

¶1. Partial orderings and complete Boolean algebras.

Notation I use the letters A, B, C, D, E for CBAs, and Scott's arithmetical notation [BVM p 3] for the Boolean operations. I write $|B|$ for the underlying set of B , and $\emptyset, \mathbb{1}$ for the minimum and maximum elements. So

$$B = \langle |B|, \cdot, +, \emptyset, \mathbb{1} \rangle.$$

\leq is the associated partial ordering of elements of $|B|$ given by $b \leq c \iff b \cdot c = b$. Superscript B will be used if necessary; ordinarily I shall not distinguish the $\emptyset, \mathbb{1}$ of one algebra from those of another.

Let B be a CBA (which is a set.) Working in ZF we can construct a class term V^B which defines a collection of sets that in a reasonable sense is a B -valued model of ZF and is maximal in that sense.

D 6100
[BVM]

For $\alpha \in \text{On}$ (the class of ordinals) define

$$V_\alpha^B = \{u \in B^{\text{dom}(u)} \mid \forall \xi < \alpha \text{ dom}(u) \subseteq V_\xi^B\},$$

$$V^B = \bigcup \{V_\alpha^B \mid \alpha \in \text{On}\},$$

Then

$$V^B = \{u \in B^{\text{dom}(u)} \mid \text{dom}(u) \subseteq V^B\},$$

so that in Scott's phrase, a Boolean valued set is a Boolean valued function defined on Boolean valued sets.

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[6101]

Associate with V^B a language, assumed to be "derived in V ", which has

a class-forming operator E

variables: e, \dot{e}, \dots

quantifiers: \forall, \exists ("for all" "for some")

brackets: $[,]$

connectives: \wedge, \vee, \neg

two two-place predicate symbols: \equiv, \in

and for each $x \in V^B$ a name \dot{x} for x . We shall assume that $\dot{\dot{x}} = \dot{x}$. Call this language L^B . I shall write " $\phi \in L^B$ " to mean " ϕ is a well-formed formula of L^B ".

Note that I use the same signs \forall, \exists for quantifiers as for the language of set theory; but I follow D. Jensen in writing (frequently) $\dot{x}, \dot{f} \dots$ for elements of V^B . L^B is an object language, and I accordingly place \circ over the signs for sundry concepts to indicate that they are to be written out in L^B ; e.g.

$\dot{\dot{T}}$ is the L^B -term $E\dot{e}[e \equiv e]$

whereas \dot{T} is the universal class, and $\dot{\dot{T}}$ (to be introduced later) is the image of T in the canonical embedding of V in V^B .

D 6101 The evaluation in B of the sentences of

[BVM] the language L^B is defined according to the following principles: define two functions $E(\cdot, \cdot)$, $I(\cdot, \cdot)$ from $V^B \times V^B$ to B by recursion:

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[6102]

$$E^B(u, v) = \sum_{y \in \text{dom}(v)} (v(y) \cdot I^B(u, y))$$

$$I^B(u, v) = \prod_{x \in \text{dom}(u)} (u(x) \Rightarrow E^B(x, v)) \cdot \prod_{y \in \text{dom}(v)} (v(y) \Rightarrow E^B(y, u)).$$

Then the evaluation function $\llbracket \cdot \rrbracket^B$ is given by

$$\llbracket u \in v \rrbracket^B = E^B(u, v)$$

$$\llbracket u = v \rrbracket^B = I^B(u, v)$$

$$\llbracket \text{And}(\dot{a}, \dot{b}) \rrbracket^B = \prod_{u \in V^B} \llbracket \dot{a}(u) \rrbracket^B$$

$$\llbracket \dot{a} \wedge \dot{b} \rrbracket^B = \llbracket \dot{a} \rrbracket^B \cdot \llbracket \dot{b} \rrbracket^B$$

$$\llbracket \neg \dot{a} \rrbracket^B = -\llbracket \dot{a} \rrbracket^B$$

$$\llbracket u \in \text{Ex}(\dot{a}) \rrbracket^B = \llbracket \dot{a}(u) \rrbracket^B$$

This definition cannot be carried out in ZF for all \dot{a} simultaneously : but for each $\dot{a}(x)$ the function $x \mapsto \llbracket \dot{a}(x) \rrbracket^B$ may be defined.

The fundamental theorem of forcing in term of BVMs is:

T 6102 Let Ω be an axiom of ZF (including [BVM] the axioms of predicate logic). $ZF \vdash \llbracket \dot{\Omega} \rrbracket^B = 1$.

Further the property of having B-value 1 is preserved under deduction, and so if $ZF \vdash \alpha$, $\llbracket \dot{\alpha} \rrbracket^B = 1$. Finally the following may be proved in ZF:
 $A \in C \rightarrow \llbracket A \in C \rrbracket = 1$.

[14]

[6103]

V^B may be regarded as an extension of the universe V , for there is a natural embedding of V in V^B : define, for $x \in V$,

$$\text{D 6103} \quad \check{x} = \{\langle \check{y}, 1 \rangle \mid y \in x\}.$$

[BVM page 9]

Then $\check{x} \in V^B$: indeed, writing $\mathcal{D} = \{0, 1\}$ the CBA of two elements, which is a regular subalgebra of B we see that $\check{x} \in V^2 \subseteq V^B$. The mapping $\check{\cdot}$ has the following properties:

$$(i) \forall u \in V^2 \forall y \in V [\check{u} \equiv \check{y}]^B = 1$$

$$(ii) x \in y \longleftrightarrow [\check{x} \in \check{y}]^B = 1$$

$$(iii) x = y \longleftrightarrow [\check{x} \equiv \check{y}]^B = 1,$$

$$(\text{and } x \neq y \longleftrightarrow [\check{x} \equiv \check{y}]^B = 0).$$

Thus V and V^2 are "isomorphic."

It is convenient to add a predicate letter $\check{V}(\cdot)$ to the language L^B ; extend the definition of evaluation by

$$\text{D 6104} \quad [\check{V}(v)]^B = \sum_{u \in V}^B [\check{v} \equiv \check{u}]^B, \quad \text{for } v \in V^B$$

I shall often write " $v \in \check{V}$ " for " $\check{V}(v)$ ".

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[6105]

\check{V} denotes the "standard" universe and evidently we have the schema of ZF:

$$[\check{V} \text{ is an inner model of } \text{ZF}]^B = 1;$$

and $\text{ZF} \vdash \text{AC} \rightarrow [\dot{\text{AC}}^{\check{V}}]^B = 1.$

D 6105 A subalgebra B' of a cBA B is regular (Scott calls it complete) \leftrightarrow for every nonempty $X \subseteq |B'|$ for which $\sum^{B'} X$ exists,

$$\sum^{B'} X = \sum^B X.$$

In symbols:

$$B' \subseteq_{\text{reg}} B.$$

T 6106 (ZF) If B' is also complete, then
[BVM page 8]

$$\check{V}^{B'} \subseteq \check{V}^B$$

and for $x, y \in \check{V}^{B'}$,

$$[\dot{x} \in \dot{y}]^{B'} = [x \in y]^B$$

$$[\dot{x} = \dot{y}]^{B'} = [x = y]^B.$$

The following, contained essentially in BVM pp 14 and 15, is useful to keep in mind:

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[6107]

T 6107 ZF+ Let $\mathbb{B}' \subseteq_{reg} \mathbb{B}$, and \mathbb{B}', \mathbb{B} both cBAs.

Let $x \in V^{\mathbb{B}'}$; then

$$[\forall y \in x \dot{\alpha}(y)]^{\mathbb{B}} = \prod_{y \in V^{\mathbb{B}'}} [\dot{\alpha}(y)]^{\mathbb{B}} \cdot [\dot{\alpha}(y)]^{\mathbb{B}}$$

In particular, in the case $\mathbb{B}' = \mathcal{D}$:

$$\text{if } x \in V, \text{ then } [\forall y \in x \dot{\alpha}(y)]^{\mathbb{B}} = \prod_{y \in x} [\dot{\alpha}(y)]^{\mathbb{B}}$$

The further development of the general theory of BVMs is given in the appendix BVM. Scott uses AC, but only, as far as we are concerned, to prove on the one hand that $[\dot{\alpha}]^{\mathbb{B}} = 1$, and on the other the following very useful rule which he calls the Maximum Principle (page 24):

T 6108 ZF+AC \vdash For any $\dot{\alpha}(v)$ of $L^{\mathbb{B}}$, there is a $v \in V^{\mathbb{B}}$ such that

$$(\dot{\alpha}(v))^{\mathbb{B}} = [\forall v \dot{\alpha}(v)]^{\mathbb{B}}$$

Some remarks about Boolean reasoning: often a proof that $[\dot{\alpha}]^{\mathbb{B}} = 1$ for some $\dot{\alpha}$ and \mathbb{B} will fall into two parts. First it is shown that $[\dot{\beta}]^{\mathbb{B}} = 1$ by arguing from the peculiar properties of \mathbb{B} , for some other sentence $\dot{\beta}$; and thereafter the argument is purely set-theoretical in that it can be shown that $ZF \vdash \dot{\beta} \rightarrow \dot{\alpha}$, so that $[\dot{\alpha}]^{\mathbb{B}} = 1$ then follows by the general theorem T 6102. When a piece of reasoning is purely set-theoretical in this sense, I shall say: "Reason in $V^{\mathbb{B}}$:"; and give an informal proof in the traditional mathematical style, with free

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[6109]

variables abounding. It is to be understood that these statements are to be translated into sentences (with no free variables) of \mathcal{L}^B , and then my argument to be construed as the assertion that each of these has B -value \mathbb{I} . When reasoning informally, I continue to write \dot{x}, \dot{y} etc for the elements in V^B , but write \equiv for \equiv and \in instead of Σ . Thus if I say "In V^B , $\dot{x} = \dot{y}$ " I mean that $[\dot{x} \equiv \dot{y}]^B = 1$; the two Boolean-valued functions \dot{x} and \dot{y} may well not be the same in V . (Cf BVM page 5, half way down). It is indeed possible to factor the model V^B so that $[\dot{x} \equiv \dot{y}] = 1 \rightarrow x = y$, but then the inclusion $V^B \subseteq V^{B'}$ in T 6106 becomes an embedding, which is less convenient.

I now consider partial orderings. With only trivial exceptions, all the partial orderings considered have a greatest element which will always be denoted by \mathbb{I} or \mathbb{I}^P , and have no minimal elements. Further conventions are mentioned shortly.

D 6109 (ZF) Let $P = \langle |P|, \leq \rangle$ be a partial ordering.

A set $X \subseteq |P|$ is dense in $P \iff$

$\forall p \in |P| \forall p' \in X \quad p' \leq p. \quad X \subseteq \underline{\leq\text{-closed}} \iff$

$\forall p \in X \quad \forall p' \in |P| \quad p' \leq p \rightarrow p' \in X. \quad p, p' \in |P|$

are compatible $\iff \forall p'' [p'' \leq p \wedge p'' \leq p']. \quad$ [otherwise incompatible].

" $p \leq p'$ " may be read " p is stronger than p' ." All the partial orderings encountered, with minor exceptions, have the property that $\forall p \forall p', p'' (p' \leq p \wedge p'' \leq p \wedge p' \leq p'' \text{ and } p'' \text{ are incompatible})$.

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[6110]

A partial ordering P is densely imbedded in $P' = \langle |P'|, \leq' \rangle$ iff there is a 1-1 map ϕ of $|P|$ onto a dense subset of $|P'|$ such that $p_1 \leq p_2 \iff \phi(p_1) \leq' \phi(p_2)$.

Write P_0 for partial ordering. When the properties of the partial ordering \leq of a BA B are discussed, it is convenient to exclude the element 0 . Therefore I specify that an assertion of the form " B qua P_0 has the property Q " is to be read as

"the partial ordering $\langle |B| \setminus \{0\}, \leq^B \cap (|B| \setminus \{0\}) \rangle$ has the property Q "; I shall only use the phrase "qua P_0 " in this special sense. Thus " P is dense in B qua P_0 " (usually shortened to " P is dense in B ") means $\forall b \in B \setminus \{0\} \exists p \in |P| (p \leq b)$ and $|P| \subseteq |B| \setminus \{0\}$.

" B is a dense subalgebra of C " means " B is a subalgebra of C and B qua P_0 is dense in C qua P_0 ". The following point occurs frequently:

+ 6110 ZF \vdash Let C' be a dense subalgebra of a cBA C . Then $C' \subseteq_{reg} C$.

Proof: Let $0 \neq x \in |C'|$, and let $b = \sum^C x$, $b' = \sum^{C'} x$.

Then $b \leq b'$; if $b \neq b'$, then $b' - b \neq 0$, so by density there is a $b'' \in |C'|$ s.t.

$0 \neq b'' \leq b' - b$; as $b'' \leq b'$, there is a $b''' \in C$ s.t. $b'' \cdot b''' \neq 0$; but that contradicts $b'' \cdot b = 0$ and $b'' \leq b$. QED.

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[6111]

D 6111 In the situation of T 6110, \mathbb{C} is said to be a regular minimal completion (r.m.c. for short) of \mathbb{C}' and, as will be seen later, is characterised up to isomorphism by the properties

\mathbb{C}' is a dense subalgebra of \mathbb{C}

\mathbb{C} is complete.

The importance of partial orderings is that in practice \mathbb{B} is only indirectly defined. Consider the original proof by Cohen of the independence of $V = L$. He wished to construct a real $x \in \omega$, and did so by considering conditions on x . A condition was a finite set of statements of the form $\tilde{n} \in \tilde{x}$ or $\neg \tilde{n} \in \tilde{x}$ (where \tilde{y} is a name for y and $n \in \omega$), and no condition P contained both $\tilde{n} \in \tilde{x}$ and $\neg \tilde{n} \in \tilde{x}$. P was defined to be a stronger condition than Q (abbreviated to $P \preceq Q$) if $Q \subseteq P$. Thus the set $|P|$ of conditions was partially ordered by \preceq , and the empty set was the maximum element.

That situation is perfectly general, there being a natural correspondence between cBAs \mathbb{B} and POs \mathbb{P} , which I shall now discuss.

T 6112 ZF \vdash Let $X = \langle X, \tau \rangle$ be a topological space, and let $|\mathbb{B}|$ be the collection of all regular open subsets of X : that is, $Y \in |\mathbb{B}|$ iff $Y = \text{int cl } Y$, for $Y \subseteq X$. Endow $|\mathbb{B}|$ with the operations defined by

$$- Y = \text{int}(X \setminus Y)$$

$$Y + Z = \text{int cl } (Y \setminus Z)$$

Then $\mathbb{B} = \langle |\mathbb{B}|, +, - \rangle$ is a cBA.

For if $\{x_i \mid i \in I\}$ is a set of elements of $|\mathbb{B}|$, the set $\text{int cl} \bigcup_{i \in I} x_i$

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[6113]

is $\in |\mathbb{B}|$ and is the supremum of $\{x_i \mid i \in I\}$ in the Boolean sense.

\mathbb{B} is the regular open algebra of \mathcal{X} . A little elementary topology shows that

$$T 6113 \quad Y \circ Z = Y \cap Z, \text{ and so } Y \leq Z \rightarrow Y \leq Z.$$

Let now $\mathbb{P} = \langle |\mathbb{P}|, \leq \rangle$ be a partially ordered set. For $p \in \mathbb{P}$ set

$$D 6114 \quad O_p^{\mathbb{P}} = \{p' \in |\mathbb{P}| \mid p' \leq p\}.$$

Then $O_p^{\mathbb{P}} \cap O_{p'}^{\mathbb{P}}$ is either empty or equal to $\bigcup \{O_q^{\mathbb{P}} \mid q \in O_p^{\mathbb{P}} \cap O_{p'}^{\mathbb{P}}\}$, and so we may define a topology $\tau_{\mathbb{P}}$ on $|\mathbb{P}|$ by taking the family $\{O_p^{\mathbb{P}} \mid p \in |\mathbb{P}|\}$ as a basis. Associate with \mathbb{P} the regular open algebra \mathbb{B} of the space $\langle |\mathbb{P}|, \tau_{\mathbb{P}} \rangle$. $O_p^{\mathbb{P}}$ will also be written $O_p^{\mathbb{B}}$ or plain O_p . \mathbb{B} is the algebra over \mathbb{P} . Remark that

$$\begin{aligned} \Sigma^{\mathbb{B}} \mathcal{X} &= \{p \mid \bigwedge p' \leq p \vee p'' \leq p' \vee q \in \mathcal{X} \text{ } p'' \leq q\} \\ -(\Sigma^{\mathbb{B}} \mathcal{X}) &= \{p \mid \neg(p' \leq p \vee q \in \mathcal{X} \text{ } p' \leq q)\}. \end{aligned}$$

T 6115 $ZF \vdash$ (i) Set $\mathbb{P}^* = \{O_p \mid p \in |\mathbb{P}|\}$ and $\mathbb{P}^* = \langle \mathbb{P}^*, \subseteq \rangle$. Then $\mathbb{P}^* \cong \mathbb{P}$ as POs, and \mathbb{P}^* is dense in \mathbb{B} (qua PO).

(ii) Conversely let \mathbb{C} be a BA, and set $\mathbb{P} = \langle \mathbb{C} \setminus \{0\}, \leq \rangle$; \mathbb{B} , the algebra over \mathbb{P} is an r.m.c. of \mathbb{C} , and the mapping

$$b \mapsto O_b^{\mathbb{P}} \quad (b \in |\mathbb{C}|)$$

is a regular embedding of \mathbb{C} in \mathbb{B} .

(iii) If in (ii) \mathbb{C} is complete then that mapping is onto, and so the two cBAs are isomorphic.

(iv) If \mathbb{P}_1 is dense in \mathbb{P}_2 , then the algebra over \mathbb{P}_1 is isomorphic to the algebra over \mathbb{P}_2 .

All these, save (i) which is immediate from the definitions and T 6113, will be proved as corollaries to T 6117.

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[6116]

Consider Cohen's original argument again, or the version of forcing expounded by Jensen in [4]. There is a set of conditions \mathbb{P} with a partial ordering \leq , and associated with it is a ramified language \mathcal{L} , and a forcing relation, \Vdash , between the conditions and sentences of \mathcal{L} , with the properties:

$$F_1 \quad \bigwedge_{P \in \mathbb{P}} \forall \alpha \vee P' \leq P \quad P \Vdash \alpha \text{ or } P \Vdash \neg \alpha$$

$$F_2 \quad \bigwedge_{P \in \mathbb{P}} \bigwedge_{P' \in \mathbb{P}} \bigwedge_{\alpha} (\alpha(P \Vdash \alpha) \text{ and } P \leq P' \rightarrow P' \Vdash \alpha)$$

$$F_3 \quad \bigwedge_{P \in \mathbb{P}} \bigwedge_{\alpha} (\alpha \Vdash \neg \alpha \rightarrow \neg(P \Vdash \neg \alpha))$$

where α ranges over sentences of \mathcal{L} . Usually the elements of \mathbb{P} are conditions on a new object to be added to the universe, and α has a name for this new object, and a name for each element of the universe. Thus in Cohen's original construction, the conditions are finite sets of statements about a new subset of ω , for which \mathcal{L} has the name \dot{x} . Jensen shows in [4] that from the properties F 1 - 3 the fundamental theorem of forcing may be established in the form "If α is an axiom of ZF, then $\dot{1} \Vdash \alpha$ ", where $\dot{1}$ is the maximal element of \mathbb{P} .

It is a nuisance though to deal with ramified languages. Instead, let \mathbb{P} be the partially ordered set of conditions, and \mathbb{B} the algebra over \mathbb{P} . Form $\mathcal{L}^{\mathbb{B}}$ and $\mathcal{V}^{\mathbb{B}}$: now the language $\mathcal{L}^{\mathbb{B}}$ is not ramified. For $P \in \mathbb{P}$ and $\dot{\alpha} \in \mathcal{L}^{\mathbb{B}}$, define

$$D 6116 \quad P \Vdash \dot{\alpha} \leftrightarrow o_P^{\mathbb{B}} \leq [\dot{\alpha}]^{\mathbb{B}}$$

Then \Vdash satisfies the three axioms F 1, 2, 3.

In showing that $[\dot{\alpha}]^{\mathbb{B}} = \dot{1}$ it is only necessary to show $\{o_p | p \Vdash \dot{\alpha}\}$ is dense in \mathbb{B} . I shall often argue about $\mathcal{V}^{\mathbb{B}}$ and use $\mathcal{L}^{\mathbb{B}}$ and \mathbb{P} but not mention \mathbb{B} . Note that by definition of \Vdash , \mathbb{B} ,

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[6117]

$$P \Vdash \neg \dot{\alpha} \leftrightarrow \neg \forall Q \leq P Q \Vdash \dot{\alpha}$$

$$P \Vdash \dot{\alpha} \leftrightarrow \exists P' \leq P \forall P'' \leq P' P'' \Vdash \dot{\alpha}$$

$$P \Vdash \lambda e \dot{\alpha} \leftrightarrow \lambda t \in V^B P \Vdash \dot{\alpha}(t).$$

The following lemma will clarify the relation of P and B :

T6117 2F+ Let B be a cBA, and P a subset of $|B|$ s.t.

$$(i) \emptyset \notin P, \mathbb{I} \in P;$$

If \mathcal{X} and \mathcal{Y} are nonempty subsets of P then

$$(ii) \sum^B \mathcal{X} - \sum^B \mathcal{Y} \neq \emptyset \rightarrow \forall p \in P \ p \leq \sum^B \mathcal{X} - \sum^B \mathcal{Y};$$

$$(iii) -\sum^B \mathcal{Y} \neq \emptyset \rightarrow \forall p \in P \ p \leq -\sum^B \mathcal{Y};$$

$$(iv) p \in P \text{ and } p \circ \sum^B \mathcal{X} \neq \emptyset \rightarrow \exists p' \in P (p' \leq p \circ \sum^B \mathcal{X}).$$

Let C be the algebra over $P = \langle P, \leq \rangle$.

Then there is a regular embedding $\pi: C \rightarrow B$
such that for $p \in P$,

$$\pi(O_p^C) = p.$$

Proof: Throughout, \mathcal{X}, \mathcal{Y} are variables for non-empty subsets of P ;

p, p' are variables for elements of P .

Sublemma: If $\sum^C \{O_p \mid p \in \mathcal{X}\} = \sum^C \{O_p \mid p \in \mathcal{Y}\}$,

$$\text{then } \sum^B \mathcal{X} = \sum^B \mathcal{Y}.$$

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Proof of the sublemma

If not, then either $\sum^B \mathcal{X} - \sum^B \mathcal{Y} \neq 0$ or $\sum^B \mathcal{Y} - \sum^B \mathcal{X} \neq 0$; wlog suppose the first. By (ii) there is a $p_0 \leq \sum^B \mathcal{X} - \sum^B \mathcal{Y}$. Then

$$\neg \forall p_1 \leq p_0 \forall p_2 \leq p_1 \forall q \in \mathcal{Y} (p_2 \leq q)$$

(for otherwise $p_0 \cdot q \neq 0$) and so

$$(a) \quad O_{p_0} \cdot^C \sum^C \{o_q | q \in \mathcal{Y}\} = O_{p_0} \cap \sum^C \{o_q | q \in \mathcal{Y}\} = \emptyset.$$

But for any $\lambda p_2 \leq p_0 \forall p \in \mathcal{X}$ ($p_2 \cdot p \neq 0$), so by (iv)
(setting the \mathcal{X} in (iv) = $\{p_2\}$),

$$\lambda p \leq p_0 \forall p_2 (p_2 \leq p_1 \cdot p).$$

But that shows that $O_{p_0} \subseteq \sum^C \{o_q | q \in \mathcal{X}\}$, and so

$$(b) \quad O_{p_0} \leq^C \sum^C \{o_q | q \in \mathcal{X}\}.$$

(a) and (b) together contradict the hypothesis of the sublemma.

Proof of T 6.117.

Now define π as follows: let $c \in \mathbb{C}$, $c \neq \emptyset$; select any \mathcal{X} s.t.

$$c = \sum^C \{o_p | p \in \mathcal{X}\};$$

then set $\pi(c) = \sum^B \mathcal{X}$.

Set also $\pi(\emptyset) = \emptyset$.

(By the sublemma, the choice of \mathcal{X} for given c is irrelevant).

[24] for p take $\mathcal{X} = \{p\}$,

Then (A) $\pi(O_p) = p$.

(B) Let $x \in \mathbb{C} \setminus \{0\}$. Pick \mathcal{X} so that

$$x = \bigcup \{O_q \mid q \in \mathcal{X}\}$$

(possible, as x is open). Then

$$x = \{p \mid \forall q \in \mathcal{X} \quad q \geq p\}$$

$${}^c x = \{p \mid \neg \forall p_1 \leq p \forall q \in \mathcal{X} \quad p_1 \leq q\}.$$

Set $b = \sum {}^B \mathcal{X} = \pi(x)$.

I assert that $-{}^c x = \{p \mid p \cdot b = 0\}$; for

(1) $p \notin -{}^c x \rightarrow \forall p_1 \leq p \forall q \in \mathcal{X} \quad p_1 \leq q$

$$\rightarrow p_1 \leq b \text{ for such a } p_1$$

$$\rightarrow p \cdot b \neq 0.$$

(2) $p \cdot b \neq 0 \rightarrow$ by (iv), as $b = \sum {}^B \mathcal{X}$,

there is a $p' \leq p \cdot b$; for this p' ,

$$\forall p_2 \in \mathcal{X} \quad (p' \cdot p_2 \neq 0); \text{ so } \forall p_3 \text{ s.t. } p_3 \leq p' \cdot p_2$$

for this p_3 $\underline{p_3 \leq p}$, and $p_3 \leq p_2 \in \mathcal{X}$, so $p \notin -{}^c x$.

$$\therefore -{}^c x = \bigcup \{O_p \mid p \cdot b = 0\} = \sum \{O_p \mid p \cdot b = 0\},$$

$$\text{and so } \pi(-{}^c x) = \sum \{p \mid p \cdot b = 0\}.$$

Set $\mathcal{Y} = \{p \mid p \cdot b = 0\}$. Now $\sum {}^B \mathcal{Y} \leq -{}^B b$.

[ACTUALLY, $\mathcal{Y} = -{}^c x$].

If $b = 1$, then $-{}^B b = 0$, so $\sum {}^B \mathcal{Y} = 0$, so that

$$-{}^c x = {}^B \mathcal{Y}, \quad x = {}^B \mathcal{C}, \text{ and } \pi(-{}^c x) = -\pi(x).$$

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So suppose $b \neq 0$. Then by (iii) V_p ($p \cdot b = 0$), and so \mathcal{Y} is not empty.

If $\sum \mathcal{Y} \neq -b$, then

$$\sum^B x_i \cup \sum^B y_j = \sum^B (x_i \cup y_j) \neq 1,$$

$$\text{and so } V_p - p \cdot \sum^B (x_i \cup y_j) = 0, \text{ by (iii)}$$

$$\therefore p \cdot \sum^B x_i = 0; \text{ i.e. } p \cdot b = 0 \text{ and so } p \in \mathcal{Y},$$

$$\text{contradicting } p \cdot \sum \mathcal{Y} = 0.$$

Thus $\sum \mathcal{Y} = -b$, and so $\pi(-c_x) = -\sum^B \pi(x_i)$.

(c) Let $\{y_i | i \in I\} \subseteq \mathbb{C}$.

$$\text{Let } x = \sum^C \{y_i | i \in I\}.$$

Pick x_i s.t. $y_i = \bigcup \{0_p | p \in x_i\}$.

[possible as the y_i are open].

Then

$$x = \sum^C \{0_p | p \in \bigcup_{i \in I} x_i\}$$

$$\text{and } \pi(x) = \sum^B \bigcup \{x_i | i \in I\}$$

$$= \sum_{i \in I}^B \sum^B x_i,$$

$$= \sum_{i \in I}^B \pi(y_i).$$

So π is a regular homomorphism.

Further π is 1-1, for if $c \neq 0$, choose p s.t. $0_p \subseteq c$; then $\pi(c) \ni \pi(0_p) = p \neq 0$.

QED.

[26]

(6118)

T 6118 2F \vdash (i) Let $|P|$ be a dense subset of $|B|$ containing \mathbb{I} ; then the algebra over $|P|$ is isomorphic to B .

(ii) Any isomorphism $\varphi: B \xrightarrow{\sim} B'$ extends to an isomorphism of their r.m.c.'s.

Proof (i) Let C be the algebra over $|P|$: take $P = |P|$ in T 6117; then hypotheses (i) — (iv) are easily checked using the definition of density; let π be the embedding constructed above. π is onto, for let $b \in B$, $b \neq 0$; by density there is an $X \subseteq |P|$ s.t. $b = \sum^C B_X$.

$$\text{Then } b = \pi \left(\sum^C \{0_p \mid p \in X\} \right).$$

(ii) by (i), (as an algebra is dense, quā P_0 , in its r.m.c.) and composition of embeddings.

It is now easy to check T 6115 (ii) — (iv); (ii) holds as B is dense in itself; (iii) again as an algebra is dense in any r.m.c. (iv) because density is a transitive relation.

An important property enjoyed by certain cBAs is that of homogeneity.

D 6119 A cBA \mathbb{D} is homogeneous $\iff \forall d, d' \in \mathbb{D} \setminus \{\mathbb{0}, \mathbb{1}\}$, there is an automorphism ϕ of \mathbb{D} s.t. $\phi(d) = d'$.

By the Corollary, (BVM p. 49.), ϕ lifts to an automorphism of $V^\mathbb{D}$: $\phi(\mathbb{0}) = \mathbb{0}$ & $\phi(\mathbb{1}) = \mathbb{1}$, $\phi|V^2$ is

[27)

[6120]

the identity; and hence the important

T6120 2F+ Let \mathbb{D} be a homogeneous cBA;
 $x_1, \dots, x_k \in V$. Then for all 2F-formulae
 $\sigma(x_1, \dots, x_k)$ with no other free variables,

$$[\sigma(\check{x}_1, \check{x}_2, \dots, \check{x}_k)]^{\mathbb{D}} = 0 \text{ or } 1.$$

Proof: If $b = [\sigma(\check{x}_1, \dots, \check{x}_k)]^{\mathbb{D}} \neq 0 \text{ or } 1$,
then there is a ϕ s.t. $\phi(b) = -b$.

But then

$$\begin{aligned} -b &= \phi([\sigma(\check{x}_1, \dots, \check{x}_k)]^{\mathbb{D}}) \\ &= [\sigma(\phi(\check{x}_1), \dots, \phi(\check{x}_k))]^{\mathbb{D}} \\ &= [\sigma(\check{x}_1, \dots, \check{x}_k)]^{\mathbb{D}} = b \quad \text{*}. \end{aligned}$$

REMARK I have extended the language by adding \check{V} ,
and therefore before the proof of the corollary is
complete, must note that

$$\phi([x \in \check{V}]) = [\phi(x) \in \check{V}],$$

which holds as the elements \check{y} of V^2 are fixed by ϕ .

D6121 For $d \in |\mathbb{D}|$ write $(d) =$ (ideal generated by d).

There is a useful criterion for homogeneity which I
shall now state.

D6122 For $d \in |\mathbb{D}| \setminus \{0\}$, define an algebra $\mathbb{D}/d = \langle D_d, \cdot_d, -_d \rangle$
by $D_d = (d) \Rightarrow d_1 \cdot d_2 = d_1 \circ d_2 ; -_d d_1 = d \circ -_d d_1$
Then $0_d = \emptyset$, and $1_d = d$.

[28]

[6123]

T6123 Then \mathbb{D}/d is a cBA, and is isomorphic both to the algebra over $\{d' \neq 0 \mid d' \leq d\}$, and to $\mathbb{D}/(-d)$.

T6124 Suppose $d_1 \leq d_2$ and $\mathbb{D}/d_1 \cong \mathbb{D}$. Then $\mathbb{D} \cong \mathbb{D}/d_2$.

T6124 is really the Schröder-Bernstein theorem applied to the cardinal algebra of isomorphism types of CBAs. For a proof, see the appendix of Sikorski [12].

Now let \mathbb{D} be atomfree (which is always the case in applications of forcing).

T6125 $\text{ZF} + \mathbb{D}$ is homogeneous $\iff 1d \neq \emptyset, \mathbb{D} \cong \mathbb{D}/d$.

Proof: \rightarrow . Then $1d_1, d_2 \neq \emptyset, \mathbb{D}/d_1 \cong \mathbb{D}/d_2$; so

$$\mathbb{D} \cong \mathbb{D}/d_1 + \mathbb{D}/-d_1 \cong \mathbb{D}/d_2 + \mathbb{D}/-d_2.$$

Pick $d_2, d_3 \neq \emptyset$ s.t. $d_1 = d_2 + d_3$, $d_2 \circ d_3 = \emptyset$.

Then $\mathbb{D}/d_1 \cong \mathbb{D}/d_2 + \mathbb{D}/d_3 \cong \mathbb{D}/d_2 + \mathbb{D}/-d_2 \cong \mathbb{D}$.

\leftarrow . Let $d_1, d_2 \neq \emptyset, \mathbb{D}/d_1 \cong \mathbb{D}/d_2$
 $\phi: \mathbb{D}/-d_1 \cong \mathbb{D}/-d_2$.

$$\text{Set } \chi(d) = \psi(d \cdot d_2) + \phi(d \circ -d_2).$$

χ is an automorphism of \mathbb{D} with $\chi(d_1) = d_2$.

$\psi \in \mathbb{D}$

TT 6124 and 5 yield

T6126 $\text{ZF} +$ If $\emptyset \notin P \subseteq |\mathbb{D}|$ is dense in \mathbb{D} and for all $d \in P$, $\mathbb{D} \cong \mathbb{D}/d$, then \mathbb{D} is homogeneous.

Finally some standard facts about BVMs in the theory $\text{ZF} + \text{AC}$.

[29]

[6127]

D 6127 (ZF+AC) Let \beth be an uncountable cardinal, B a cBA. B satisfies the \beth chain condition (\beth -c-c) \iff every subset of pairwise disjoint (i.e. $\inf = 0$) non-0 elements of $|B|$ is of cardinality $< \beth$.

The countable chain condition is then the \aleph_1 -c-c.

T 6128 (ZF+AC) Let B be a cBA, α a cardinal, δ a limit ordinal; and $\beth = \text{cf}(\delta)$. If B satisfies the \beth -c-c, then

$$[\text{cf}(\delta) = \beth]_B^B = 1.$$

Proof. Let $0 \neq b =_{\text{df}} [f: \beta \rightarrow \delta]_B^B$, where $\beta < \delta$, $f \in V^\beta$. For each $\gamma < \delta$, the set

$$X_\gamma =_{\text{df}} \{b \cap [f(\gamma) = \delta] \mid \delta < \alpha\}$$

is of pairwise disjoint elements; so

$$Y_\gamma =_{\text{df}} \{\delta < \alpha \mid b \cap [f(\gamma) = \delta] \neq \emptyset\}$$

is of cardinality $< \beth$ and so is bounded below α .

Let $\beta_\gamma = \sup Y_\gamma$. The $\{f_\gamma \mid \gamma < \beta\}$ is of cardinality $= \bar{\beta} < \beth = \text{cf}(\alpha)$, and so is also bounded below α , by η say.

Let $\eta < \zeta < \alpha$. Then $b \leq [\text{Range}(f) \subseteq \zeta]_B^B$

$$\leq [\text{Rg}(f) \text{ is bounded below } \zeta]_B^B$$

Hence for every $f, \beta < \delta$, $[f: \beta \rightarrow \delta \rightarrow \text{Rg}(f) \text{ is bounded below } \zeta]_B^B = 1$; the rest of the proof uses the definition of $[\]$ and

[36]

[6129]

See theorem (BVM page 28) that in a Boolean-valued sense, the ordinals in V^B are those in V ;

that is, that $\{\dot{x} \in \text{On} \mid \dot{x}\}^B = \sum_{\alpha \in \text{On}} [\dot{x} = \dot{\alpha}]^B$. QED

T6129 ZF+AC \vdash If α is a ^{regular}₊ cardinal and B is a CBA satisfying the α -c.c. (a fortiori, if $|B| < \alpha$) then $[\dot{\alpha} \text{ is a cardinal}]^B = 1$.

Proof. If α is regular, then $c_f(\alpha) = \alpha$, and by T6128,

$$[\dot{c}_f(\dot{\alpha}) = \dot{\alpha}]^B = 1.$$

QED.

T6130 ZF+AC \vdash Let κ, λ be cardinals, and $2^\kappa = \lambda$.
Let $|B| \leq \kappa$. Then

$[\text{the power set of } \kappa \text{ has cardinality } \lambda]^B = 1$,
which may be written as $[2^\kappa = \lambda]^B = 1$.

Proof. Suppose not. By T6129,

$$1 = [\dot{\lambda} \text{ is a cardinal, } \dot{\lambda}^+ \text{ is a cardinal, and } \dot{\lambda} = \dot{\lambda}^+].$$

Then ex hyp. there is an $\dot{f} \in V^B$ such that

$$b = \text{def } [\dot{f}: \dot{\lambda}^+ \xrightarrow{\text{1-1}} \dot{S}(\kappa)]^B \neq \emptyset.$$

Define the map $\dot{\xi}: \dot{\lambda}^+ \rightarrow (|B|)^{\kappa}$ by

$$\dot{\xi}(p) = \{ \langle [\dot{\alpha} \in \dot{f}(p)]^B \cap b, \dot{\alpha} \rangle \mid \alpha < \kappa \},$$

for $p < \lambda^+$

[Remember that $f(x) = y$ means $\langle y, x \rangle \in f$].

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$$\begin{aligned}\xi(\beta) = \xi(\beta') &\rightarrow \forall \alpha < \kappa \left(b \cap [\alpha \in \dot{f}(\beta)] = b \cap [\alpha \in \dot{f}(\beta')] \right) \\ &\rightarrow \forall \alpha < \kappa \quad b \leq [\alpha \in \dot{f}(\beta) \leftrightarrow \alpha \in \dot{f}(\beta')] \\ &\rightarrow b \leq [\forall \alpha (\alpha \in \dot{f}(\beta) \leftrightarrow \alpha \in \dot{f}(\beta'))] \\ &= [\dot{f}(\beta) = \dot{f}(\beta')];\end{aligned}$$

but $b \leq [\dot{f} \text{ is } 1-1]$, so

$$\begin{aligned}0 \neq b &\leq [\dot{f} \text{ is } 1-1] \cdot [\dot{f}(\beta) = \dot{f}(\beta')] \\ &\leq [\beta = \beta'] \quad \text{which} = 0 \text{ or } 1;\end{aligned}$$

but cannot = 0 as $b \neq 0$, and so $[\beta = \beta'] = 1$,
 $\Rightarrow \beta = \beta'$. $\therefore \xi \text{ is } 1-1$, and so

$$\frac{|\mathcal{B}|^\kappa}{(|\mathcal{B}|)^\kappa} \geq 2^+.$$

But $|\mathcal{B}|^\kappa \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^\kappa = 2 < 2^+$
 \star .