On a generalisation of Ramsey's theorem.

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¶0. Statement of the results

The first partition theorem of Ramsey states that for any positive integer n and for any partition of the set of unordered n-tuples of natural numbers into two, there is an infinite set X of natural numbers such that any two n-tuples formed from elements of X lie in the same half of the partition; and may be proved without the axiom of choice.

Ramsey's theorem has so many applications that it is natural to hope that it remains true when "n-tuples" is replaced by "infinite subsets". (Let us call that the case $n = \infty$.) An application of the axiom of choice shows that it does not: Professor Scott, in his Seminar on partition theorems held at Stanford in California in the autumn quarter of 1967, raised the problem of refuting the case $n = \infty$ without using choice.

The main purpose of this paper is to show that under a hypothesis stronger than that of the consistency of Zermelo-Fraenkel set theory but nevertheless widely believed, Scott's problem is insoluble.

Throughout this paragraph I use the letters x, y and z as variables ranging over the set of infinite subsets of ω, the set of natural numbers.

D 6000 (ZF) A set P of subsets of ω is a <u>Scott family</u> (SF for short) iff $\bigwedge x \bigvee y \subseteq x \quad x \in P \longleftrightarrow y \notin P$.

Thus Ramsey's theorem is true for the case $n = \infty$ iff there is no Scott family: in the partition notation of Erdös and Rado, Ramsey's theorem is $\bigwedge n(\omega \longrightarrow (\omega)^n_2)$ and the case $n = \infty$ is $\omega \longrightarrow (\omega)^{\omega}_2$.

I shall now state the theorem that almost answers Scott's question.

T 6001 The consistency of the theory ZF + AC + "there is a strongly inaccessible cardinal" implies that of the theory ZF + DC + "there are no Scott families".

The implication may be proved in elementary arithmetic. Here DC is Tarski's principle of dependent choices:

> if R is a relation on a non-empty set X such that $\bigwedge v \in X \bigvee w \in X\ vRw$ then there is a mapping $f: \omega \longrightarrow X$ such that $\bigwedge i < \omega \left[f(i)\ R\ f(i+1)\right]$

and is equivalent in ZF to the weak axiom of choice, DC^{ω}, defined on page 20 of the Survey.

As an intermediate step I show that

T 6002 the consistency of the theory ZF + AC + "there is a strongly inaccessible cardinal" implies that of the theory ZF + AC + "no Scott family is definable, even if parameters for real and ordinal numbers are allowed";

a device of McAloon is then used to derive T 6001.

I shall comment later in this paragraph on the proof of T 6002; first I shall review the earlier results on SFs.

D 6003 (ZF) $x \sim_{f} y \longleftrightarrow$ the symmetric difference $x \mathbin{\Delta} y$ is finite.

Then $\sim_{f}$ is an equivalence relation; so define

D 6004 $\quad x/_{f} = \{y \mid x \sim_{f} y\}$, and let $\mathcal{F} = \{x/_{f} \mid x \subseteq \omega\}$.

A method of obtaining SFs is given by the next lemma:

T 6005 ZF $\vdash$ If there is a function $g: \mathfrak{F} \longrightarrow S(\omega)$ such that $\bigwedge u \in \mathfrak{F}\ g(u) \in u$, then there is an SF.

Proof: Let g be such a function. Define

$$P_g = \left\{ x \subseteq \omega \,\middle|\, \overline{\overline{x \Delta g(x/\xi)}} \text{ is even} \right\}.$$

Then if $n \in x$, $x \setminus \{n\} \notin P_g$ iff $x \in P_g$, so that P_g is indeed an SF, q.e.d.

Given a well-ordering of $S(\omega)$, such a g may be constructed, so

T 6006 ZF + AC $\vdash$ There is an SF.

But it has been shown by Feferman [2] that Con(ZF) implies Con(ZF + there is no such choice function for $\mathfrak{F}$). (I know not whether there is an SF in Feferman's model, in which Solovay has shown DC to hold. I conjecture that there is.) Indeed it may be shown in ZF that no "simply" definable set P can be an SF. That statement may be made precise by using the projective hierarchy of sets of reals introduced by Lusin, for the levels of which I shall use the notation of Shoenfield [14, Chapter 7, page 175] and Addison. (The projective hierarchy is also defined in Kuratowski's book [6, page 361]). Some remarks about the notation are in order, as I use slight variations of that described in Shoenfield's book.

The projective hierarchy is most conveniently investigated not in the real line itself, but in either the Baire space of functions from ω to ω or the Cantor space of functions from ω to $2 = \{0,1\}$; and of these the first is easier, as there Kleene's Normal Form theorem for $\underset{\omega\omega}{\Pi}{}^1_1$ sets holds. Now Shoenfield defines "projective" for the Baire space, but his definitions work equally well for the other.

As I am now concentrating on subsets of ω, I am compelled to avoid the Baire space.

From now on, a <u>real</u> always means a subset of ω.

Let 2^ω be the set of all reals. The topology on 2^ω is defined by taking as a basis all sets of the form

$$N_u = \{X \subseteq \omega / \bigwedge m < n \ (m \in X \text{ iff } u(m) = 1)\}$$

where $n \in \omega$ and $u: n \to 2$. (Remember that $n = \{0, 1, \ldots, n-1\}$).
The letters X, Y, and Z will now be used as variables for arbitrary subsets of ω. A <u>recursive</u> predicate of reals and numbers is one built up from the predicates $m + n = k$, $m \cdot n = k$, $m \in X$, $m < n$, and the functions f_1, f_2 defined by

$$f_1(m) = \text{the highest power of 2 dividing } m,$$
$$f_2(m) = \text{the highest power of 3 dividing } m,$$

by composition and the least-number operator μ (cf Shoenfield's book, page 109).

An <u>arithmetical</u> predicate of say X and n is one which is equivalent in set theory to one of the form

$$\bigwedge m_1 \bigvee m_2 \ldots R(m_1, m_2, \ldots, X, n)$$

where R is recursive and all the quantifiers are of number variables.

Then as in Shoenfield page 174 a Σ^1_n predicate of say X, m is defined as one equivalent in set theory to one of the form

$$\bigvee Y_1 \bigwedge Y_2 \ldots \bigvee\!\!\!\!\bigwedge Y_n \, R(Y_1, Y_2, \ldots, Y_n, X, m)$$

where R is arithmetical and the quantifiers alternate. If a real parameter is allowed in R, then the predicate is $\underset{\sim}{\Sigma}^1_n$ (bold face.)

A set $\mathcal{X}$ of reals is $\underset{\sim}{\Sigma}^1_n$ if the predicate $\ulcorner X \in \mathcal{X} \urcorner$ is $\underset{\sim}{\Sigma}^1_n$, &c; a $\underset{\sim}{\Pi}^1_n$ set is the complement of a $\underset{\sim}{\Sigma}^1_n$ set, and a $\underset{\sim}{\Delta}^1_n$ set one which is both $\underset{\sim}{\Pi}^1_n$ and $\underset{\sim}{\Sigma}^1_n$. Then the $\underset{\sim}{\Delta}^1_1$ sets are precisely the Borel sets, $\underset{\sim}{\Sigma}^1_1$ the analytic sets (A sets in Kuratowski); $\underset{\sim}{\Pi}^1_1$ = CA; $\underset{\sim}{\Sigma}^1_2$ = PCA, and so on. (cf Shoenfield [14, page 185]).

In ¶5 the projective hierarchy on the Baire space ω^ω will also be used; I shall then write $\underset{\sim}{\Pi}^1_1(\omega^\omega)$ etc for distinctness.

Two remarks:

6007 These definitions can and are assumed to be formalised in set theory, by setting up an appropriate language and defining a satisfaction relation between the formulae of the language and the structure $\langle 2^\omega, \omega, \in, +, \circ, <, f_1, f_2, 0 \rangle$.

6008 I have used the phrase "equivalent in set theory" above. I shall usually work in ZF + some form of the axiom of choice, (at least DC); now if DC is assumed, more predicates become expressible in (say) $\underset{\sim}{\Sigma}^1_1$ form. Shoenfield (loc cit page 173 (i),(ii),(iii)) gives a number of invaluable rules (which I shall call <u>Shoenfield's rules</u>, not that they are his invention) for coaxing predicates into the right shape, which work also for 2^ω. It should be stressed that two of his rules in (iii) p. 173 require some form of the axiom of choice (e.g. DC) in their proof: in his notation they are

$$\forall x \exists \alpha P(\alpha, x) \longleftrightarrow \exists \alpha \forall x P((\alpha)_x, x)$$
$$\exists x \forall \alpha P(\alpha, x) \longleftrightarrow \forall \alpha \exists x P((\alpha)_x, x).$$

The crux of the proof runs: if $\forall x \exists \alpha\ P(\alpha, x)$, <u>then to each</u> x <u>pick an</u> α; code these together as a function β, and then $\forall x P((\beta)_x, x)$. Thus in my notation, a set of the form

6009 $$\{X \mid \bigwedge n \bigvee Y\ R(n,Y,Z,X)\}$$

where R is arithmetical is, assuming DC, $\underset{\sim}{\Sigma}^1_1$. I emphasize therefore that assertions about $\underset{\sim}{\Sigma}^1_n$ sets proved in ZF are understood to be about those sets for which a $\underset{\sim}{\Sigma}^1_n$ definition has been exhibited.

I shall now state the known facts about SFs.

Four people have independently proved theorems which all are more or less the assertion that

T 6010 ZF $\vdash$ No SF is an open subset of the space 2^ω.

The first of these was Nash-Williams [9]; the others Ehrenfeucht, Cohen and Galvin [3], all done at Stanford or Berkeley in 1967/8. Cohen proved T 6010 preliminary to showing that

T 6011 ZF $\vdash$ there is an x not recursive in any $y \subseteq x$ with $x \setminus y$ infinite,

a result obtained independently by Soare [15].

Then in early February of this year, Přikrý (at Berkeley) improved T 6010 to

T 6012 (Přikrý) ZF $\vdash$ no SF is Borel;

finally Silver (also at Berkeley), using Přikrý's result, established the following:

T 6013 (Silver) ZF $\vdash$ No SF is $\underset{\sim}{\Sigma}^1_1$;

T 6014 (Silver) ZF + MC $\vdash$ No SF is $\underset{\sim}{\Sigma}^1_2$; and

T 6015 (Silver) ZF + (✱) $\vdash$ No SF is $\underset{\sim}{\Sigma}^1_2$.

T 6013 and T 6014 are best possible for their theories: for by T 3003 and T 6005, ZF + V = L $\vdash$ there is a Δ^1_2SF, and by T 6005 and T 3226, ZF + V = L^μ $\vdash$ there is a Δ^1_3SF.

Here MC says there is an ordinal $\kappa > \omega$ with an ultrafilter D on it (that is, D $\subseteq$ S(κ)) containing no one-point sets and such that the intersection of fewer than κ sets in D is always in D. (*) and V = L^μ are the axioms given in the survey at D 1236 and after T 2019.

I shall now comment on the proof of T 6002, and on the plan of the paper.

D 6016 (ZF) A <u>transitive model</u> of a set of sentences $\mathfrak{X}$ is a collection M of sets such that u $\in$ v $\in$ M $\rightarrow$ u $\in$ M, and all sentences in $\mathfrak{X}$ are true in $\langle M, \in \rangle$.

I shall use the notion of a transitive model in two distinct ways: first in which M is a set, in which case there is no difficulty in formalising D 6016 in ZF; but I shall also, in discussions of a transitive model M include the possibility that M is a proper class, when I shall call M an <u>inner model</u>. (Caution: in the survey I use "model" to mean only the first case.) I mean then either that M is defined by a ZF-formula M($\mathfrak{e}$) with one free variable such that it is provable in ZF that all sentences of are true when all their quantifiers and terms are relativised to M: that is, ZF $\vdash \mathfrak{A}^M$, for each $\mathfrak{A}$ in $\mathfrak{X}$; or that a predicate letter M ($\mathfrak{e}$) with one free variable has been added to the language of ZF, and all the relativised sentences $\mathfrak{A}^M$ ($\mathfrak{A} \in \mathfrak{X}$) have been added as axioms.

(If M is an inner model, I shall write u $\in$ M for M(u). $\mathfrak{A}^M$

is obtained (when M is a set or an inner model) by replacing $\bigwedge x \mathfrak{B}$ by $\bigwedge x \in M \mathfrak{B}^M$; $\bigvee x \mathfrak{B}$ by $\bigvee x \in M(\mathfrak{B}^M)$, $\mathrm{E}x \mathfrak{B}$ by $\mathrm{E}x\, [x \in M \wedge \mathfrak{B}^M]$ in $\mathfrak{A}$. I often write "$\mathfrak{A}$ is true in M" for $\mathfrak{A}^M$.

When necessary, I shall distinguish the two cases by saying $M \in V$ or $M \notin V$, but in general an assertion "ZF $\vdash$..." about transitive models is to be understood as being both a theorem of ZF (in the case $M \in V$; when M is a bound variable) and a schema for predicates $M(\hat{e})$.

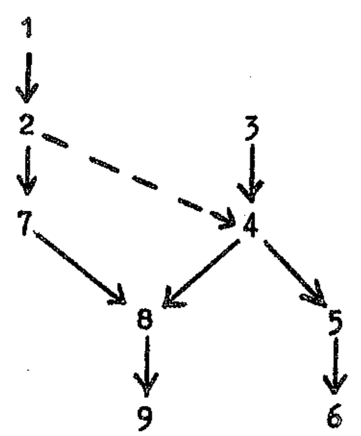
I shall make much use of the method of forcing, invented by Cohen to solve the independence of the Continuum Hypothesis: its fundamental concept is (probably) that of a partially ordered set; that of the associated model theory a complete Boolean algebra, cBA for short. [The theory of Boolean-valued models, which I shall use, is developed in the appended lecture notes (cited as BVM) by Scott.] In ¶ 1, the principal definitions are repeated, the relations between partial orderings and cBAs derived from a technical lemma which has a further application in ¶ 7, and two standard results on cardinalities in Boolean-valued models given. The theory of generic filters and complete homomorphisms is treated in ¶ 2, where two theorems taken from notes of Jensen and first used in ¶ 7, on sub- and quotient algebras, are proved. ¶ 3 establishes a combinatorial lemma, of which T 6010 is an immediate corollary. In ¶ 4 a notion $\mathbb{P}$ of forcing is introduced, and is shown, using the methods of ¶ 3, to have the following property:

T 6017 ZF $\vdash$ Let M be a transitive model of ZF + DC, and let x be $\mathbb{P}$-generic over M. Then every infinite subset of x is also $\mathbb{P}$-generic over M.

T 6017 is the step-ladder from which all the plums are picked.

The next two paragraphs digress from the road to T 6002. ¶5 continues the investigation of the properties of $\mathbb{P}$-generic reals. First T 6017 is applied to yield a very short proof of T 6014; the observation that M need only satisfy a certain finite subset of the axioms of ZF for T 6017 to hold leads to almost as short a proof of T 6013, which yields an improvement of T 6011, which in its turn shows that an x $\mathbb{P}$-generic over L is not of minimal L-degree (D 1105). ¶6 discusses briefly two other notions of forcing, one used by Silver, I gather, in showing his theorems TT 6013,4,5, and shows that $\mathbb{P}$-generic reals are not (Cohen) generic, random Sacks or Silver. (cf DD 1103,4,9 and 1111).

The last three paragraphs establish T 6002 and T 6001. A theorem of Jensen is proved in ¶7, which is then employed to investigate the appropriate cBA. I prove T 6002 in ¶8 using T 6017 and the results of ¶7. McAloon's observation that the definable class of sets hereditarily definable-with-ordinal-and-real-parameters is an inner model of ZF + DC then yields T 6001, in ¶9, which closes with further comments on the methods used, and a list of open problems. The diagram illustrates the logical relationships of the paragraphs.



(¶4 uses only the most standard results from ¶2.)

The model used to prove T 6001 is McAloon's simplification of Solovay's model in which all sets of reals are Lebesgue measurable (T 3307). Solovay has never released his account of his work, promised for some years with the title "The measure problem I: a model of set theory in which all sets of reals are Lebesgue measurable" and I am indebted to Prof. Silver of Berkeley for describing Solovay's argument to me. Conversations about Silver's sketch with Dr. Jensen have been most helpful, as will be apparent: in particular, Jensen remarked that Solovay's analysis of the first model (for T 6002; actually first constructed by Lévy) could be more readily presented using his own lemmata on Boolean algebras which collapse cardinals. Persons familiar with Solovay's proof of T 3307 will recognise that in ¶8 I do with $\mathbb{P}$-generic reals, using T 6017, what he did with random reals. Thus ¶¶1,2 and 7 are largely devoted to expounding Solovay's work.

Interest in this problem was widely encouraged by Prof. Friedman who lectured at Berkeley on his attempts to define an SF; the work of Ehrenfeucht, Cohen and Příkrý, and my lemma T 6005, are all the result of his enthusiasm. My own stimulus was the proofs of T 6010 and T 6001; I realised that Cohen's method could be generalised, and in February obtained the weaker results of ¶5. At the same time Příkrý proved T 6012. Silver stated his theorems in a letter to me in February. I returned to the problem in June , when I noticed that $\mathbb{P}$-generic reals are not of minimal L-degree, and Dr. Jensen told me about his own observations, stated in ¶¶4 and 5. I then realised that a consistency proof for "no SFs" could be obtained from Solovay's model were T 6017 true: I proved T 6017 on July 7th and noticed that it gave easy proofs of two of Silver's theorems. Thus the original parts of this paper are ¶¶3,4,5,6,8 and 9 (except where stated); though the basic methods are not mine, I think I can reasonably claim to have got more from them than other workers. The heart of the argument is ¶¶ 3 and 4; toute la reste est littérature.