Set Theory of the Continuum

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WHAT IS MAC LANE MISSING?

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A sociologist observing the 1989/90 Logic Year at the Berkeley Mathematical Sciences Research Institute would have judged it to be a typical gathering of mathematicians, exchanging ideas, running seminars to chip away at current problems, and writing papers and books. But there was one speaker who from time to time would tell the others that they were working on the wrong problems in the wrong subject. This was not the result of a momentary aversion: Professor Mac Lane has for at least twenty years been saying that "set theory is obsolete," that "measurable cardinals are bizarre," and so on, and he has written one large book ([2]) and many articles in order to present his view of mathematics.

It is the purpose of this essay to examine his stance, and to suggest that insofar as Mac Lane urges the unity of mathematics, he is to be supported, but insofar as he secretly desires the uniformity of mathematics, he is to be opposed.

Perhaps one should begin with a few reflections on the psychology of mathematics. One of the remarkable things about mathematics is that I can formulate a problem, be unable to solve it, pass it to you; you solve it; and then I can make use of your solution. There is a unity here: we benefit from each other's efforts. In this regard mathematicians interact much more than do (say) historians or composers.

But if I pause to ask *why* you have succeeded where I have failed to solve a problem, I find myself faced with the baffling fact that you have thought of the problem in a very different way from me: and if I look around the whole spectrum of mathematical activity the huge variety of styles of thought becomes even more evident.

Is it desirable to press mathematicians all to think in the same way? I say not: if you take someone who wishes to become a set theorist and force him to do (say) algebraic topology, what you get is not a topologist but a neurotic. Uniformity is not desirable, and an attempt to attain it, by (say) manipulating the funding agencies, will have unhealthy consequences.

The purpose of foundational work in mathematics is to promote the unity of mathematics; the larger hope is to establish an ontology within which all can work in their different ways.

What, then, is Mac Lane's ontology? This seems to admit a clear answer. In his book *Mathematics: Form and Function* he urges the claims of a system he calls ZBQC, which initials stand for Zermelo with Bounded Quantification and Choice, to supply all that he needs to do the mathematics he wants to do.

The axioms of this system are Extensionality, Null Set, Pairing, Power Set, Union, Infinity, Comprehension for Δ_0 formulæ, Regularity (*i.e.* Foundation) and Choice.

This system provides for the existence of the real numbers, and for ω types over them, thus yielding the complex numbers, functions from reals to reals, functionals and so on.

That this system represents a natural portion of mathematics may be seen from the way in which it keeps reappearing, first as the simple theory of types, and more recently as topos theory, with each of which it is equiconsistent. A natural model for it is $V_{\omega+\omega}$.

It is plain from Mac Lane's book that this system indeed supports a large amount of mathematics, more than I shall ever learn. Why then need we go outside it?

I suggest that an area ill supported by Mac Lane's system ZBQC is that of iterative constructions. We know from the work of Cantor onwards that there are processes which need more than ω steps to terminate; of which examples may be found even within traditional areas of mathematics. For example, within the space of continuous functions on $[0,1]$, the class of differentiable functions forms a set which is not a Borel set but is naturally expressible as the union of $\aleph_1$ Borel sets; and this has implications for the problem of building the anti-derivative of a given function.

So therefore let us look for a moment at abstract recursion theory and ask how easily it sits within Mac Lane's system.

A well-established axiomatic framework for abstract recursion theory is the system of Kripke-Platek.

Theorem 1. *If Consis(ZBQC) then Consis(ZBQC + KP).*

The intuition behind the proof of theorem 1 is this: just as $V_{\omega+\omega}$ is a natural model for ZBQC, so $H_{\beth_\omega}$, the collection of sets which are members of transitive sets of cardinality less than $\beth_\omega$ is a natural model of ZBQC + KP; moreover each transitive set in the second model is isomorphic to some well-founded extensional relation which is a member of $V_{\omega+\omega}$. Hence the second model can be regarded as coded within the first, the building bricks being well-founded extensional relations with designated elements. To get a relative consistency proof one has to convert this semantic argument into a syntactic manipulation.

With slightly more trouble one may establish

Theorem 2. *If Consis(ZBQ) then Consis(ZBQC + KP + V=L).*

Here ZBQ is ZBQC with the axiom of choice omitted. The proof is similar to that of theorem 1, but here the building bricks are fragments of the constructible hierarchy defined along well-orderings.

Thus ZBQC has *via* suitable coding a reasonable capacity for recursive constructions; and this would support Mac Lane's thesis that it is a reasonable basis for much of mathematics. However it will, as is clear from the work of Harvey Friedman, fail to support many constructions: it will not be able to prove Borel determinacy, which requires the iteration of the power set operation through all countable ordinals; similarly it will not be able to prove Borel diagonalization.

Set theory is so rich a theory that it has been claimed for much of this century to be the foundation of mathematics. In ontological terms this claim is not unreasonable; but Mac Lane resists. I would guess that his reason is not so much that he objects to the ontology of set theory but that he finds the set-theoretic cast of mind oppressive and feels that other modes of thought are more appropriate to the mathematics he wishes to do.

One must acknowledge that ideas from category theory provide a smooth way to handle a large amount of material. However to reject a claim that set theory supplies a universal mode of mathematical thought and of mathematical existence need not compel one to declare set theory entirely valueless.

Let us therefore set aside set theory's claim to be a foundation of the whole of mathematics, it being misguided to define the worth of a subject solely in terms of its serviceability to other areas of mathematics. Instead let us define set theory to be the study of well-foundedness. As such, it is a worthy object of study; and it can scarcely be said that this is a subject of little content !

From this point of view, Mac Lane's view that "measurable cardinals are bizarre" becomes hard to defend. May we suppose him to mean that he sees no need to think about them and therefore resents a suggestion that he should think about them?

In terms of the study of well-foundedness, measurable cardinals are natural objects: just as ZBQC has resurfaced in many forms, so do measurable cardinals keep bobbing up in unexpected contexts. The hypothesis that they exist, or the hypothesis that in some inner model there are measurable cardinals may be construed as saying that in certain circumstances the direct limit of well-founded structures is well founded. Other large cardinal axioms may also be interpreted as assertions of this general kind. These hypotheses seem worthy of study: well-foundedness is important, being central to the general enterprise of constructing objects by recursion, and it is natural to ask when well-foundedness is preserved under direct limits. These questions are interesting in their own right.

This might be a good moment to challenge one of Mac Lane's opinions, which I believe to rest on a misconception. On page 359 of his book he writes, after reflecting on the plethora of independence results, that "for these reasons 'set' turns out to have many meanings, so that the purported foundation of all of Mathematics upon set theory totters." Elsewhere, on page 385, he remarks that "the Platonic notion that there is somewhere *the* ideal realm of sets, not yet fully described, is a glorious illusion."

I would suggest a contrary view: independence results within set theory are generally achieved either by examining an inner model of the universe (an inner model being a transitive class containing all ordinals) or by utilizing forcing to build a larger universe of which the original one is an inner model. The conception that begins to seem more and more reasonable with the advance of the inner model program on the one hand and a deeper understanding of iterated forcing on the other is that within one enormous universe there are many inner models, and the various "independence arguments" may be reworked to give positive information about the way the various inner models relate to each other. Far from undermining the unity of the set-theoretic view, the various techniques available for building models actually promote that unity.

In a more diplomatic mood, Mac Lane writes on page 407:

> Neither organization is wholly successful. Categories and functors are everywhere in topology and in parts of algebra, but they do not as yet relate very well to most of analysis. Set theory is a handy vehicle, but its constructions are artificial. ... We conclude that there is as yet no simple and adequate way of conceptually organizing all of Mathematics.

Let me now consider briefly whether there can be a single foundation for Mathematics. In probing this question I have found myself coming to a

view that can be traced back certainly to Plato, namely that there are *two* primitive mathematical intuitions; which might be called the geometrical and the arithmetical; or, alternatively, the spatial and the temporal.

Plato did not have the advantage of modern research into the functions of the left and right half of the brain; this work suggests that the temporal mode (which would include recursive constructions) is handled in the left brain, whereas the spatial mode is handled in the right.

What can each mode of thought contribute to the understanding of the other? I believe, a lot.

Can either be reduced to the other? I should say not; certain formal translations exist, but the underlying intuitions do not translate; and these obstructions show themselves as paradoxes such as that of Banach-Tarski.

Let me refer to my contention that there are these two modes, neither reducible to the other, as positing an essential *bimodality* of mathematical thought.

In earlier pieces I have remarked how Mac Lane's choice of axioms agrees with that made by Bourbaki, at least initially; Liliane Beaulieu has recently remarked that Bourbaki's initial choice of topics was influenced by consideration of the needs of physicists (see [1]); this in turn suggests that Bourbaki attaches greater importance to the descriptive powers of mathematics than to the constructive, and prompts a speculative question: what need is there for a theory of recursion in physics?

There is certainly a need for a theory of recursion in mathematics. The recursion theorem itself is the heart of logic; it is the watershed where processes become objects. In descriptive set theory it takes the shape of the Coding Theorem of Moschovakis, and is thus the source of the strength of the axiom of determinacy.

My sense of the bimodality of mathematics is such that to suppress the ordinals or other frameworks on which to carry out recursions is to suppress half one's mathematical consciousness. I wonder therefore what physicists might be missing by using only the Bourbaki-Mac Lane portion of mathematics in their modeling. Might it be that physical time might fruitfully be modeled by an ordering other than the reals, for example by $\mathbf{R} \times \omega_2$, so that a leap ahead by ω_1 corresponds to some discontinuous event?

Such speculation prompts a further question: is it necessary for all the mathematical concepts invoked in physical explanation to have a direct physical meaning? Or might it be desirable to have abstract concepts which have the merit of making the physics easier to understand without having a perceptible physical interpretation?

But physics aside, the unity of mathematics is a desirable aim; and I would suggest as a modest first step that working in ZBQC + KP rather than ZBQC would encourage awareness of the temporal side of mathematics as well as the spatial side.

Mac Lane's set theory is weak in constructive power, but strong in manipulating the objects naturally arising in geometry. The reverse, as I expect Mac Lane would agree, is true of set theory. I suggest that category theory is as natural a framework for spatial mathematics as set theory is for temporal. I suggest therefore that we should seek an organization of mathematics that will allow the two fundamental intuitions room to develop and to interact; in doing so, we should move away from the regrettable situation so pithily described by Augustus de Morgan over a century ago and still, sadly, to be found today:

> We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact science are mathematics and logic: the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two.

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IS MATHIAS AN ONTOLOGIST?

Saunders Mac Lane

I am glad to see that Adrian Mathias has taken me to task. Yes, I once gave a lecture with the flamboyant title, "Set theory is obsolete." In this and in a few other contentious articles, I have violated one of the cardinal principles of mathematical activity: Mathematicians do not make pronouncements; they prove theorems. My apologies.

Mathias also argues correctly that there are at least two modes of mathematical thought: the geometrical and the arithmetical. I doubt that this has much to do with the two halves of the brain because I would include at least two more modes: the algebraical and the analytical.

My "one large book" (*Mathematics, Form and Function*, Springer, 1986), is said just to present "my" view of mathematics. I had a wider aim. The first ten chapters try to summarize many of the basic constructions of mathematics up through manifolds, mechanics, complex analysis and topology, in a form that might be of use to beginning mathematicians, including those with no interest in foundations, ontology, or philosophy.

That shaky subject of foundations does then appear in Chapter XI of the book, where I discuss ZBQC (Zermelo set theory with bounded quantifiers). I claim that this does better fit what most mathematicians do because their quantifiers are almost always bounded. As Mathias notes, this system ZBQC is not adequate for Borel determinacy or even for a good theory of ordinals. For that there are other foundations. But I see no need for a single foundation—on any one day it is a good assurance to know what the foundation of the day may be—with intuitionism, linear logic or whatever left for the morrow.

Yesterday, when I wrote that chapter, I suspected that the Kripke-Platek approach might be somehow used. I am delighted to see Mathias propose this, and I hope that he will publish his relative consistency results. The only sources I found yesterday on KP were so buried in technicalities that I failed to see this possibility.

Incidentally, that was one of my earlier flamboyant criticisms: logicians have isolated themselves too much from the rest of mathematics and often present the technique and not the meaning of their theorems. I am now inclined to apologize to my friends the logicians—other branches of mathematics, including some categorists, are even more isolated, and the algebraic geometers are accomplished experts at obscuring their ideas behind mountains of technique.

Mathias seems to claim that having just one foundation promotes the unity of mathematics. I disagree; it is still the case that most mathematicians don't think much about foundations. Real unity is fine, and unity is promoted more by cross connections, especially the unexpected ones. For example, categorical coherence theorems for tensored categories cropped up in Tanaka duality for groups and then in conformal field theory. Again, set theoretical forcing turned out to be related to Kripke semantics for intuitionistic logic, then to Kripke-Joyal semantics for topoi and then to sheafification for Grothendieck topologies. This latter connection seems to me illuminating, but is one as yet little noted by logicians.

In this case, the neglect of this remarkable connection may arise because the available categorical presentations are obscure. A forthcoming book by Mac Lane and Moerdikj on topos theory will, I hope, serve to rectify this situation.

A final word about foundations: my flashy title "Set theory is obsolete" was intended to draw attention to that remarkable observation by F.W. Lawvere: axiomatics for sets is no longer the only effective way to a foundation—one may instead start with axioms on functions—that is on the category of sets.

The last chapter of my "big book" deals with the philosophy of mathematics, with the hope of perhaps reviving this moribund field. My first claim was that too many philosophers of mathematics pay too little heed to what there really is in mathematics. This applies in particular to Wittgenstein and Lakatos, but for now I take on the biggest living target. My learned and articulate friend Van Quine has claimed that ontology is served by observing that "to be" is to be existentially quantified. I disagree, and I also doubt if Van realizes that the existential quantified is a left adjoint—an important observation, again due to Lawvere.

My last chapter attempted to use the earlier survey of the content of the mainstream of mathematics to draw some philosophical conclusions. Today, I would put my view as follows: Mathematics is that branch of science in which the concepts are protean: each concept applies not to one aspect of reality, but to many. The real numbers are both analytical and geometrical, natural numbers are both cardinal and ordinal, and so on in many, many cases. Mathematical form fits varied substance.

This view, if correct, has consequences. For example, the familiar set theoretic explanation of the ordered pair is a convenience and not an ontology. The same idea is formulated differently by observing that a product $A \times B$ is something with projections to the objects A and B which are "universal;" in this case the ordered pair has been swallowed by the syntactic order. Again, a real number is not a Dedekind cut; that cut is just one possible model of a protean idea of the reals.

Long ago, mathematicians recognized that "Space" was not unique. There was the Euclidean plane and the hyperbolic one, as well as elliptic planes. Now there are many types of space—Hausdorff, metric, uniform and so on, each with various contacts with different realities. Much the same now applies to sets. The notions arise variously from finite sets, infinite sets, combinatorial properties of sets, sets as extensional representations of properties, and so on. ZFC had different models. Mathias observes that one model of sets is often inner with respect to another. I am not persuaded that this circumstance argues for the existence of "One enormous universe." Evidently, what one has is different universes, perhaps with different axioms, and connected with each other. These differences match the different purposes of set theory. Moreover, the connections by the inner model relation can be described with sheaf theory more clearly by observing that the new model may consist of sheaves for a suitable "site" of the given model and that then there often is a geometric morphism form one model to the other (For definition, MacLane-Moerdijk, *loc. cit.*). This view of the matter does give a better understanding because it ties the relations between different models of set theory to the continuous functions between different models of space. This promotes the unity of mathematics.

Mathias asks "What, then, is Mac Lane's ontology?" Since mathematics is protean, I can answer easily: Ontology has to do with the nature of the reality at issue. Each mathematical notion is protean, thus deals with different realities, so does not have an ontology.

In closing, may I count my advantages. About 1940, when Bertrand Russell lectured at the mathematical colloquium at Harvard, I was in a position to berate him for his ignorance of the progress in foundational studies. In the 1970's when I was a member of the National Science Board, I was able to tell my colleagues that Kurt Gödel was the greatest logician since Aristotle; soon thereafter, Gödel was awarded the National Medal of Science...I admire Gödel's accomplishments, but I suspect that it is futile to wonder now what he imagined to be the "real" cardinal of the continuum. Those earnest specialists who still search for that cardinal may call to mind that infamous image of the philosopher—a blind man in a dark cellar looking for a black cat that is not there.

Set theory, like the rest of mathematics, is protean, shifting and working in different ways for different uses. It is subordinate to mathematics and not its foundation. The unity of mathematics is real and depends on wonderful new connections which arise all around us. I urge my friends in logic to look around.

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