \neq \mathcal{X} \subseteq |\mathbb{B}|$ .

$$\begin{aligned} [\mathcal{X}^{\check{\vee}} \subseteq \dot{F}]_{\mathbb{B}} &= [\bigwedge_{x \in \mathcal{X}} x^{\check{\vee}} \varepsilon \dot{F}]_{\mathbb{B}} \\ &= \prod_{b \in \mathcal{X}} [b^{\check{\vee}} \varepsilon \dot{F}]_{\mathbb{B}} \quad \text{by T6107.} \\ &= \prod_{b \in \mathcal{X}} b = \prod \mathcal{X} = [\bigvee \mathcal{X} \varepsilon \dot{F}]_{\mathbb{B}}, \end{aligned}$$

$$\text{so that } [\mathcal{X}^{\check{\vee}} \subseteq \dot{F} \iff \bigvee \mathcal{X} \varepsilon \dot{F}]_{\mathbb{B}} = 1;$$

$$\text{it follows that as } [b^{\check{\vee}} \varepsilon \dot{F} \wedge b'^{\check{\vee}} \varepsilon \dot{F}]_{\mathbb{B}} = b \cdot b' = [b \cdot b' \varepsilon \dot{F}]_{\mathbb{B}};$$

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[6212]

$$[b^{\vee} \in \dot{F} \wedge b'^{\vee} \in \dot{F} \leftrightarrow b \cdot b' \in \dot{F}]^{\mathbb{B}} = \mathbb{1}$$

for all  $b, b'$ ; so

$$[\dot{F} \text{ is a } \check{V}\text{-complete filter on } \check{\mathbb{B}}]^{\mathbb{B}} = \mathbb{1}.$$

$$\begin{aligned} \text{As } b = \neg\neg b, \quad [b^{\vee} \in \dot{F}]^{\mathbb{B}} &= \neg [\neg b \in \dot{F}]^{\mathbb{B}} \\ &= [\neg b \notin \dot{F}]^{\mathbb{B}} \end{aligned}$$

$$\text{so } [b^{\vee} \in \dot{F} \leftrightarrow \neg b \notin \dot{F}]^{\mathbb{B}} = \mathbb{1},$$

$$\text{so } [\dot{F} \text{ is an ultrafilter}]^{\mathbb{B}} = \mathbb{1} \quad \text{QED.}$$

I assert now that

$$T6212 \quad F = \phi_F(\dot{F})$$

$$\text{for } [x \in \dot{F}]^{\mathbb{B}} = \sum_{b \in |\mathbb{B}|} [x \equiv b^{\vee}] \cdot b,$$

$$\begin{aligned} \text{so } \phi_F(x) \in F &\leftrightarrow \forall b \in \mathbb{B} [x \equiv b^{\vee}] \cdot b \in F \\ &\leftrightarrow \forall b \in \mathbb{B} \quad b \in F \wedge [x \equiv b^{\vee}] \in F \\ &\leftrightarrow \forall b \in \mathbb{B} \quad b \in F \wedge \phi_F(x) = b \\ &\leftrightarrow \phi_F(x) \in F. \end{aligned}$$

That  $F \in N$  follows immediately from T6212, and as  $N$  is a ZF model,  $M[F] \subseteq N$ ; as every element of  $N$  is constructed from  $F$  and elements of  $M$ ,  $N \subseteq M[F]$ . (A rigorous proof proceeds by induction on rank).

$$\text{Thus } N = M[F].$$

I shall call the  $\dot{F}$  of T6211 the canonical

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[6213]

$\check{V}$ -complete ultrafilter on  $B$ . Remark that if  $B \subseteq_{\text{reg}} C$ ,  
and  $\dot{F}_C \in$  the canonical  $\check{V}$ -complete ultrafilter on  $C$ , then

$$[\dot{F} \equiv \check{B} \cap \dot{F}_C]^C = \mathbb{1};$$

for by the definition of  $\dot{F}$ ,  $\dot{F} \in V^C$  and for  $b \in |B|$ ,

$$[b \varepsilon \dot{F}]^C = b = [b \varepsilon \dot{F}_C]^C;$$

$$\approx [b \varepsilon |B| \rightarrow (b \varepsilon \dot{F} \leftrightarrow b \varepsilon \dot{F}_C)]^C = \mathbb{1};$$

$$\text{by T6107, } [\bigwedge b \in |B| b \varepsilon \dot{F} \leftrightarrow b \varepsilon \dot{F}_C]^C = \mathbb{1}.$$

T6210 has its Boolean counterpart:

T6213 ZF+ Let  $B$  be a cBA,  $\dot{F}$  the canonical  
 $\check{V}$ -complete ultrafilter on it: then

$$[\check{V}[\dot{F}] \equiv \dot{V}]^B = \mathbb{1}.$$

Proof. Remember that  $V^B \subseteq V$ ; therefore for  
 $\check{x} \in V^B$  the element  $\check{\check{x}} \in V^B$  is defined.

Following D6207 there is a term  $\dot{\phi}_{\dot{F}}$  of  $\mathcal{L}^B$   
such that  $[\dot{\phi}_{\dot{F}} : \check{V}^B \rightarrow \dot{V}]^B = \mathbb{1}$ ,

$$\text{and } [\bigwedge \check{x} \varepsilon \check{V}^B \dot{\phi}_{\dot{F}}(\check{x}) \equiv \{\dot{\phi}_{\dot{F}}(y) \mid [\check{x} \varepsilon y]^B \varepsilon \dot{F}\}]^B = \mathbb{1}$$

(by  $[\cdot \varepsilon \cdot]$  I mean the image in  $V^B$  under the  
embedding  $\check{V}$  of the function  $E^B(\cdot, \cdot)$  of D6101)

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But for  $\dot{x}, \dot{y} \in \mathcal{V}^{\mathbb{B}}$ ,

$$\begin{aligned} [\dot{\phi}_{\dot{F}}(\dot{y}) \in \dot{\phi}_{\dot{F}}(\dot{x})]^{\mathbb{B}} &= [\underbrace{[\dot{y} \in \dot{x}]^{\mathbb{B}} \in \dot{F}}]^{\mathbb{B}} \\ &= [\dot{y} \in \dot{x}]^{\mathbb{B}}, \end{aligned}$$

so that  $[\dot{\phi}_{\dot{F}}(\dot{y}) \in \dot{\phi}_{\dot{F}}(\dot{x}) \leftrightarrow \dot{y} \in \dot{x}]^{\mathbb{B}} = \mathbb{1}$ ;

by induction on the rank of  $\dot{x}$ ;

$$[\dot{x} \equiv \dot{\phi}_{\dot{F}}(\dot{x})]^{\mathbb{B}} = \mathbb{1},$$

$$\text{and so } [\exists \dot{y} \in \mathcal{V}^{\mathbb{B}} \dot{\phi}_{\dot{F}}(\dot{y}) \equiv \dot{x}]^{\mathbb{B}} = \mathbb{1}.$$

Q.E.D.

REMARK.

The equation at the top of this page is a special case of the general theorem that

$$[\dot{\Omega}(\dot{x}_1, \dots, \dot{x}_k)]^{\mathbb{B}} = [\underbrace{[\dot{\Omega}(\dot{x}_1, \dots, \dot{x}_k)]^{\mathbb{B}} \in \dot{F}}]^{\mathbb{B}},$$

as by T6213 and T6209,

$$\begin{aligned} [\dot{\Omega}(\dot{x}_1, \dots, \dot{x}_k) \leftrightarrow \text{the } b \in |\mathbb{B}| \text{ such that} \\ \text{in } \mathcal{V}, (b \equiv \underbrace{[\dot{\Omega}(\dot{x}_1, \dots, \dot{x}_k)]^{\mathbb{B}}}) \text{ is in } \dot{F}]^{\mathbb{B}} &= \mathbb{1}. \end{aligned}$$

(Here  $[\dot{\Omega}(\cdot, \cdot, \dots)]^{\mathbb{B}}$  is the function  $\langle \dot{x}_1, \dots, \dot{x}_k \rangle \mapsto [\dot{\Omega}(\dot{x}_1, \dot{x}_2)]^{\mathbb{B}}$   
and  $\underbrace{[\dot{\Omega}(\cdot, \cdot, \dots)]^{\mathbb{B}}}$  is its  $\in \mathcal{V}^{\mathbb{B}}$  by  $\mathcal{V}$ .)

In general, the existence of such filters cannot be proved. An important exception is provided by the

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[6214]

T 6214 (Rasiowa and Sikorski [11]) ZF  $\vdash$  Let  $\{X^i \mid i < \omega\}$  be a countable family of countable subsets of a cBA  $B$ , and let  $b \in B \setminus \{0\}$ . Then there is a filter  $F$  which preserves the sup of each  $X^i$  and the inf of each  $Y^i$  and contains  $b$ .

Hence

T 6215 Let  $S^M(|B|)$  be countable. Then for all  $b \in |B| \setminus \{0\}$  there is an  $M$ -complete ultrafilter  $F$  containing  $b$ .

Proof Let  $F$  be given by applying T 6214 to the set of all sups and infs in  $M$ ; and s.t.  $b \in F$ . It remains only to check that  $F$  is an ultrafilter. Let  $b \in B$ .

As  $b + -b = 1 \in F$ ,  $b$  or  $-b$  is in  $F$ . Q.E.D.

The following theorem elucidates the connection between dense subsets of a partial ordering and complete filters on the associated algebra.

Let  $M$  be a transitive model of ZF, and  $P = \langle |P|, \leq \rangle \in M$  a partial ordering. Let  $B \in M$  be the algebra over  $P$  in the sense of  $M$ . Let  $F \subseteq |P|$ . The map  $p \mapsto 0_p^B$  embeds  $P$  densely in  $B$ . Define

$$6216 \quad \widetilde{F} = \{b \in B \mid \forall p \in F \ 0_p^B \leq b\}.$$

D 6217  $F$  is an  $M$ -generic filter on  $P \iff$

$$(i) \ \bigwedge p_1, p_2 \in F \ \exists p_3 \in F \ (p_3 \leq p_1 \wedge p_3 \leq p_2)$$

$$(ii) \ \bigwedge p_1 \in F \ \bigwedge p_2 \in |P| \ (p_1 \leq p_2 \rightarrow p_2 \in F)$$

$$(iii) \ \bigwedge \Delta \in M \ (\Delta \text{ a dense and } \leq\text{-closed subset of } |P| \rightarrow F \cap \Delta \neq \emptyset).$$

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[6218]

The word 'filter' is used here for the obvious analogy; moreover

T6218 If  $F$  is an  $M$ -generic filter on  $\mathbb{P}$  then  
 $\tilde{F}$  is an  $M$ -complete ultrafilter on  $\mathbb{B}$ .

Proof. (a)  $\tilde{F}$  is a filter by 6217 (i), (ii). It is an ultrafilter as  $A =_{df} \{p \mid 0_p \leq b \text{ or } 0_p \leq -b\}$  is dense and  $\leq$ -closed, and so  $\forall p \in F \cap A$ ;  $\therefore b$  or  $-b \in \tilde{F}$ , and not both, by 6217 (i).

(b) Let  $0 \neq \mathcal{X} \in M$ ,  $\mathcal{X} \subseteq |\mathbb{P}|$ ,  $b =_{df} \sum_{\mathcal{X}} \mathcal{X} \in \tilde{F}$ .

The set  $D =_{df} \{p \mid 0_p \leq -b \text{ or } \forall b' \in \mathcal{X} \ 0_p \leq b'\}$  is dense and  $\leq$ -closed; and so for some  $p$ ,  $p \in D \cap F$ , and as  $0_p \not\leq -b$  (as  $b \in \tilde{F}$ ),  $\forall b' \in \mathcal{X} \ b' \in \tilde{F}$ .

QED

The converse of this theorem also holds:

T6219 Let  $G$  be an  $M$ -complete ultrafilter on  $\mathbb{B}$ .

Define  $F = \{p \mid 0_p \in G\}$ ; then  $F$  is an  $M$ -generic filter on  $\mathbb{P}$  and  $G = \tilde{F}$ .

Proof. If  $\Delta \in M$ ,  $\Delta$  dense, then  $\sum^{\mathbb{B}} \{0_p \mid p \in \Delta\} = \mathbb{1}$ , and  $\therefore \forall p \in \Delta$  s.t.  $0_p \in G$ ; and  $\therefore$  this  $p \in F \cap \Delta$ .

$F$  satisfies  $\text{D}6217$  (i), (ii) as  $G$  is a filter.

Clearly  $\tilde{F} \subseteq G$ . Now if  $b \in G$ ,  $\forall \mathcal{X} \subseteq |\mathbb{P}|$  s.t.

$b = \sum^{\mathbb{B}} \{0_p \mid p \in \mathcal{X}\}$ , so  $\forall p \in \mathcal{X} \ 0_p \in G$ ; and

then such a  $p \in F$ , and so  $b \in \tilde{F}$ .

QED

These arguments show incidentally that

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[6220]

43  $F$  intersects every dense  $\Delta \iff F$  intersects every dense and  $\leq$ -closed  $\Delta$ .

Notice that if  $F$  is  $M$ -generic for  $\mathbb{P}$ , then  $M[F] = M[\tilde{F}]$ , by their interdefinability.

REMARK. Let  $\mathbb{B}$  be an (incomplete) BA  $\in M$ . Let  $\mathbb{B}'$  be in  $M$  an r.m.c. of  $\mathbb{B}$ . Define a filter  $F$  on  $\mathbb{B}$  to be  $M$ -generic  $\iff$  it is  $M$ -generic for  $\mathbb{B}$  qua  $\mathcal{P}0$ . Then the above arguments show that  $F$  is  $M$ -generic on  $\mathbb{B}$  iff it is the basis of an  $M$ -complete ultrafilter on  $\mathbb{B}'$ .

The proof of the next group of theorems (T 6220 to T 6227) are taken from notes of Jensen.

T6220  $\Sigma F \vdash$  Let  $\mathbb{B} \subseteq_{\text{reg}} \mathbb{B}'$  be cBAs.

Let  $\dot{F} \in \mathcal{V}^{\mathbb{B}}$  be the canonical  $\check{V}$ -complete ultrafilter on  $\mathbb{B}$ . Then

$$[\dot{F} \text{ is the basis of a } \check{V}\text{-complete filter on } \mathbb{B}']^{\mathbb{B}} = \mathbb{1}.$$

Corollary:

T 6221 Let  $\mathbb{B} \subseteq_{\text{reg}} \mathbb{A}$ : then  $\mathcal{V}^{\mathbb{B}} \subseteq \mathcal{V}^{\mathbb{A}}$  and so  $\dot{F}, \mathbb{B}' \in \mathcal{V}^{\mathbb{A}}$ , and  $[\dot{F} \text{ is the basis of a } \check{V}\text{-complete filter on } \mathbb{B}']^{\mathbb{A}} = \mathbb{1}$ .

Proof By absoluteness: the quantifiers of the  $L^{\mathbb{B}}$ -sentence concerned are all bound restricted to elements of  $\mathcal{V}^{\mathbb{B}}$ .

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[6222]

Proof of T6220.Define  $\dot{F}^+ \in \mathcal{V}^B$  by

$$\text{dom}(\dot{F}^+) = \{b' \vee \mid b' \in |B'|\}.$$

$$D6222 \quad \dot{F}^+(b' \vee) = \sum^B \{b \in |B| \mid b \leq b'\}.$$

(By regularity,  $\dot{F}^+(b' \vee)$  then equals  $\sum^{B'} \{b \in |B| \mid b \leq b'\}$ .)

Then as in 6211 it is seen that

$$\dot{F}^+(b' \vee) = \llbracket b' \varepsilon \dot{F}^+ \rrbracket^B.$$

I assert that for  $b \in |B|$ ,  $b' \in |B'|$ ,

$$b \leq \llbracket b' \varepsilon \dot{F}^+ \rrbracket^B \iff b \leq b'.$$

$$\text{for } b \leq b' \rightarrow b \leq \sum^B \{b \mid b \leq b'\}$$

$$\text{and } \llbracket b' \varepsilon \dot{F}^+ \rrbracket^B = \sum^{B'} \{b \in |B| \mid b \leq b'\} \leq b'.$$

Now for  $b'_1, b'_2 \in |B'|$ ,

$$\sum^B \{b \mid b \leq b'_1\} \cdot \sum^B \{b \mid b \leq b'_2\} = \sum^B \{b \mid b \leq b'_1 \cdot b'_2\},$$

(as the left hand side =  $\sum^B \{b \cdot c \mid b \leq b'_1 \wedge c \leq b'_2\}$ .)

$$\text{and so } \llbracket b'_1 \cdot b'_2 \varepsilon \dot{F}^+ \iff b'_1 \varepsilon \dot{F}^+ \wedge b'_2 \varepsilon \dot{F}^+ \rrbracket^B = \mathbb{1}.$$

As  $\llbracket 0 \varepsilon \dot{F}^+ \rrbracket = \mathbb{1}$ ,

$$\llbracket \dot{F}^+ \text{ is a filter on } |B'| \rrbracket^B = \mathbb{1}$$

Let now  $A' \in |B'|$ ; and set  $b = \llbracket A' \varepsilon \dot{F}^+ \rrbracket^B$ ;

$$\begin{aligned} \text{then } b &= \llbracket \bigwedge a' \varepsilon A' \quad a' \varepsilon \dot{F}^+ \rrbracket^B \\ &= \prod_{a' \varepsilon A'} \llbracket a' \varepsilon \dot{F}^+ \rrbracket^B, \end{aligned}$$

and so for all  $a' \varepsilon A'$ ,  $b \leq \llbracket a' \varepsilon \dot{F}^+ \rrbracket^B$ ,

and so  $b \leq a'$ ;

therefore  $b \leq \prod_{a' \varepsilon A'} a'$ , and so  $b \leq \llbracket \prod A' \varepsilon \dot{F}^+ \rrbracket$ .

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[6223]

So for all  $A'$   $\left[ \bigvee A' \in \dot{F}^+ \rightarrow \prod A' \in \dot{F}^+ \right]^B = \mathbb{1}$ ,

and therefore  $\left[ \dot{F}^+ \text{ is } \check{V}\text{-complete} \right]^B = \mathbb{1}$ .

By D 6222,  $\left[ b' \in \dot{F}^+ \leftrightarrow \forall b \in |B| \ b \leq b' \right]^B = \mathbb{1}$ ,  $Q \in P$ .

so  $\left[ \dot{F}^+ \equiv \{ b' \in |B'| \mid \forall b \in \dot{F} \ b \leq b' \} \right]^B = \mathbb{1}$ .

QED

When considering cBAs  $B \subseteq_{\text{cg}} B'$  the following function  $h: |B'| \rightarrow |B|$  is useful:

D 6223 For  $b' \in |B'|$ ,  $h(b') = \prod^B \{ b \in |B| \mid b' \leq b \}$ .

Then  $h(b') = \prod^{B'} \{ b \in |B| \mid b' \leq b \}$ ; and (cf D 6222)

$$\left[ b' \in \dot{F}^+ \right]^B = \sum^B \{ b \mid b \leq b' \} = -\prod^B \{ -b \mid b \leq b' \}$$

$$= -\prod^B \{ b \mid -b' \leq b \} = -h(-b');$$

so that  $h(b') = - \left[ (-b') \in \dot{F}^+ \right]^B$ , so

$$\begin{aligned} \text{T 6224 } h(b') &= \left[ b' \text{ is not in the ideal dual to } \dot{F}^+ \right]^B \\ &= \left[ b' \text{ is not in the kernel of the natural projection } B' \rightarrow B'/\dot{F}^+ \right]^B. \end{aligned}$$

It is not asserted that  $h$  is a homomorphism, but it has the property

$$\text{T 6225 } b \cdot b' = 0 \rightarrow b \cdot h(b') = 0,$$

$$\text{for } b \cdot b' = 0 \rightarrow b' \leq -b \rightarrow h(b') \leq -b \rightarrow b \cdot h(b') = 0$$

$$\text{T 6226 } b \in |B| \rightarrow h(b) = b.$$

Let now  $B, B', \dot{F}, \dot{F}^+, h$  be as above, and  $\dot{F}' \in \check{V}^{B'}$  the canonical  $\check{V}$ -complete ultrafilter on  $B'$ . Let  $\dot{f} \in \check{V}^B$  be such that

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$[f$  is the natural projection  $\check{B}' \rightarrow \check{B}' / \check{f}^+ ]^B = 1$ .

Write  $\check{F}' / \check{f}^+$  for an element of  $V^B$  s.t.

$$[\check{F}' / \check{f}^+ \equiv \{f(d) \mid d \in \check{F}'\}]^{B'} = 1.$$

REMARK

$$\text{as } b' \geq [\check{b}' \in \check{F}^+]^{B'} \leq b' = [\check{b}' \in \check{F}']^{B'}$$

$$[\check{F}^+ \in \check{F}']^{B'} = 1.$$

T6227 ZFT  $[\check{F}' / \check{f}^+$  is generic for  $\check{B}' / \check{f}^+$  over  $\check{V}[\check{F}]]^{B'} = 1$ .

Proof. By T6213,  $[\check{V}[\check{F}] \equiv \check{V}]^B = 1$ ,

so that in  $V^{B'}$ ,  $V^B = V[\check{F}]$ :

$$\text{that is, } [x \in \check{V}[\check{F}]]^{B'} = \sum_{y \in V^B}^{B'} [x \equiv y]^{B'}$$

for  $x \in V^{B'}$ . It is required to prove that

$$[\bigwedge D \in \check{V}[\check{F}] \quad D \text{ dense and } \leq\text{-closed in } \check{B}' / \check{f}^+ \\ \rightarrow \check{F}' / \check{f}^+ \cap D \neq \check{0}]^{B'} = 1.$$

Let therefore  $b \in B$ ,  $\check{D} \in V^B$  s.t.

$b = [\check{D}$  is dense and  $\leq$ -closed in  $\check{B}' / \check{f}^+ ]^B$ ; it must be shown that  $b \leq [\check{D} \cap \check{F}' / \check{f}^+ \neq \check{0}]^{B'}$ .

If  $b = 0$ , there is nothing to prove. Suppose  $b \neq 0$ , and set  $D^* = \{b' \in |B'| \mid b \leq [f(\check{b}') \neq 0 \rightarrow f(\check{b}') \in \check{D}]^{B'}\}$

$$= \{b' \in |B'| \setminus \{0\} \mid b \cdot h(b') \leq [f(\check{b}') \in \check{D}]^{B'}\} \cup \{0\}$$

I assert that  $D^*$  is dense and  $\leq$ -closed in  $B'$ .



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$$[\dot{F}' \text{ is } \check{V}\text{-generic on } \check{B}' \text{ qua } \hat{P}0]^{B'} = \mathbb{1},$$

$$[\forall b' \in |\check{B}'| \quad b' \in \dot{F}' \wedge b' \in \check{D}^*]^{B'} = \mathbb{1},$$

$$\text{and } \therefore b = [b \in \dot{F} \wedge \forall b' \in |\check{B}'| \quad b' \in \dot{F}' \wedge b' \in \check{D}^*]^{B'};$$

$$[b' \in \dot{F}']^{B'} = b' \leq h(b') = [h(b') \in \dot{F}]^{B'} \\ = [h(b') \in \dot{F}]^{B'};$$

$$\text{so } b \leq [\forall b' \in |\check{B}'| (b' \in \dot{F}' \wedge b' \in \check{D}^* \wedge h(b') \in \dot{F} \wedge b' \in \check{D}^*)]^{B'}$$

$$\leq [\forall b' \in |\check{B}'| (f(b') \in \check{D} \wedge b' \in \dot{F}')]^{B'}$$

(by definition of  $\check{D}^*$ )

$$= [\dot{F}' / \dot{F}^+ \cap \check{D} \neq \emptyset]^{B'}.$$

QED

$$T 6228 \text{ ZFT } [\check{V} \equiv \check{V}[\dot{F}][\dot{F}' / \dot{F}^+]]^{B'} = \mathbb{1}.$$

Proof. By T6213 it suffices to show that

$$[\check{V}[\dot{F}'] \equiv \check{V}[\dot{F}][\dot{F}' / \dot{F}^+]]^{B'} = \mathbb{1}.$$

Reason in  $\check{V}^{B'}$ : as  $\dot{F} = |\check{B}'| \cap \dot{F}'$ ,

and  $\dot{F}^+ = \{b' \in |\check{B}'| \mid \forall b \in \dot{F} \quad b \leq b'\}$ ,

the left hand side contains the right. Conversely

the projections

$$|\check{B}'| \longrightarrow |\check{B}'| / \dot{F}^+ \longrightarrow (|\check{B}'| / \dot{F}^+) / (\dot{F}' / \dot{F}^+) = \mathbb{2}$$

compose to give a complete homomorphism:  $\check{B}' \rightarrow \mathbb{2}$

with kernel the ideal dual to  $\dot{F}'$ , which is  $\therefore \in \check{V}[\dot{F}][\dot{F}' / \dot{F}^+]$

QED