The Strength of Mac Lane Set Theory

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To Saunders Mac Lane on his ninetieth birthday

Abstract

Saunders Mac Lane has drawn attention many times, particularly in his book *Mathematics: Form and Function*, to the system ZBQC of set theory of which the axioms are Extensionality, Null Set, Pairing, Union, Infinity, Power Set, Restricted Separation, Foundation, and Choice, to which system, afforced by the principle, TCo, of Transitive Containment, we shall refer as MAC.

His system is naturally related to systems derived from topos-theoretic notions concerning the category of sets, and is, as Mac Lane emphasizes, one that is adequate for much of mathematics.

In this paper we show that the consistency strength of Mac Lane's system is not increased by adding the axioms of Kripke–Platek set theory and even the Axiom of Constructibility to Mac Lane's axioms; our method requires a close study of Axiom H, which was proposed by Mitchell; we digress to apply these methods to subsystems of Zermelo set theory Z, and obtain an apparently new proof that Z is not finitely axiomatisable; we study Friedman's strengthening $\mathsf{KP}^{\mathcal{P}} + \mathsf{AC}$ of $\mathsf{KP} + \mathsf{MAC}$, and the Forster–Kaye subsystem KF of MAC, and use forcing over ill-founded models and forcing to establish independence results concerning MAC and $\mathsf{KP}^{\mathcal{P}}$; we show, again using ill-founded models, that $\mathsf{KP}^{\mathcal{P}} + V = L$ proves the consistency of $\mathsf{KP}^{\mathcal{P}}$; turning to systems that are type-theoretic in spirit or in fact, we show by arguments of Coret and Boffa that KF proves a weak form of Stratified Collection, and that $\mathsf{MAC} + \mathsf{KP}$ is a conservative extension of MAC for stratified sentences, from which we deduce that MAC proves a strong stratified version of KP; we analyse the known equiconsistency of MAC with the simple theory of types and give Lake's proof that an instance of Mathematical Induction is unprovable in Mac Lane's system; we study a simple set theoretic assertion — namely that there exists an infinite set of infinite sets, no two of which have the same cardinal — and use it to establish the failure of the full schema of Stratified Collection in Z; and we determine the point of failure of various other schemata in MAC.

The paper closes with some philosophical remarks.

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§		Ext^{y}	$\varnothing$	$\{x,y\}$	$\bigcup x$	$x \setminus y$	$\mathcal{P}(x)$	Sep^{n}	Foun^{dn}	TCo	ω	Choice	Repl^{mt}	Coll^{n}	$V{=}L$	$ON{=}\aleph_\omega$	Ax H	RAS
	S_0	✓	✓	✓	✓	✓												
6	S_1	✓	✓	✓	✓	✓	✓											
8	KF	✓	✓	✓	✓	[✓]	✓	a-Δ_0[a-$\Delta_0^{\mathcal{P}}$]					[a-$\Delta_0^{\mathcal{P}}$]					
	M_0	✓	✓	✓	✓	[✓]	✓	$\Delta_0[\Delta_0^{\mathcal{P}}]$					[a-$\Delta_0^{\mathcal{P}}$]					
1	M_1	✓	✓	✓	✓	[✓]	✓	$\Delta_0[\Delta_0^{\mathcal{P}}]$	set	✓			[a-$\Delta_0^{\mathcal{P}}$]					
8	KFI	✓	✓	✓	✓	[✓]	✓	a-Δ_0[a-$\Delta_0^{\mathcal{P}}$]			AxInf		[a-$\Delta_0^{\mathcal{P}}$]					
	M	✓	✓	✓	✓	[✓]	✓	$\Delta_0[\Delta_0^{\mathcal{P}}]$	set	✓	✓							
10	ZBQC	✓	✓	✓	✓	[✓]	✓	$\Delta_0[\Delta_0^{\mathcal{P}}]$	set		✓	✓						
	MAC	✓	✓	✓	✓	[✓]	✓	$\Delta_0[\Delta_0^{\mathcal{P}}$, a-$\Sigma_1]$	set	✓	✓	✓		[oa-Σ_1]				
10	MOST	✓	✓	✓	✓	[✓]	✓	Σ_1	set$[\Sigma_1,\Pi_1]$	✓	✓	✓	$[\Sigma_1]$	$\Delta_0[s\text{-}\Sigma_1]$			[✓]	
	KP	✓	✓	✓	✓	[✓]		$\Delta_0[\Delta_1]$	Π_1	[✓]				$\Delta_0[\Sigma_1]$				
	KPI	✓	✓	✓	✓	[✓]		$\Delta_0[\Delta_1]$	Π_1	[✓]	✓			$\Delta_0[\Sigma_1]$				
4	KPL	✓	✓	✓	✓	[✓]		$\Delta_0[\Delta_1]$	Π_1	[✓]	✓	[✓]		$\Delta_0[\Sigma_1]$	✓			
6	KPR	✓	✓	✓	✓	[✓]	[✓]	$\Delta_0[\Delta_1]$	Π_1	[✓]	✓			$\Delta_0[\Sigma_1]$				✓
6	$\mathsf{KP}^{\mathcal{P}}$	✓	✓	✓	✓	[✓]	✓	$\Delta_0[\Delta_1^{\mathcal{P}}]$	$\Pi_1^{\mathcal{P}}$	[✓]	✓			$\Delta_0^{\mathcal{P}}[\Sigma_1^{\mathcal{P}}]$				[✓]
5	$\mathsf{Z}_{\mathfrak{k}}$	✓	✓	✓	✓	[✓]	✓	$\Sigma_{\mathfrak{k}}$	set		✓							
5	$\mathsf{KZ}_{\mathfrak{k}}$	✓	✓	✓	✓	[✓]	✓	$\Sigma_{\mathfrak{k}}$	$\Pi_1[\Sigma_{\mathfrak{k}},\Pi_{\mathfrak{k}}]$	[✓]	✓			$\Delta_0[(s)\text{-}\Sigma_1]$			$[\mathfrak{k} \geqslant 1]$	
5	$\mathsf{KLZ}_{\mathfrak{k}}$	✓	✓	✓	✓	[✓]	✓	$\Sigma_{\mathfrak{k}}$	$\Pi_1[\Sigma_{\mathfrak{k}},\Pi_{\mathfrak{k}}]$	[✓]	✓	[✓]		$\Delta_0[s\text{-}\Sigma_1]$	✓		$[\mathfrak{k} \geqslant 0]$	
5	$\mathsf{KLMZ}_{\mathfrak{k}}$	✓	✓	✓	✓	[✓]	✓	$\Sigma_{\mathfrak{k}}$	$\Pi_1[\Sigma_{\mathfrak{k}},\Pi_{\mathfrak{k}}]$	[✓]	✓	[✓]		$\Delta_0[s\text{-}\Sigma_1]$	✓	✓	$[\mathfrak{k} \geqslant 2]$	
7	Z	✓	✓	✓	✓	[✓]	✓	full	set		✓		[strat]					
	ZC	✓	✓	✓	✓	[✓]	✓	full	set		✓	✓	[strat]					
5	KLMZ	✓	✓	✓	✓	[✓]	✓	full	Π_1[full]	[✓]	✓	[✓]	[strat]	$\Delta_0[s\text{-}\Sigma_1]$	✓	✓	[✓]	
	ZF	✓	✓	✓	✓	[✓]	✓	full	set[full]	[✓]	✓		[full]	✓		[•]	[✓]	[✓]
	ZFC	✓	✓	✓	✓	[✓]	✓	full	set[full]	[✓]	✓	✓	[full]	✓		[•]	[✓]	[✓]

Notes

1. Unexplained terms are defined in the text of the paper. Most systems are introduced in section 0. Under § is given the number of a later section where the given system is discussed.
2. The entries [✓] and [•] mean respectively that the statement is derivable or refutable in the given system; not all facts of this nature appear in the chart.
3. In M_0, TCo implies the existence of the transitive closure of a set.
4. The equivalence AC $\iff$ WO of the Axiom of Choice and the Well-ordering Principle is provable in M_0, using the power set axiom and the existence of Cartesian products.
5. Save for KFI, those systems with an axiom of infinity take it as asserting the existence of the infinite von Neumann ordinal ω. AxInf is the statement that there is a Dedekind infinite set.
6. s-Σ_1 collection is strong Σ_1 collection; it and Axiom H are provable in $\mathsf{KZ}_{\mathfrak{k}}$ for $\mathfrak{k} \geqslant 1$, and in $\mathsf{KLZ}_{\mathfrak{k}}$ for $\mathfrak{k} \geqslant 0$. Σ_1 collection is provable in KZ_0; indeed in KP.
7. a-Σ_1 means stratified Σ_1; o-Σ_1 means strong Σ_1; oa-Σ_1 collection means strong stratified Σ_1 collection.
8. RAS is the assertion that $\forall\zeta\ \{x \mid \varrho(x) < \zeta\} \in V$, where $\varrho(x)$ is the set-theoretical rank of x.
9. Quine's *New Foundations*, NF, is mentioned briefly in sections 0 and 7, and two versions—TSTI, TST; TKTI, TKT—of the simple theory of types with and without infinity are discussed in section 8.

0: Introduction

We describe here the scope and results of this paper in the semi-formal style, customary among logicians, that makes no visual distinction between levels of language; at the end of this introduction we shall summarise certain more rigorous conventions that will be used, with occasional relaxations, in later sections.

Our underlying logic is classical throughout the paper. We suppose that set theory is formalised with two primitive notions, $\in$ and $=$, a class-forming operator $\{\ \ |\ \ \ldots\ \ \}$, and apart from the usual quantifiers $\forall$ and $\exists$, *restricted quantifiers* $\forall x\,{:}{\in}\, y$ and $\exists x\,{:}{\in}\, y$, (in which x and y are distinct variables) for which the following axioms are provided. for each formula $\mathfrak{A}$ of the language:

$$\forall x\,{:}{\in}\, y\ \ \mathfrak{A} \Longleftrightarrow \forall x(x \in y\ \Rightarrow\ \mathfrak{A})$$
$$\exists x\,{:}{\in}\, y\ \ \mathfrak{A} \Longleftrightarrow \exists x(x \in y\ \&\, \mathfrak{A})$$

Formulæ in which all quantifiers occurring are restricted are called Δ_0. A Δ_0 formula prefixed by a single unrestricted existential or universal quantifier is called Σ_1 or Π_1 respectively.

Denote by S_0 the system with axioms of *Extensionality, Null Set, Pairing, Union, Difference* ("$x \setminus y \in V$"); by S_1 the system S_0 plus *Power Set* ("$\{y \mid y \subseteq x\} \in V$"); by M_0 the system S_1 plus the scheme of *Restricted Separation* ("$x \cap A \in V$", for each Δ_0 class A), known also as Δ_0 *Separation* and as Σ_0 *Separation*; by M_1 the system M_0 + *Foundation* ("Every non-empty set x has a member y with $x \cap y$ empty") + *Transitive Containment* ("Every set is a member of a transitive set"): using the Axiom of Pairing, that may be seen to be equivalent to saying that each set is a subset of a transitive set; by M the system M_1 + *Infinity*, the latter taken in the form $\omega \in V$ asserting that there exists an infinite von Neumann ordinal; and by MAC the system M plus the *Axiom of Choice*, which we may take either as the assertion AC of the existence of selectors for sets of non-empty sets, or as the assertion WO that every set has a well-ordering, since M_0 suffices for Zermelo's 1904 proof of their equivalence.

We shall also consider the system KF studied recently by Forster and Kaye [B3], which differs from M_0 in that Separation is admitted only for formulæ that are both Δ_0 and also stratifiable in the sense, recalled below, of Quine. For that system, $\omega \in V$ is an unsuitable, because unstratifiable, formulation of the axiom of infinity; a suitable one is AxInf, the assertion that there is a Dedekind-infinite set, that is, a set in bijection with a proper subset of itself; for two other versions, define a set to be *finite* if it has a well-ordering $\leqslant$ whose reflection $\geqslant$ is also a well-ordering, and then formulate the axiom of infinity either as ExInf, the assertion that there is a set which is not finite, or as InfWel, the assertion that there exists a well-ordering $\leqslant$ whose reflection $\geqslant$ is not.

Fortunately, KF proves, with the help of the Power Set axiom, the equivalence of those three formulations, so formally we need not decide between them. Forster and Kaye favour AxInf; the present author does not. We write KFI for the system KF with any one formulation added.

KF proves the equivalence of WO and the stratifiable form of AC that asserts the existence of a function choosing a one-element subset, rather than an element, of each non-empty set in a given family.

We abbreviate *Transitive Containment* as TCo. We write $\mathcal{P}(x)$ for the power set $\{y \mid y \subseteq x\}$ of x. We call an axiom of the form $x \cap A \in V$ a *separation axiom*, reserving the term *comprehension axiom* for an axiom, such as occurs in type theory, of the form $\{x \mid \Phi(x)\} \in V$, where intersection with a set is not required. The Axiom of Foundation, as given above, will sometimes be called *Set Foundation* for emphasis, to distinguish it from schematic versions for certain classes, such as the scheme of Π_1 *Foundation*, introduced below, or the full scheme of *Class Foundation*, whereby foundation is assumed for all classes, which latter scheme we sometimes call Π_∞ *Foundation*.

Π_1 Separation and Σ_1 Separation are of course equivalent in any system containing the Axiom of Difference. We use both names.

It will be remembered that Zermelo set theory, Z, results by dropping TCo from M and adding the full unrestricted Separation scheme, which we sometimes call Π_∞ Separation. We write ZC for $\mathsf{Z}+\mathsf{AC}$. In $\mathsf{Z}+\mathsf{TCo}$ the Axiom of Foundation is self-improving to the scheme of Class Foundation, in that for each class A it is provable that $\exists y(y \in A) \implies \exists y(y \in A\ \&\ y \cap A = \varnothing)$; but Jensen and Schröder [C2] and Boffa [C3], [C4] have shown that there are instances of the scheme of Class Foundation which are not theorems of ZC, and consequently that TCo is not provable in ZC. The system ZBQC introduced by Mac Lane on page 373 of his book [L1],which is in our notation $\mathsf{M}_0 + \mathsf{Foundation} + \omega \in V + \mathsf{AC}$, being a subsystem of ZC, therefore cannot prove TCo; thus our system MAC, which is $\mathsf{ZBQC} + \mathsf{TCo}$, is a proper extension of ZBQC, albeit, by the discussion of §1, equiconsistent with it.

In a companion, but much easier, paper, *Slim Models of Zermelo Set Theory* [E1], the author gives a method for building models of Z with strong failures of Δ_0 Collection. That paper will be cited at points in the present work. Its methods easily give a proof of the fact, known to Drabbe [E2] and Boffa [E3], that $\omega \in V$ is not derivable from (say) InfWel even in Zermelo's system without its axiom of infinity, and is thus stronger than any of the stratifiable versions of the Axiom of Infinity considered above.

Mac Lane, perhaps surprisingly in view of his expressed distrust of von Neumann ordinals, gives the axiom of infinity for ZBQC as the assertion of the existence of a set that contains $\varnothing$ and with every element x its successor $x \cup \{x\}$; but then the formula $\omega \in V$ can be derived in his system. We shall see in the model building of §2 that ω would re-emerge even were we to replace his formulation of infinity by a stratifiable one.

Zermelo–Fraenkel set theory, ZF, is the result of adding to Z either all instances of *Replacement*

$$\forall x :\in a\ \exists! y \mathfrak{R}(x,y) \Rightarrow \exists b \forall y (y \in b \Longleftrightarrow \exists x :\in a\ \mathfrak{R}(x,y))$$

or of *Collection*

$$\forall x \exists y\, \mathfrak{A} \Rightarrow \forall u \exists v \forall x :\in u\ \exists y :\in v\ \ \mathfrak{A}$$

where the variables u, v have no occurrence in the wff $\mathfrak{A}$. ZFC is $\mathsf{ZF}\ + \mathsf{AC}$.

We shall at times examine variants of those, namely *strong Collection*,

$$\forall u \exists v \forall x :\in u\ ((\exists y \mathfrak{A}) \Rightarrow \exists y :\in v\ \ \mathfrak{A})$$

and *strong Replacement*

$$\forall x :\in a\ \exists^{\leqslant 1} y\, \mathfrak{R}(x,y) \Rightarrow \exists b \forall y (y \in b \Longleftrightarrow \exists x :\in a\ \mathfrak{R}(x,y)).$$

In those formulations of Replacement, $\exists! y$ and $\exists^{\leqslant 1} y$ abbreviate the assertions "there is exactly one y" and "there is at most one y": intuitively Replacement says that the image of a set by a total function is a set, and strong Replacement makes the same assertion for partial functions.

Replacement for a given formula $\mathfrak{R}$ can easily be derived from Collection for the same $\mathfrak{R}$, but the converse can fail, so that in some sense Collection is stronger. For example we shall in §7 prove Coret's theorem that strong stratifiable Replacement is provable in Z and shall show in §9 that stratifiable Collection is not. Nevertheless full Collection can be derived from full Replacement in Z.

Replacement and Collection are natural in systems studying recursive definitions: a well-developed subsystem of ZF focusing on this concern is that developed independently by Kripke, Platek and others in the 1960's.

The system KP may be given thus: Extensionality, Null set, Pairing, Union, Restricted Separation, plus Π_1 foundation ($A \neq 0 \Rightarrow \exists x :\in A\ \ x \cap A = 0$, for each Π_1 class A) and Restricted (or Δ_0 or Σ_0) Collection ("$\forall x \exists y\, \mathfrak{A} \Rightarrow \forall u \exists v \forall x :\in u\ \exists y :\in v\ \ \mathfrak{A}$" for each Δ_0 wff $\mathfrak{A}$ in which the variables u, v do not occur.) As shown below, Transitive Containment is provable in KP, but it should be noted that *strong* Δ_0 Collection, ("$\forall u \exists v \forall x :\in u\ ((\exists y \mathfrak{A}) \Rightarrow \exists y :\in v\ \ \mathfrak{A})$" for $\mathfrak{A}$ a Δ_0 wff), is not. We emphasize that the Power Set axiom is not among those of KP, nor has KP an axiom of infinity; we shall write KPI for the system $\mathsf{KP} + \omega \in V$.

A system we shall call *Mostowski set theory*, or MOST for short, will be of particular interest to us. It is the result of adding strong Δ_0 Collection to MAC: an easily equivalent axiomatisation is $\mathsf{ZBQC} + \mathsf{KP} + \Sigma_1$ $\mathsf{Separation}$, and other axiomatisations will be given in §3.

KP is to some extent "orthogonal" to Z, as it builds up the height of the universe without making it very fat. Many formulations of it omit our strengthening of foundation for sets to foundation for Π_1 classes, because they are concerned with *admissible sets*, that is, with transitive models of KP, for which even Π_∞ Foundation is automatically true. Gödel's axiom of constructibility, $V = L$, can be handled very naturally in KPI; we shall write KPL for the system KPI $+ V = L$. Discussion of constructibility is almost certainly impossible in KF, for, as we shall see in §7, the global choice function available in L is irredeemably unstratifiable; and awkward in M or Z, which lack suitable axioms of replacement.

However it is possible sufficiently to simulate in M the generation of L by Gödel's functions, as expounded in [A3], to prove, in arithmetic:

Theorem 1. *If* M *is consistent then so is* MOST $+V = L$.

Proving Theorem 1 by keeping that simulation purely within M, which was our original approach, is most cumbersome. In §2 we shall introduce the hypothesis we call Axiom H, or, simply, H, following Cole [H2], who credits Mitchell [H1] with first noticing its importance. As a first estimate of the strength of this axiom we show that M + H proves that $\forall\kappa\exists\kappa^+$, meaning that there is no greatest initial (von Neumann) ordinal. H emerges as the pivotal principle of the paper, and enables us in §4 to present, besides the original working in M, two much simpler proofs of Theorem 1 working in M + H.

We begin the paper by giving in §1 a simple but illustrative construction which derives the consistency of MAC from that of ZBQC. The main part of §2 is devoted to establishing the consistency of H relative to M without any use of the axiom of choice. The proof easily adapts to yielding other relative consistencies, giving, in arithmetic:

Theorem 2. *The consistency of* M *implies that of* M + H*; the consistency of* MAC *that of* MAC + H*; and the consistency of* Z *that of* Z + TCo + H.

In §3 we examine Axiom H in detail, and show in particular that

Theorem 3. *All axioms of* MOST *are theorems of* MAC + H*. Conversely* H *is derivable without use of the Axiom of Choice in* M + KP + Σ_1 Separation.

Note the one-sided reliance on the Axiom of Choice in the proof of that theorem, and its corollary that MOST may be given as MAC + H.

From Theorems 2 and 3 we have

Theorem 4. *The consistency of* MAC *implies that of* MOST*, and of* ZC *that of* ZC + KP.

In §5 we look at level-by-level results for subsystems of Zermelo. For $\mathfrak{k}$ a natural number of the meta-language, we write $\mathsf{Z}_\mathfrak{k}$ for Z with separation only for $\Sigma_\mathfrak{k}$ classes, $\mathsf{KZ}_\mathfrak{k}$ for $\mathsf{Z}_\mathfrak{k}$ with KP added, and $\mathsf{KLZ}_\mathfrak{k}$ for $\mathsf{Z}_\mathfrak{k}$ with KPL added. Our results are not quite as expected. Corresponding to the above theorems we have

Theorem 5. *For each* $\mathfrak{k} \geq 2$*, if* $\mathsf{Z}_{\mathfrak{k}+1}$ *is consistent, so is* $\mathsf{Z}_\mathfrak{k}$ + TCo + H*;* $\mathsf{Z}_\mathfrak{k}$ *together with either* KP *or* H *proves the truth of* $\mathsf{KLZ}_\mathfrak{k}$ *in* L*; hence if* $\mathsf{Z}_{\mathfrak{k}+1}$ + AC *is consistent, so is* $\mathsf{Z}_\mathfrak{k}$ + AC + KP*, if* $\mathsf{Z}_{\mathfrak{k}+1}$ *is consistent, so is* $\mathsf{KLZ}_\mathfrak{k}$*, and if* Z *is consistent, so is* KLZ.

We show that after the first two, these systems are of strictly increasing strength:

Theorem 6. KLZ_0 *proves* H*, and hence* KLZ_0 *and* KLZ_1 *are the same system. The truth of* KZ_1 *in* L *is provable in* KZ_1 *and in* Z_0 + H*, but not in* KZ_0*.* Z_0 *and* Z_1 *are equiconsistent, although they are not the same system.*

Theorem 7. *For each* $\mathfrak{k} \geq 2$*,* $\mathsf{KLZ}_\mathfrak{k}$ *proves the consistency of* $\mathsf{KLZ}_{\mathfrak{k}-1}$.

That implies a result that has been known since work of Wang and others in the fifties, of which our proof is apparently new:

Theorem 8. Z *is not finitely axiomatisable.*

We begin the sixth section with a definition and discussion of the class, introduced by Takahashi, of $\Delta_0^\mathcal{P}$ formulæ and of the corresponding variant $\mathsf{KP}^\mathcal{P}$ of KP, of which the transitive models are Harvey Friedman's *power admissible sets*, and of which M is a subsystem. With the help of the Gandy basis theorem and ideas of constructibility and forcing in the context of non-standard models of $\mathsf{KP}^\mathcal{P}$ we obtain two theorems and a new proof of a third:

Theorem 9. $\mathsf{KP}^{\mathcal{P}} + V = L$ *proves the consistency of* $\mathsf{KP}^{\mathcal{P}}$.

Theorem 10. (H. Friedman) $\mathsf{KP}^{\mathcal{P}} + \mathsf{AC}$ *does not prove the existence of an uncountable von Neumann ordinal, nor even a non-recursive von Neumann ordinal.*

Theorem 11. $\mathsf{KP}^{\mathcal{P}} + \mathsf{AC} + \forall\kappa\exists\kappa^+$ *does not prove Axiom H.*

In §7 we turn to a refreshingly different system of set theory, NF, the *New Foundations* of Quine, which rests on the notion of a *stratifiable* formula, and emphasizes set formation by Comprehension at the expense of well-foundedness. Here a *stratifiable* formula is a formula of the language of set theory in which integer *types* may be assigned to the individual variables of a formula so that (i) different occurrences of the same variable are assigned the same type, while (ii) in every subformula $x \in y$ the type assigned to y is one higher than the type assigned to x, whereas (iii) in every subformula $x = y$ the types assigned to x and y are equal.

Once types have been assigned, of course, the resulting *stratified formula* is a formula of the language of type theory; we may speak of it as a *stratification* of the original formula. A given stratifiable formula will have many different stratifications. The non-logical axioms of NF are the Axiom of Extensionality and the scheme of stratifiable Comprehension:

$$\{x \mid \Phi(x)\} \in V$$

for each stratifiable formula Φ.

Here is one essential difference: in systems like ZF, the function $F(x) = \{y \in x \mid y \notin y\}$ is a set for every x, and it is provable — by Russell — that for each x, $F(x)$ is a set which is not a member of x. In NF, the universe V is a set, so $V \in V$ and any function G defined for all sets will have the property that $G(V) \in V$: the property $x \notin x$ not being stratifiable, the function F is simply not available in NF.

Now Quine's theory is a theory of types in disguise; and it may be argued that the parts of mathematics supported by Mac Lane set theory, with its emphasis on the power set operation and on the restricted Separation scheme, are also type-theoretic in spirit.

Support for that view comes from the fact that M_0 is equiconsistent with the simple theory of types and with the system KF of Forster and Kaye mentioned above, and from the result of Coret that every instance of strong stratified Replacement is provable in Zermelo set theory. For the first part of Coret's argument KF is sufficient:

Theorem 12. *Let G be a class defined by a stratifiable formula. Then* KF *proves that if G is a partial function then* $\forall u \exists v G``u \subseteq v$.

Coret, working in Z, is of course able to conclude using the Separation schema that the image $G(u)$ of a set by an arbitrary stratifiable function G is a set. In our context, his argument proves

Theorem 13. *All instances of the scheme of strong stratifiable* $\Delta_0^{\mathcal{P}}$ *Replacement are provable in* KF*; all instances of the scheme of strong stratifiable* Π_1 *Replacement are provable in* MAC.

The orthogonality of KP to MAC is underlined by

Theorem 14. *Every stratifiable sentence provable in* MOST *is provable in* MAC*; every stratifiable sentence provable in* $\mathsf{ZC} + \mathsf{KP}$ *is provable in* ZC *alone; and similarly* $\mathsf{M}_1 + \mathsf{H}$, $\mathsf{M} + \mathsf{H}$ *and* $\mathsf{Z} + \mathsf{H}$ *are conservative over* M_1, M *and* Z*, respectively, for stratifiable sentences.*

From Theorems 12 and 14 follows

Theorem 15. *All instances of the scheme of strong stratifiable* Σ_1 *Collection, and therefore also of the schemes of strong stratifiable* Σ_1 *Replacement, stratifiable* Σ_1 *Separation, stratifiable* Π_1 *Foundation and stratifiable* Σ_1 *Foundation, are provable in* MAC.

which further elucidates the type-theoretic character of MAC.

In §8 we, following a method of Forster and Kaye, give detailed proofs of the known results that both KF and M_0 are equiconsistent with the simple theory of types, and that both KFI and M are equiconsistent with the simple theory of types with an axiom of infinity. Indeed we cover two presentations of type theory, the classical one of Kemeny's thesis and the more recent one employed by Forster and his co-workers.

Our equiconsistency proofs, however, are not optimal, in this sense: in each case, one direction is much harder than the other, and our proofs of the harder direction are formalisable in second-order but not in first-order arithmetic, relying as they do on an axiom of infinity in the meta-theory. To weaken the meta-theory to primitive recursive arithmetic would require techniques of proof theory beyond the province of this paper.

The final mathematical section, §9, is devoted to establishing limitations of KF, MAC and Z. We begin with an example showing that the derivation of Collection from Replacement is non-trivial:

Theorem 16. *The consistency of* Z *implies the consistency of* ZC + H+ *a failure of stratifiable* $\Delta_0^{\mathcal{P}}$ *and of stratifiable* Π_1 *Collection: indeed there are instances of stratifiable* $\Delta_0^{\mathcal{P}}$ *Collection and* Π_1 *Collection that over* MAC *prove the consistency of* Z.

We give various examples of such failures: underlying them all is the stratifiable sentence "there is an infinite set all of whose members are infinite and no two of whose members have the same cardinal" which is provable in ZF, but not in Z, and which shows that Coret's theorem cannot be improved to "ZF is a conservative extension of Z for stratifiable wffs."

We employ an unstratifiable variant of the formulæ used for Theorem 16 to prove

Theorem 17. Z + H, *if consistent, proves neither the scheme of* $\Delta_0^{\mathcal{P}}$ *Replacement nor the scheme of* Π_1 *Replacement.*

The failure of a scheme of induction in systems such as MAC was first noticed in the 1970's:

Theorem 18. (Lake) *There is a set-theoretical formula* $\Phi(n)$ *with one free variable, ranging over natural numbers, such that* M *proves both* $\Phi(0)$ *and* $\forall n\!:\in\omega\ [\Phi(n) \implies \Phi(n{+}1)]$ *but* MAC *does not prove* $\forall n\!:\in\omega\ \Phi(n)$.

Lake's result establishes other limitations to the strength of MAC. By a different argument, inspired by an unpublished Bounding Lemma of Forster and Kaye, we show that

Theorem 19. MAC *cannot prove that each initial segment of the function* $n \mapsto \omega + n$ *is a set; consequently the scheme of* Π_1 *Foundation is not provable in* MAC.

We pause to give an illustration drawn from algebra of the inadequacy of MAC and Z for certain natural mathematical constructions, and then conclude with some open problems and a summary, which we do not repeat *in toto* here, of the success or failure of various systems in proving schemes of Separation, Foundation, Replacement and Collection: for example,

Theorem 20. *Any one of the following schemes, if added to* MAC, *would prove the consistency of* MAC*: and all are therefore unprovable in* MOST*:*

stratifiable $\Sigma_1^{\mathcal{P}}$ *Separation; stratifiable* Σ_2 *Separation;*
stratifiable Π_2 *Foundation; stratifiable* $\Pi_1^{\mathcal{P}}$ *Foundation.*

In section 10 we outline the philosophical motivation of the paper and consider briefly the implications for the philosophy of mathematics of the similarities and differences between the two systems ZBQC and MOST.

The reader will have seen that there is a concern in this paper not only with comparing certain weak systems, usually subsystems of Zermelo–Fraenkel set theory, by means of relative consistency results, but also with establishing various theorems within those weak systems, some of which theorems assert the consistency of, or even the existence of well-founded models of, other weak systems.

Thus, at times, and particularly in §5, we shall find ourselves in the dangerous part of logic where to blur distinctions between languages is to jeopardise meaning.

Now if A is a subsystem of B we know that strictly it is ungrammatical nonsense to say that B proves the consistency of A; but it may well be true that B proves the consistency of a system A that from a certain standpoint may be claimed to be a copy of A. Hence we must make the familiar, but often suppressed, distinction between an object language — in our example, that in which the system A is formulated — and its meta-language, in our example that of the system B. In the interests of readability, we shall keep this distinction before the reader by the occasional use of a few typographical conventions rather than by a flood of super-precise notation.

In fact, we regard ourselves as working at three levels of language: there is the language of set theory, formulated with $\in$ and $=$; we shall call that *the language of discourse* or *the $\in$-language*; the range of its variables is *the domain of discourse.* The language of discourse is the vehicle of our flow of mathematical reasoning. We generally use fraktur letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ or capital Greek letters Φ, Θ, Ψ to indicate formulæ of that language.

One level higher, there is the *meta-language*, English, in which we may comment on relationships between different systems formulated in the language of discourse, as we do in Theorem 2; or on properties of particular systems, as in Theorem 8 and Theorem 17. At the bottom level there are *object languages*, of which the symbols and formulæ are all particular sets, that is, members of the domain of discourse.

Our conventions for such object languages are these: the letters $\mathit{\Theta}$,$\mathit{\Phi}$, $\mathit{\Psi}$, ϑ, φ, are reserved for formulæ of some language that we are in our set theory discussing. Though, when in model-theoretic mood, we may consider different object languages, there is one of particular importance for us, which we call the ϵ-language. That language resembles the language of set theory; its equality symbol is denoted by $\mathtt{=}$, its membership symbol by ϵ, its connectives and quantifiers by $\wedge$, $\neg$, $\vee$, $\longrightarrow$, $\longleftrightarrow$, $\bigvee$, $\bigwedge$, corresponding to the symbols of the set theory that we are talking, & , ¬, or , $\Longrightarrow$, $\Longleftrightarrow$, $\exists$, $\forall$. Its formal variables are indicated by Fraktur letters $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \ldots$ at the end of the alphabet.

Corresponding to the notion of a Δ_0 formula of the $\in$-language, we have a corresponding concept of a formula of the ϵ-language with only restricted quantifiers, which we shall call a $\dot{\mathit{\Delta}}_0$ formula. We may in fact *define* $\dot{\mathit{\Delta}}_0$ to be the set of all such formulæ and write "if $\varphi \in \dot{\mathit{\Delta}}_0$, then ....".

A $\dot{\mathit{\Sigma}}_{\mathfrak{k}}$ formula is a formula of the ϵ-language that consists of an arbitrary $\dot{\mathit{\Delta}}_0$ formula prefaced by the unique string, which we write $\dot{\mathcal{Q}}_{\mathfrak{k}}$, of length $\mathfrak{k}$ of formal unrestricted quantifiers $\bigvee$, $\bigwedge$, that starts with the existential quantifier $\bigvee$ and strictly alternates existential and universal quantifiers.

$\dot{\mathit{\Pi}}_{\mathfrak{k}}$ formulæ are those formulæ of the ϵ-language that consist of $\dot{\mathit{\Delta}}_0$ formulæ prefaced by the alternating string $\dot{\mathcal{R}}_{\mathfrak{k}}$ of length $\mathfrak{k}$ of unrestricted quantifiers starting with the universal quantifier $\bigwedge$.

Formulæ in the $\in$-language can be copied into the ϵ-language: a useful device to indicate such copying is to place a dot over a symbol or formula: $\dot{\Phi}$; $x \dot{\subseteq} y$. Thus if $\mathfrak{A}$ is a Δ_0 $\in$-formula, $\dot{\mathfrak{A}}$ will be a $\dot{\mathit{\Delta}}_0$ ϵ-formula.

This transfer from one language to a lower one will work also for proofs; therefore discussions of manipulations of formulæ will tend to be presented as a discussion in the meta-language of formulæ of the $\in$-language, in the knowledge that such discussions can be usually be transported to form discussions in the $\in$-language of manipulations of formulæ of the ϵ-language.

An English phrase, or name of an axiom, in nine-point sansserif font, such as every von Neumann ordinal is recursive or Foundation, denotes some reasonable formalisation in the $\in$-language: the same word or phrase in nine-point typewriter font denotes an appropriate formula of the ϵ-language: `Foundation`, `every von Neumann ordinal is recursive`. Thus if Φ is the formula ranks are sets, both $\dot{\Phi}$ and `ranks are sets` specify the same ϵ-formula.

A similar convention applies to the names of schemes and of systems: Class Foundation, `Class Foundation`, ZF, `ZF`, ZBQC, `ZBQC`.

With these conventions, the statement of Theorem 7 would be more correctly given thus:

For each $\mathfrak{k} \geq 2$, $\mathsf{KLZ}_{\mathfrak{k}}$ *proves* $\mathsf{Consis}(\mathtt{KLZ}_{\mathfrak{k}-1})$.

The use of Roman font, or, in statements of theorems, slanted font, does not in itself indicate a particular level of language, though sometimes a particular level will be indicated by the context. Thus the phrase "the Axiom of Foundation" will often denote exactly the same formula of the $\in$-language as the single word Foundation, the choice of font sometimes being influenced more by aesthetic than by mathematical considerations; and Theorem 7 might also be given as

For each $\mathfrak{k} \geq 2$, $\mathsf{KLZ}_{\mathfrak{k}}$ *proves the consistency of* $\mathtt{KLZ}_{\mathfrak{k}-1}$.

In §§8 and 9, we discuss various systems of type theory, and will introduce there the requisite extensions of the above conventions.

We shall of course consider models of our various theories which are themselves sets, but the demands of our subject dictate that we must look as well at *inner models*, such as L, or the class W_0 defined in §1, which are subcollections of the universe but are not sets; and also at what we might call *outer models* of the kind produced by forcing, or by the construction of §2 that yields what we call *the* H-*model*; such constructions portray a universe that goes beyond the initial universe of discourse and again is not a set.

In other contexts when we speak of a model, we shall mean a structure say $\mathbf{N} = (N, R)$, where N is a set, and R a relation on it. When discussing the truth predicate $\models$, which may hold between a model $\mathbf{N}$ and a formula φ of an appropriate object-language, an expression such as $(N, R) \models \varphi[x]$ means that $x \in N$ and φ is true in (N, R) when its (implicitly, unique) free variable is interpreted as denoting x; and similarly for a finite list of members of the model corresponding to a finite list of variables of φ. The exact correspondence between variables and interpretations will usually not be indicated. However, when a formal formula is explicitly given, such as $\mathfrak{x} \mathrel{\epsilon} \mathfrak{y}$ we may write $a \mathrel{\epsilon} b$ rather than $\mathfrak{x} \mathrel{\epsilon} \mathfrak{y}[a, b]$.

We use the unadorned symbol $\models$ only for the truth predicate for a structure that is a set; as we eschew the inaccurate convention of using $\models$ also for truth in proper classes, we must, for both inner and outer models, employ other devices to capture truth. One such device for inner models is that of *relativisation*, which we indicate by superscripts: thus $(\mathfrak{A})^L$ is the formula that results from the formula $\mathfrak{A}$ when it is rewritten to restrict all variables to range over only members of the class L. Sometimes in the interest of clarity or typographical elegance we shall write "$\mathfrak{A}$ is true in L", or "relative to L, $\mathfrak{A}$ is true", rather than "$(\mathfrak{A})^L$".

We have a second device for inner models: partial approximations to $\models$ for classes. We shall introduce a definition, symbolised by $\overset{\mathfrak{k}}{\models}$, of truth for $\dot{\Sigma}_{\mathfrak{k}}$ formulæ in the universe, and another, symbolised by $\overset{\mathfrak{k}}{\underset{L}{\models}}$, of truth in L for $\dot{\Sigma}_{\mathfrak{k}}$ formulæ. Our definitions exploit the fact that provided we can define the transitive closure of a set, we can give a truth definition $\overset{0}{\models}$ for $\dot{\Delta}_0$ formulæ of the ϵ-language; strings of unrestricted quantifiers are then added "by hand" so that there is no quantification over such strings within the language of discourse, but only in the meta-language.

1: Adding Foundation and Transitive Containment

Our first observation is very simple: the system S_0 is strong enough to define an inner model, W_0 say, as the union of all well-founded transitive sets, and prove that relative to W_0, Foundation and Transitive Containment and the axioms of the system S_0 are true; indeed if we adopt the convention whereby if T is a system of set theory, T^+ will be the system T + set Foundation + Transitive Containment — for example, M_0^+ is M_1, and Z_0^+ is M — then for many natural systems T, the class W_0 defined below will, provably in T, be an inner model of T^+.

In preparation for the more complex model-building in later sections, it will be helpful to set that out in some detail. Define the class of transitive well-founded sets as follows:

$$F_0 =_{\text{df}} \{a \mid \bigcup a \subseteq a \text{ and } \forall x \subseteq a (x \neq \varnothing \Rightarrow \exists y \in x\, y \cap x = \varnothing)\}$$

and the union of that class by

$$W_0 =_{\text{df}} \{\alpha \mid \exists a {:\in} F_0 \;\; \alpha \in a\}.$$

It is immediate from the definition that W_0 is transitive.

1·0 Lemma *If $\bigcup a \subseteq a$, any $\in$-minimal element of $x \cap a$ is an $\in$-minimal element of x.*

The two closure principles 1·1 and 1·4 are the core of the relative consistency proof. The property given in 1·3 is often written "W is supertransitive".

1·1 $\mathcal{P}$-closure of F_0: *If $a \in F_0$ and $\forall x {:\in} b \; x \subseteq a$ then $a \cup b \in F_0$.*

1·2 COROLLARY $a \in F_0 \implies \mathcal{P}(a) \in F_0$.

1·3 COROLLARY $(A \subseteq b \in W_0$ & A *a set*$) \implies A \in W_0$.

1·4 $\bigcup$-CLOSURE OF F_0: $((\forall a{:}{\in} A\ a \in F_0)$ *and* A *a set*$) \implies \bigcup A \in F_0$.

1·5 REMARK The system S_0 is strong enough to develop the very elementary theory of (von Neumann) ordinals, such as the statement that given two ordinals ζ and η either one is a member of the other or they are equal.

Certain results later in this section must be proved in stronger systems such as S_1 or M_0; the reader will notice that, for some, KF will suffice; but we must defer to a later paper a detailed study of the sufficiency or otherwise of KF for the various model-building techniques of the present paper.

1·6 REMARK About the principle of $\bigcup$-closure: it is enough, if all one wants is to prove Zermelo-hood, to have it for A an unordered pair. In that form it is used in the definition of a *fruitful* class in the essay *Slim Models* [E1] on the class of hereditarily finite sets in Zermelo set theory. In its present form, the statement implies that $x \subset W \implies x \in W$, and so W will contain all ordinals and be a proper class. In this form, too, the axioms of ZF would hold in W_0 if true in V.

The two closure principles enable us to check easily that any of the axioms of M_0 are true in W_0 provided they are true in V. The truth of the Power Set axiom follows from the supertransitivity of W_0, and the truth of Δ_0 Separation follows from the absoluteness of Δ_0 statements for transitive sets or classes. Further TCo and Foundation are true in W_0.

We write $(\mathfrak{A})^0$ for the formula that results from $\mathfrak{A}$ when all variables are bound to range over W_0: thus a quantifier $\exists z \ldots$ must be replaced by $\exists z \exists w\ z \in w \,\&\, \bigcup w \subseteq w \,\&\, \forall v{:}{\subseteq} w\ (v \neq \varnothing \implies \exists u{:}{\in} v\ u \cap v = \varnothing)\ \&\ \ldots$.

1·7 PROPOSITION (S_0) $(\mathsf{TCo})^0$

1·8 PROPOSITION (S_0) $(\mathsf{Foundation})^0$

1·9 PROPOSITION $(\mathsf{S}_0 + \mathsf{Power\ Set})$ $(\mathsf{Power\ Set})^0$

1·10 THEOREM SCHEME *If* $\mathfrak{A}$ *is an axiom of* M_0, *then* $\vdash_{\mathsf{M}_0} (\mathfrak{A})^0$.

Hence, provably in arithmetic,

1·11 METATHEOREM *If* M_0 *is consistent, so is* M_1.

1·12 PROPOSITION (S_0) $\omega \in V \implies (\omega \in V)^0$.

If we extend our theory to Z, we have another

1·13 THEOREM SCHEME *If* $\mathfrak{A}$ *is an axiom of* Z^+, *then* $\vdash_{\mathsf{Z}} (\mathfrak{A})^0$.

Hence, provably in arithmetic,

1·14 METATHEOREM *If* Z *is consistent, so is* Z^+.

To prove these last, we need only note that by the supertransitivity of W_0, it is enough to show that for each formula $\mathfrak{A}$, $p \in W_0 \,\&\, x \in W_0 \implies x \cap \{y \in W_0 \mid (\mathfrak{A}(y,p))^0\} \in V$ is provable in Z. That follows, of course, from the axiom scheme of Separation.

The rewriting of a formula $\mathfrak{A}$ to yield $(\mathfrak{A})^0$ potentially introduces more quantifiers: fortunately each new one will be absorbed into the next until we come down to the matrix. Thus $(\mathfrak{A})^0$ for $\mathfrak{A}$ Σ_1 will be Σ_2, and, inductively, will *prima facie* be $\Sigma_{\mathfrak{k}+1}$ or $\Pi_{\mathfrak{k}+1}$ if $\mathfrak{A}$ is respectively $\Sigma_{\mathfrak{k}}$ or $\Pi_{\mathfrak{k}}$.

However the Axiom of Foundation is among those of Z_0, and **in its presence** the definition of F_0 simplifies to being the class of all transitive sets, and of W_0 to being the union of all transitive sets. Hence $(\mathfrak{A})^0$ will in Z_0 be equivalent to a formula of the same quantifier level, and we have for each $\mathfrak{k} \geqslant 0$:

1·15 THEOREM SCHEME *If* $\mathfrak{A}$ *is an axiom of* $\mathsf{Z}_{\mathfrak{k}}^+$, *then* $\vdash_{\mathsf{Z}_{\mathfrak{k}}} (\mathfrak{A})^0$.

Hence, provably in arithmetic,

1·16 METATHEOREM *If* $\mathsf{Z}_{\mathfrak{k}}$ *is consistent, so is* $\mathsf{Z}_{\mathfrak{k}}^+$.

1·17 REMARK In the model to be built in §2, a corresponding obstacle seems to be insuperable. We shall obtain (Δ_0 Separation)^1 from Δ_0 Separation, because for $\mathfrak{A}$ Δ_0 $(\mathfrak{A})^1$ will be Δ_0 relative to some appropriate set, and we obtain (full Separation)^1 from full Separation because there is then no need to count quantifiers, but in between we are one step out.

1·18 We record some observations concerning TCo and both set and class forms of the Axiom of Foundation.

1·19 REMARK In S_0, the assertions that every set is a member of a transitive set and that every set is a subset of a transitive set are equivalent.

I am grateful to Professor Jané of Barcelona for pointing out the following to me:

1·20 PROPOSITION *Transitive Containment is provable in* KP.

Proof : Consider $\{x \mid \neg\exists y\ (x \in y\ \&\ \bigcup y \subseteq y\}$. This is Π_1: suppose it not empty, and let a be a minimal element. Then $\forall x \mathbin{:\in} a\ \exists y\ (x \in y\ \&\ \bigcup y \subseteq y\}$. By Δ_0 Collection, $\exists b \forall x \mathbin{:\in} a\ \exists y \mathbin{:\in} b\ (x \in y\ \&\ \bigcup y \subseteq y\}$. By Δ_0 Separation, $b \cap \{y \mid \bigcup y \subseteq y\}$ is a set, call it c. Then $\bigcup c$ is a set, which is transitive, being the union of transitive sets, and $a \subseteq c$. So $c \cup \{a\}$ is a transitive set of which a is a member, contradicting the choice of a. Hence every set is a member of a transitive set. ⊣ (1·20)

1·21 PROPOSITION *In* Z^+, *class Foundation is provable.*

Proof : let A be a non-empty class, x one of its members. Using TCo, let $x \in y$, with y transitive. Using Separation, form the set $y \cap A$, which is non-empty as x is a member. Apply set Foundation to $y \cap A$. ⊣ (1·21)

1·22 COROLLARY *In* $\mathsf{Z} + \Delta_0$ *Collection the following are equivalent:*

i) TCo + *set Foundation;*
ii) Π_1 *Foundation;*
iii) Π_∞ *Foundation.*

Proof : (ii) implies (i) by 1·20. (iii) trivially implies (ii). The proof of the previous Proposition completes the circle. ⊣ (1·22)

1·23 REMARK To adopt for a moment Boffa's notation in [C4], it is known, and three different proofs are given in [C2], [C3] and [C4], that the scheme $\mathcal{D}$ of Class Foundation is not provable in ZC (including the axiom, D, of Set Foundation), assuming, of course, the consistency of the latter system. It follows from 1·21 that TCo is not a theorem of ZC.

Boffa in [C3] and [C4] indeed gives two proofs of a stronger result, that TCo is not a theorem of $\mathsf{ZC} + \mathcal{D}$.

That the relative consistency results presented in this section will extend easily to systems including the well-ordering principle follows from the next two propositions.

1·24 PROPOSITION (M_0) *The Cartesian product of two sets is a set.*

Proof : $x \times y = \{\langle a, b\rangle \mid a \in x\ \&\ b \in y\}$, where, with the usual definition of ordered pair, $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. So if $a \in x$ and $b \in y$, both $\{a\}$ and $\{a, b\}$ are subsets of $x \cup y$ and so members of $\mathcal{P}(\bigcup\{x, y\})$; hence $\langle a, b\rangle$ is a subset of $\mathcal{P}(\bigcup\{x, y\})$ and so a member of $\mathcal{P}(\mathcal{P}(\bigcup\{x, y\}))$; and hence $x \times y \subseteq \mathcal{P}(\mathcal{P}(\bigcup\{x, y\}))$. An application of Δ_0 Separation now suffices to prove that $x \times y \in V$. ⊣ (1·24)

1·25 PROPOSITION (M_0) WO $\Longrightarrow$ (WO)^0.

Proof : X be a set in W_0, say $X \in a \in F_0$. Let $R \subseteq X \times X$ be a well-ordering of X. Then R is a subset of $a \times a$ which in turn is a subset of $\mathcal{PP}(a)$ and therefore a member of $\mathcal{PPP}(a)$, which is a member of F_0 by Corollary 1·2. Hence R is in the inner model W_0 and will still be a well-ordering there. ⊣ (1·25)

1·26 REMARK The plausible implication AxInf $\Longrightarrow$ (AxInf)^0 is not provable: we shall in section 8 construct a model where it would fail.

1·27 REMARK In M_0 + TCo, the *transitive closure* $tcl(x)$ of a set x — the minimal transitive set including the said set — may be shewn to exist: given x, let $x \subseteq y$ with y transitive; form the set $\mathcal{P}(y) \cap \{z \mid x \cup \bigcup z \subseteq z\}$, using Δ_0 Separation and Power Set; that set is non-empty (y being a member); its intersection is the desired transitive closure.

1·28 REMARK Transitive closures may be also be proved to exist in KP, (without the Axiom of Infinity but using Π_1 Foundation and Transitive Containment), by the recursive definition

$$tcl(x) = x \cup \bigcup\{tcl(y) \mid y \in x\}.$$

With some effort we may see that that recursion also succeeds in M_1: we give the details, as the definability of *tcl* will be important in later sections. The reader will forgive the anticipatory use of the symbol $\Delta_0^{\mathcal{P}}$: the concept of a $\Delta_0^{\mathcal{P}}$ formula will be introduced in §6.

1·29 PROPOSITION *The predicates $y \in tcl(x)$ and $u = tcl(x)$ are, provably in M_1, equivalent to formulæ that are Σ_1 and Π_1, and also, in the case of $u = tcl(x)$, $\Delta_0^{\mathcal{P}}$.*

Proof :

$$w \in tcl(x) \Longleftrightarrow \underbrace{\forall u(\bigcup u \subseteq u \supseteq x \implies w \in u)}_{\Pi_1}$$

$$\Longleftrightarrow \underbrace{\exists n \exists f\, \mathrm{Fn}(f)\ \&\ 0 < n = \mathrm{Dom}\,(f) < \omega\ \&\ \forall k{:<}n\ f(k{+}1) \in f(k)\ \&\ f(0) \in x\ \&\ f(n{-}1) = w}_{\Sigma_1}$$

$$u = tcl(x) \Longleftrightarrow \underbrace{x \subseteq u\,\&\,\bigcup u \subseteq u\,\&\,\forall z{:\subseteq}u\,(x \cup \bigcup z \subseteq z \implies u = z)}_{\Delta_0^{\mathcal{P}} \text{ and } \Pi_1}.$$

Here is a Σ_1^M expression, using the axiom of infinity:

$$u = tcl(x) \Longleftrightarrow \exists f f(0) = x\ \&\ \forall n{:\in}\omega\ f(n+1) = \bigcup f(n)\ \&\ u = \bigcup_{n \in \omega} f(n)$$

Without $\omega \in V$, we must build attempts at the recursion as in KP. Our definition will be:

$$u = tcl(x) \Longleftrightarrow$$

$$\underbrace{\exists v\Big(x \in v\,\&\,\bigcup v \subseteq v\,\&\,\exists g\big(\mathrm{Fn}(g)\,\&\,\mathrm{Dom}\,(g) = v\,\&\,\forall y{:\in}v\ g(y) = y \cup \bigcup\{g(z) \mid z \in y\}\ \&\ g(x) = u\big)\Big)}_{\Sigma_1}$$

Let us call a function g with transitive domain satisfying the recursion equation an *attempt*. We wish to show that an attempt exists with domain any given transitive set v. So let v be transitive.

Note that for any attempt g with domain a transitive subset v' of v, we have $g : v' \longrightarrow \mathcal{P}(v)$; if not, consider $v' \cap \{x \mid g(x) \not\subseteq v\}$: that is a set by Δ_0 Separation, and so if non-empty has a minimal element, by Foundation, $\bar{x}$ say: but then $g(z) \subseteq v$ for each $z \in \bar{x}$; $\bar{x} \subseteq v$ as v is transitive; and so $g(\bar{x}) = \bar{x} \cup \bigcup\{g(z) \mid z \in \bar{x}\}$ is a subset of v after all.

So all such attempts are functions from v to $\mathcal{P}(v)$, hence are subsets of $\mathcal{P}(v) \times v$ and therefore elements of $\mathcal{P}(\mathcal{P}(v) \times v)$. This fact compensates for our lack of Δ_0 Collection.

Put $w = \mathcal{P}(\mathcal{P}(v) \times v)$ and $v \cap \{x \mid \neg\exists g{:\in}w\ x \in \mathrm{Dom}\,(g)\}$. That is a set by Δ_0 Separation, and hence if not empty has a minimal element, which we may again call $\bar{x}$.

Now we may complete the proof as in the proof of the Σ_1 recursion theorem for KP: we show that any two attempts agree on the intersection of their two domains; therefore if we form the union of all attempts in w, we get a set G which is an attempt and has domain containing all members of $\bar{x}$. We may form its restriction to $\bar{x}$, and then use that to form an attempt with $\bar{x}$ in its domain, a contradiction. $\dashv$ (1·29)

2: Adding Axiom H

We start with the following naïve idea for adding the axioms of KP to those of M, working in ZF.

Consider $V_{\omega+\omega}$. For each well-founded extensional relation in it, transitise it; the collection of all members of such sets is exactly $H(\beth_\omega)$, which is a model of Z^+ plus Σ_0-replacement plus Π_1-separation, where we write $H(\kappa)$ or H_κ for $\{x \mid \overline{\overline{tcl(x)}} < \kappa\}$ and $H_{\leqslant\kappa}$ for $\{x \mid \overline{\overline{tcl(x)}} \leqslant \kappa\}$.

Thus we have a relative consistency proof: unfortunately our metatheory — ZF in this case — is so strong as to prove the consistency of both theories outright, thus trivialising the fact of their relative consistency.

Our aim therefore, guided by the above idea, is to weaken the metatheory to elementary arithmetic.

To establish Π_1 separation in the proof of Theorem 4, we shall first establish the truth in our model of the following assertion, which we call Axiom H:

$$\forall u \exists T\big(\bigcup T \subseteq T \;\&\; \forall z(\bigcup z \subseteq z \;\&\; \overline{\overline{z}} \leq \overline{\overline{u}} \implies z \subseteq T)\big)$$

We shall call such a T *u-large.* We might express Axiom H in words as asserting the existence of "universal" transitive sets.

As a first estimate of the strength of Axiom H, we show that it yields large von Neumann ordinals, given the axiom of infinity.

2·0 PROPOSITION $(\mathsf{M}+\mathsf{H})$ *To each ordinal κ there exists a larger initial ordinal. In short, $\forall\kappa\, \exists\kappa^+$.*

Proof : Let κ be an infinite ordinal, let $u = \kappa$, and let w be as supplied by Axiom H. Consider $\theta =_{\rm df} \bigcup w \cap ON$. Since w is transitive, θ is a transitive set of ordinals and therefore an ordinal, and $\kappa \leqslant \theta$ since $\kappa \subseteq w$. If $\overline{\overline{\theta}} \leq \overline{\overline{\kappa}}$, then $\overline{\overline{\theta+2}} \leq \overline{\overline{\kappa}}$, so $\theta+2 \subseteq w$, so $\theta \in \theta$, a contradiction: thus $\overline{\overline{\kappa}} < \overline{\overline{\theta}}$. ⊣ (2·0)

2·1 REMARK Without the axiom of infinity, Axiom H is not strong, for it is true in HF, the class of hereditarily finite sets. That may be seen *via* the following amusing

2·2 PROPOSITION *If $\bigcup z \subseteq z$ and $\overline{\overline{z}} \leq n$ then $z \subseteq V_n$.*

Since $V_{\omega+\omega} \cap ON = \omega+\omega$,

2·3 PROPOSITION (ZF) *Axiom H is false in $V_{\omega+\omega}$.*

Hence (assuming the consistency of ZF), Axiom H is not a theorem of Z, of which theory $V_{\omega+\omega}$ is well-known to be a model. We sketch a proof of the following theorem of arithmetic.

2·4 METATHEOREM *If Z is consistent, so is $\mathsf{Z}+\neg\mathsf{H}$.*

Proof : We work in Z to define a transitive model, which might be a proper class, of the theory $\mathsf{Z}+\neg\mathsf{H}$. Set

$$HF =_{\rm df} \bigcup\{x \mid x \text{ is finite and transitive}\}$$
$$\mathcal{T} =_{\rm df} \{x \mid \bigcup x \subseteq x \;\&\; \exists n{:}{\in}\omega \bigcup{}^n x \subseteq HF\}$$

The desired model will be $\mathcal{M} =_{\rm df} \bigcup \mathcal{T}$: to see that $\mathcal{M}$ models $\mathsf{Z}+\neg\mathsf{H}$, show that $\mathcal{M}\cap ON = \omega+\omega$ and that $\mathcal{T}$ is otherwise fruitful in the sense of *Slim Models*, Definition 1·0. Proposition 1·2, as modified by Remark 1·1, both of that paper, completes the proof. ⊣ (2·4)

2·5 In the rest of this section we study a construction that starting from any model of M_0 yields one of $\mathsf{M}_1+\mathsf{H}$. The latter model may be considered as an extension not of the original model of M_0 but of its natural subclass, defined in the previous section, which models M_1. We reason in M_0 = axioms of extensionality, $\varnothing$, pair, union, (difference), power set, and scheme of Δ_0 Separation.

We saw in the last section that the existence of cartesian products is provable in M_0.

2·6 DEFINITION Put

$$F_1 =_{\rm df} \{(a, r) \mid r \text{ is an extensional well-founded relation on } a\}$$

and

$$W_1 =_{\rm df} \{(\alpha, a, r) \mid (a, r) \in F_1 \text{ and } \alpha \in a\}.$$

The elements of W_1 will be the ingredients of our model: we think of (a, r) as denoting, in the model we are building, a transitive set of which α names a member. We must say when two names denote the same element. To assist our intuition we write $\beta \in_r \alpha$ for $\beta \in r``\{\alpha\}$, *i.e.* $\langle\beta, \alpha\rangle \in r$, when $(a, r) \in F_1$ and β and α are in a.

2·7 Definition Given $(a, r), (b, s) \in F_1$, a map ϕ from a subset of a to a subset of b will be called a *partial isomorphism from* (a, r) *to* (b, s) if

(i) $r``\mathrm{Dom}\,\phi \subseteq \mathrm{Dom}\,\phi$;

(ii) for all $\alpha \in \mathrm{Dom}\,\phi$, $\phi(\alpha) =$ "$\{\phi(\beta) \mid \beta \in_r \alpha\}_s$"; *i.e.*, for all $\beta \in_r \alpha$, $\phi(\beta) \in_s \phi(\alpha)$, and for all $\delta \in_s \phi(\alpha)$ there is a $\beta \in_r \alpha$ with $\phi(\beta) = \delta$.

Thus the condition (ii) on ϕ is that $\{\delta \mid \delta \in_s \phi(\alpha)\} = \{\phi(\beta) \mid \beta \in_r \{\alpha\}\}$; or, more succinctly, that $s``\{\phi(\alpha)\} = \phi``r``\{\alpha\}$.

2·8 Remark Note that, provably in M_0, the property of being a partial isomorphism is Δ_0 .

2·9 Lemma *The class of partial isomorphisms from* (a, r) *to* (b, s) *is a* $\Delta_0(a, r, b, s)$ *subset of* $\mathcal{P}(a \times b)$.

2·10 Lemma *A partial isomorphism is 1-1 as far as it goes.*

Proof : If not, let α be r-minimal in $\mathrm{Dom}\,\phi$ such that for some $\alpha' \neq \alpha$, with $\alpha' \in \mathrm{Dom}\,(\phi)$, $\phi(\alpha) = \phi(\alpha')$. This exists in M_0.

Let $\beta \in_r \alpha$: then $\phi(\beta) \in_s \phi(\alpha) = \phi(\alpha')$, so there is a $\delta \in_r \alpha'$ with $\phi(\delta) = \phi(\beta)$; so $\beta = \delta$, by the minimality of α, so $\beta \in_r \alpha'$.

Conversely, let $\beta \in_r \alpha'$: there is a $\delta \in_r \alpha$ with $\phi(\delta) = \phi(\beta)$, so again by the minimality of α, $\beta \in_r \alpha$. By the extensionality of r, $\alpha = \alpha'$, and so ϕ is 1-1. ⊣ (2·10)

2·11 Lemma *Given* (a, r) *and* (b, s) *in* F_1, *any two partial isomorphisms from* (a, r) *to* (b, s) *agree on their common domain.*

Proof : Given ϕ, ψ, $r``(\mathrm{Dom}\,\phi \cap \mathrm{Dom}\,\psi) \subseteq \mathrm{Dom}\,\phi \cap \mathrm{Dom}\,\psi$; let α be an r-minimal element of $\{x \in \mathrm{Dom}\ \phi \cap \mathrm{Dom}\ \psi \mid \phi(x) \neq \psi(x)\}$ — which is easily a set. Then

$$\begin{aligned} \delta \in_s \psi(\alpha) &\Longrightarrow \text{ for some } \gamma \in_r \alpha,\ \psi(\gamma) = \delta \\ &\Longrightarrow \text{ for some } \gamma \in_r \alpha,\ \phi(\gamma) = \delta, \text{ by minimality of } \alpha, \\ &\Longrightarrow \delta \in_s \phi(\alpha); \end{aligned}$$

and conversely $\delta \in_s \phi(\alpha) \Longrightarrow \delta \in_s \psi(\alpha)$, so $\phi(\alpha) = \psi(\alpha)$ by extensionality of s. Contradiction ! ⊣ (2·11)

2·12 Lemma *There is a largest partial isomorphism from* (a, r) *to* (b, s).

Proof : The union of all partial isomorphisms from a to b is a set by Lemma 2·9; its domain is closed under $r``$, and, by the last lemma, that union is also a partial isomorphism — hence maximal and unique. ⊣ (2·12)

We write Ψ_{arbs} for the maximal partial isomorphism from (a, r) to (b, s). The following properties are easily checked.

2·13 Lemma $\Psi_{arar} = id \restriction a$; $\quad \Psi_{arbs}^{-1} = \Psi_{bsar}$; $\quad \Psi_{arbs}\Psi_{bsct} \subseteq \Psi_{arct}$.

Now define two relations on W_1:

$$(\alpha, a, r) \equiv^1 (\beta, b, s) \Longleftrightarrow_{\mathrm{df}} \alpha \in \mathrm{Dom}\,\Psi_{arbs}\ \&\ \Psi_{arbs}(\alpha) = \beta,$$

and

$$(\alpha, a, r) E^1 (\beta, b, s) \Longleftrightarrow_{\mathrm{df}} \alpha \in \mathrm{Dom}\,\Psi_{arbs}\ \&\ \Psi_{arbs}(\alpha) \in_s \beta.$$

2·14 Remark If (a, r) and (c, t) are two members of F_1 in a set A, then any partial isomorphism f between them is a member of $\mathcal{P}(\bigcup^3 A \times \bigcup^3 A)$, and hence locally the relations $\equiv^1$ and E^1 are sets.

The above Lemma may be applied to prove the

2·15 Proposition $\equiv^1$ *is an equivalence relation, and a congruence with respect to* E^1.

Proof : The first clause follows immediately from Lemma 2·13. Suppose that $(\alpha, a, r)E^1(\beta, b, s) \equiv^1 (\gamma, c, t)$. Then $\Psi_{arbs}(\alpha) \in_s \beta$, so $\Psi_{arct}(\alpha) = \Psi_{bsrt}(\Psi_{arbs}(\alpha)) \in_t \gamma$, so $(\alpha, a, r)E^1(\gamma, c, t)$.

Thus we have shown that

$$(\alpha, a, r)E^1(\beta, b, s) \equiv^1 (\gamma, c, t) \implies (\alpha, a, r)E^1(\gamma, c, t).$$

To see that

$$(\gamma, c, t) \equiv^1 (\alpha, a, r)E^1(\beta, b, s) \implies (\gamma, c, t)E^1(\beta, b, s),$$

note that, assuming the antecedent, $\Psi_{ctar}(\gamma) = \alpha$, and so $\Psi_{ctbs}(\gamma) \in_s \beta$. ⊣ (2·15)

2·16 DEFINITION Given $(a, r) \in F_1$, define $(a, r)^+ = (b, s)$ where $b = a \cup \{a\}$ and $s = (a \times \{a\}) \cup r$.

2·17 LEMMA *If* $(a, r) \in F_1$, $(a, r)^+ \in F_1$.

Corresponding to 1·1 we have

2·18 $\mathcal{P}$-CLOSURE OF F_1: "$(a, r) \in F_1$ *and* $B \subseteq \mathcal{P}(a) \Longrightarrow a \cup B \in F_1$"

2·19 REMARK The statement is expressed in inverted commas because it suggests what is happening without expressing it accurately; however, in true Wittgensteinian fashion, the meaning of the proposition will emerge during its proof.

Proof of 2·18: suppose r is a well-founded extensional relation on a. Let $B \subseteq \mathcal{P}(a)$. We must first handle an awkward point: it might be that $a \cap B \neq \varnothing$: if so, let τ be some object not in $a \cup \bigcup\bigcup a$ — easily found even without Foundation, for if z is any set, $z \cap \{x \mid x \notin x\}$ is a set by Δ_0 separation which is not a member of z, by Russell. Then $\forall\alpha :\in a\ \langle\tau, \alpha\rangle \notin a$. For $b \subseteq a$, let $\bar{b} = \{\langle\tau, \alpha\rangle \mid \alpha \in b\}$. Each $\bar{b}$ is a set, being a Δ_0 subclass of $\{\tau\} \times a$.

By choice of τ and the definition of ordered pair, $a \cap \bar{a} = \varnothing$. Let $\bar{r} = \{\langle\langle\tau, \alpha\rangle, \langle\tau, \beta\rangle\rangle \mid \langle\alpha, \beta\rangle \in r\}$, and let $\bar{B} = \{\bar{b} \mid b \in B\}$. $\bar{r}$ is a set, being a Δ_0 subclass of $\bar{a} \times \bar{a}$, and $\bar{B}$ is a set, being a Δ_0 subclass of $\mathcal{P}(\{\tau\} \times a)$.

Suppose $x \in \bar{a} \cap \bar{B}$: then for some $\xi \in a$ and $b \in B$, $\langle\tau, \xi\rangle = x = \bar{b}$: but then for some $\alpha \in b$, $\{\tau\} = \langle\tau, \alpha\rangle = \{\{\tau\}, \{\tau, \alpha\}\}$, so $\tau = \alpha \in a$, contradicting the choice of τ.

Thus $\bar{a} \cap \bar{B}$ is empty, and the $(\bar{a}, \bar{r})$, $\bar{B}$ situation is plainly isomorphic to the (a, r), B one. So we assume that that change, if necessary, has already been carried out, and that $a \cap B = \varnothing$.

For each $x \in B$, ask if there is an $\alpha \in a$ such that $\forall\beta :\in a\ (\beta \in x$ iff $\beta \in_r \alpha)$. If so, α is unique, by extensionality of r: call it α_x. $\alpha_x = \alpha_y \Rightarrow x = y$, by extensionality of $\in$.

Write $B' =_{df} \{x \in B \mid$ no such α_x exists$\}$. We form $c =_{\mathrm{df}} a \cup B'$, and we know that a and B' are disjoint.

Define a relation η on c by

$$\alpha\,\eta\,\beta \text{ if (i) } \alpha, \beta \in a \text{ and } \alpha \in_r \beta \text{ or (ii) } \alpha \in a, \beta \in B' \text{ and } \alpha \in \beta$$

The *or* here is exclusive as $a \cap B' = \varnothing$.

I assert that η is a well-founded extensional relation on c; and the existence of such $(c, \eta) \in F_1$ is the true import of the proposition.

To check extensionality: suppose β, β' are such that $\{\gamma \mid \gamma\,\eta\,\beta\} = \{\gamma \mid \gamma\,\eta\,\beta'\}$. If $\beta \in a$, $\{\gamma \mid \gamma\,\eta\,\beta\} = \{\gamma \mid \gamma \in_r \beta\}$. If $\beta \in B'$, $\{\gamma \mid \gamma\,\eta\,\beta\} = \{\gamma \mid \gamma \in \beta\} = \beta$. If β, β' are both in a or both in B', then $\beta = \beta'$ by extensionality of $\in_r$ or $\in$. If $\beta \in a, \beta' \in B$ then $\beta' = \alpha_{\beta'}$, contrary to the definition of B'.

To check well-foundedness: suppose $z \subseteq a \cup B'$. If $z \cap a \neq 0$, let α be an $\in_r$-minimal element of it. For $x \in z$, $x\,\eta\,\alpha \Rightarrow x \in a$ and so $x \in_r \alpha$, contradicting the minimality of α; thus α is an η-minimal element of z. If $z \cap a = 0$, any element of z is η-minimal. ⊣ (2·18)

We shall refer to (c, η) as $a + B$.

2·20 PROPOSITION E^1 *is set-like in the sense that given any* x *in* W_1*, there is a set* y *with* $x \in y$ *such that* $\forall z(zE^1x \implies \exists w :\in y\ z \equiv^1 w)$.

Proof : Given $x = (\alpha_0, a_0, r_0)$, let $y = \{(\alpha, a_0, r_0) \mid \alpha \in a_0\}$, which is a set, as it equals $a_0 \times \{(a_0, r_0)\}$. If $(\beta, b, s)E^1(\alpha, a_0, r_0)$, let $\gamma = \Psi_{bsa_0r_0}(\beta)$: then (γ, a_0, r_0) is in y and $(\beta, b, s) \equiv^1 (\gamma, a, r)$. ⊣ (2·20)

2·21 REMARK The equivalence classes under $\equiv^1$ will be proper classes and not sets; in our weak set theory, Scott's device for choosing subsets of these proper classes is not available to us. Hence all the details of our construction have to be treated locally.

2·22 PROPOSITION *E^1 is extensional modulo $\equiv^1$: that is, if*

$$\forall(\alpha, a, r)\ [(\alpha, a, r)E^1(\beta, b, s) \Longleftrightarrow (\alpha, a, r)E^1(\gamma, c, t)]$$

then $(\beta, b, s) \equiv^1 (\gamma, c, t)$.

Proof : Write Ψ for Ψ_{bsct}. Suppose that $\alpha \in_s \beta$, then $(\alpha, b, s)E^1(\gamma, c, t)$, so $\Psi(\alpha) \in_t \gamma$, and in particular, $s``\{\beta\} \subseteq \mathrm{Dom}\,\Psi$. Conversely, if $\delta \in_t \gamma$, $(\delta, c, t)E^1(\gamma, c, t)$, so $(\delta, c, t)E^1(\beta, b, s)$, so $\delta \in \mathrm{Dom}(\Psi^{-1})$, and $\Psi^{-1}(\delta) \in_s \beta$.

Thus $\gamma =$ "$\{\Psi(\alpha) \mid \alpha \in_s \beta\}_t$", and so by the maximality of Ψ, $\beta \in \mathrm{Dom}\ \Psi$ and $\Psi(\beta) = \gamma$; hence $(\beta, b, s) \equiv^1 (\gamma, c, t)$, as required. ⊣ (2·22)

Now a counterpart to 1·4:

2·23 $\bigcup$-CLOSURE OF F_1: $\forall A{:}\subseteq F_1\ \exists(b, s){:}\in F_1 \forall(a, r){:}\in A\ \forall\alpha{:}\in a\ \exists\beta{:}\in b\ (\alpha, a, r) \equiv^1 (\beta, b, s)$.

Proof : Given $A \subseteq F_1$, let $B =_{\mathrm{df}} \{(\alpha, a, r) \mid (a, r) \in A \,\&\, \alpha \in a\}$.† B is a set, being a Δ_0 subclass of $(\bigcup^3 A) \times A$.

Factor B by $\equiv^1$; let b be the set of $\equiv^1$-classes and let s be the relation induced by E^1 *via* the factoring. (b, s) certainly represents all the $\equiv^1$ classes represented by "members" of A, but we have to verify that s is well-founded and extensional.

That s is extensional follows from the previous Proposition, since in the present context whenever $(\delta, d, u)E^1(\beta, b, s)$, there is an $(a, r) \in A$ and an $\alpha \in a$ with $(\delta, d, u) \equiv^1 (\alpha, a, r)$. To see that s is well-founded, let $\varnothing \neq X \subseteq b$, and let $[(\alpha_0, a, r)]$ be some member of X, where for $x \in B$ we write $[x]$ for the set $\{y \in B \mid y \equiv^1 x\}$. Let $P = \{\alpha \in a \mid [(\alpha, a, r)] \in X\}$. $\alpha_0 \in P$, so P is a non-empty subset of a. Let α_1 be an r-minimal element of P. I assert that $[(\alpha_1, a, r)]$ is an s-minimal element of X.

For if not, let $[(\gamma, c, t)] \in_s [(\alpha_1, a, r)]$ where $[(\gamma, c, t)] \in X$, $(c, t) \in A$ and $(\gamma, c, t) \in B$. $(\gamma, c, t)E^1(\alpha_1, a, r)$, so let $\alpha_2 = \Psi_{ctar}(\gamma)$. Then $(\gamma, c, t) \equiv^1 (\alpha_2, a, r) \in B$. So $[(\alpha_2, a, r)] = [(\gamma, c, t)]$, and thus $\alpha_2 \in P$ and $\alpha_2 \in_r \alpha_1$, contradicting the latter's minimality.

Thus $(b, s) \in F_1$. ⊣ (2·23)

2·24 DEFINITION We shall refer to (b, s) as the "union" of A.

2·25 PROPOSITION *E^1 is well-founded.*

Proof : Let C be a non-empty subset of W_1, and let $A = \bigcup^2 C \cap \{(a, r) \mid \exists\alpha{:}\in\bigcup^2 C\ (\alpha, a, r) \in C\}$. Thus A is a set, and equals the class $\{(a, r) \mid \exists\alpha(\alpha, a, r) \in C\}$.

$A \subseteq F_1$, and so the "union" of A as supplied by our last proposition gives a member $(b, s) \in F_1$ such that

$$\forall(\alpha, a, r){:}\in C\ \exists\beta{:}\in b\ (\alpha, a, r) \equiv^1 (\beta, b, s).$$

Form the set $b \cap \{\beta \mid \exists x {:}\in C\ x \equiv^1 (\beta, b, s)\}$: that will be non-empty, and so let $\bar\beta$ be an s-minimal element of it. Any $x \in C$ with $x \equiv^1 (\bar\beta, b, s)$ will be an E^1-minimal element of C. ⊣ (2·25)

Notice that our arguments yield, without use of any form of the axiom of choice, the pivotal result that the system M_0 proves the existence of "universal" well-founded extensional relations:

2·26 PROPOSITION *Let u be a set. There is an extensional well-founded relation (b, s) such that for any extensional well-founded relation (a, r) with $\overline{\overline{a}} \leqslant \overline{\overline{u}}$, $a = \mathrm{Dom}\,\Psi_{arbs}$, and thus the restriction of s to $\mathrm{Im}\Psi_{arbs}$ is isomorphic to (a, r).*

† Remember that $(x, y, z) = (x, (y, z))$.

Proof : consider the class $B_u = \{(c,t) \mid c \subseteq u \;\&\; t \text{ is a well-founded extensional relation on } c\}$. That is, provably in M_0, a set, for each $c \in \mathcal{P}(u)$ and each $t \in \mathcal{P}(u \times u)$: thus B_u is a sub-class of $\mathcal{P}(u) \times \mathcal{P}(u \times u)$. The hardest clause defining membership of B_u is the well-foundedness of t, but that is expressible by quantification over the subsets of c, and therefore over the members of $\mathcal{P}(u)$. Hence Δ_0 separation suffices to establish the set-hood of B_u. Thus B_u is a subset of F_1; its "union" will be the required (b,s). ⊣ (2·26)

It is Proposition 2·26 that will yield the truth of Axiom H in the model we are about to build.

2·27 DEFINITION For $\mathfrak{A}$ any wff in $\in$ and $=$, let $(\mathfrak{A})^1$ be the wff where first $\in$ and $=$ are replaced by E^1 and $\equiv^1$, and then all quantifiers are restricted to range over W_1.

2·28 THEOREM SCHEME *For* $\mathfrak{A}$ *any theorem of* $\mathsf{M}_1 + \mathsf{H}$, $(\mathfrak{A})^1$ *is provable in* M_0*; and* $\mathsf{WO} \implies (\mathsf{WO})^1$ *and* $\mathsf{InfWel} \implies (\mathsf{InfWel})^1$ *are also provable in* M_0*, as, indeed, is* $\mathsf{InfWel} \implies (\omega \in V)^1$.

The proof will proceed by stages, some of which are left to the reader.

2·29 PROPOSITION *(*Extensionality*)*^{1}.

Proof : by Proposition 2·22.

2·30 PROPOSITION *(*Empty Set*)*^{1}.

2·31 PROPOSITION *(*Pairing*)*^{1}.

Proof : given (α,a,r), (β,b,s), form "$\bigcup$"$\{(a,r),(b,s)\}$; if the pair set of α and β is not there, form "$\bigcup$"$\{(a,r),(b,s)\} + \{\{\alpha,\beta\}\}$. ⊣ (2·31)

2·32 PROPOSITION *(*Union*)*^{1}.

Proof : note first that $\beta \, E^1 \, \gamma \, E^1 \, \alpha \in a \Longrightarrow \exists \beta'(\beta \equiv^1 \beta' \in a)$. It suffices, therefore, given $(\alpha,a,r) \in W_1$, to form $a + \{\{\beta \mid \exists\gamma(\gamma \, E^1 \, \alpha \;\&\; \beta \, E^1 \, \gamma\}\}$. ⊣ (2·32)

2·33 PROPOSITION *(*Difference*)*^{1}.

Proof : given $(\alpha,a,r) \in W_1$, form $(a,r) + \{\gamma \in a \mid \gamma \in_r \alpha \;\&\; \neg[(\gamma,a,r)E^1(\beta,b,s)]\}$. ⊣ (2·33)

2·34 PROPOSITION *(*Power Set*)*^{1}.

Proof : given $(\alpha,a,r) \in W_1$, form $a + \{x \mid x \subseteq \{\beta \mid \beta \in_r \alpha\}\}$. ⊣ (2·34)

2·35 PROPOSITION *(*Foundation*)*^{1}*:*

Proof : given $(\alpha,a,r) \in W_1$, let β be an r-minimal element of α, and check that (β,a,r) is an E^1-minimal element of (α,a,r). ⊣ (2·35)

2·36 PROPOSITION *(*TCo*)*^{1}*:*

Proof : given (α,a,r), form $a + \{a\}$. ⊣ (2·36)

We prove the next lemma only for formulæ with two variables, but its proof plainly works for those with more.

2·37 LEMMA *Let* $\mathfrak{A}$ *be a formula with (say) the free variables* x, y*. Let* $z = (\alpha,a,r)$ *and* q *be members of* W_1*. Let* $I_{\mathfrak{A},z,q}$ *be the class* $a \cap \{\gamma \mid \gamma \in_r a \;\&\; (\mathfrak{A})^1(z,p)\}$*. If* $I \in V$ *then* $(z \cap \{x \mid \mathfrak{A}(x,p)\} \in V)^1$.

Proof : We seek $(\beta,b,s) \in W_1$ such that

$$\forall(\delta,d,t){:}{\in}W_1 \; (\delta,d,t)E^1(\beta,b,s) \Longleftrightarrow \exists\gamma{:}{\in}a \left[\gamma \in_r \alpha \;\&\; (\delta,d,t) \equiv^1 (\gamma,a,r) \;\&\; (\mathfrak{A})^1((\gamma,a,r),p)\right].$$

If $I = I_{\mathfrak{A},z,q}$ is a set, let $(b,s) = a + \{I\}$, and let $\beta = I$. Then (β,b,s) is as desired. ⊣ (2·37)

2·38 PROPOSITION *(*Δ_0 Separation*)*^{1}.

Proof : by the Lemma, we need only prove that $I_{\mathfrak{A},z,q} \in V$, for $\mathfrak{A}$ a Δ_0 formula and z, q in W_1. But the property $\gamma \in_r \alpha$ is a $\Delta_0(r,\alpha)$ predicate of γ, and $(\mathfrak{A})^1$ will be Δ_0 relative to some "universal" set of the kind produced by Proposition 2·26 that includes copies of all the relevant members of F_1. ⊣ (2·38)

2·39 LEMMA *If* $\mathfrak{k} \geqslant 1$ *and* $\mathfrak{A}$ *is a* $\Sigma_{\mathfrak{k}}$ *formula,* $(\mathfrak{A})^1$ *will be a* $\Sigma_{\mathfrak{k}+1}$ *formula.*

Proof : E^1 and $\equiv^1$ are Δ_1 relations, and so in computing the complexity of $(\mathfrak{A})^1$ for $\mathfrak{A}$ a $\Sigma_\mathfrak{k}$ formula our only problem will be the restrictions that the members of F_1 are *well-founded* extensional relations on sets. As with the case of $(\mathfrak{A})^0$, each such restriction requires a quantifier that can be absorbed into the next one, until we reach the matrix. ⊣ (2·39)

2·40 PROPOSITION *For each meta-integer* $\mathfrak{k} \geqslant 1$*: if* $\Sigma_{\mathfrak{k}+1}$ *Separation holds, then* $(\Sigma_\mathfrak{k}\ \mathsf{Separation})^1$ *holds.*

Proof : by the two Lemmata. ⊣ (2·40)

2·41 PROPOSITION $(\mathsf{Axiom\ H})^1$

Proof : by Proposition 2·26.

2·42 PROPOSITION $\mathsf{WO} \implies (\mathsf{WO})^1$.

The proof is similar to that of Proposition 1·25. The next Proposition shows that we get an improved version of the axiom of infinity in our new model.

2·43 PROPOSITION $\mathsf{InfWel} \implies (\omega \in V)^1$.

Proof : again, essentially by Proposition 2·26. The details are spelt out in the proof of Lemma 3·0. ⊣ (2·43)

The proof of Theorem Scheme 2·28 is now complete, save for the details of the last two observations, which will be covered by the discussion of §3. Our digression into computing the quantifier complexity of $(\mathfrak{A})^1$ yields these further results:

2·44 THEOREM SCHEME *For* $\mathfrak{k} \geqslant 1$ *and* $\mathfrak{A}$ *any theorem of* $\mathsf{M} + \mathsf{H} + \Sigma_\mathfrak{k}\ \mathsf{Separation}$, $(\mathfrak{A})^1$ *is provable in* $\mathsf{M} + \Sigma_{\mathfrak{k}+1}\ \mathsf{Separation}$.

2·45 THEOREM SCHEME *For* $\mathfrak{A}$ *any theorem of* $\mathsf{Z} + \mathsf{TCo} + \mathsf{H}$, $(\mathfrak{A})^1$ *is provable in* Z.

Hence, provably in arithmetic, we have

2·46 METATHEOREM *If* M_0 *is consistent, so is* $\mathsf{M}_1 + \mathsf{H}$*; if* $\mathsf{M}_0 + \mathsf{AxInf}$ *is consistent, so is* $\mathsf{M} + \mathsf{H}$*; if* $\mathsf{Z}_{\mathfrak{k}+1}$ *is consistent, so is* $\mathsf{Z}_\mathfrak{k} + \mathsf{TCo} + \mathsf{H}$*; if* Z *is consistent, so is* $\mathsf{Z} + \mathsf{TCo} + \mathsf{H}$*; and similarly for those systems with* AC *added: in particular, if* ZBQC *is consistent, so is* $\mathsf{MAC} + \mathsf{H}$.

That completes the proof of Theorem 2 and of the first part of Theorem 5.

We should note that there is a counterpart to Proposition 2·26 for well-orderings. Let us first observe that although even in Z we cannot prove that every wellordering is isomorphic to an ordinal, we can prove in M_0 that any two well-orderings are comparable.

2·47 PROPOSITION (M_0) *Given any two well-orderings, there is an order-preserving isomorphism between one of them and a (possibly improper) initial segment of the other.*

Proof : given well-orderings $(X, \leq)$, $(Y, \leq)$, where X and Y are sets, call a function ϕ a *partial isomorphism* from X to Y if $\mathrm{Dom}(\phi)$ is an initial segment of X under $\leq$, $\mathrm{Im}(\phi)$ is an initial segment of Y under $\leq$, and for all $x \in \mathrm{Dom}(\phi)$, $\phi(x) =$ least element of Y not of the form $\phi(x')$ for any $x' < x$. As we regard functions as sets of ordered pairs, we may apply Δ_0 separation to $\mathcal{P}(X \times Y)$ to show in M_0 that

$$\{\phi \subseteq X \times Y \mid \phi \text{ is a partial isomorphism from } X \text{ to } Y\}$$

is a set, the union of which is readily verified, as in Zorn's Lemma type arguments, to be a isomorphism, either between X and an initial segment of Y or between an initial segment of X and the whole of Y. ⊣ (2·47)

2·48 PROPOSITION (M_0) *Given any well-ordering there is another of a strictly larger cardinality.*

Proof : Let $<_X$ well-order X. Consider the set of all well-orderings of subsets of X. Identify any two such if they are the same length. Form the set of equivalence classes, and well-order them by comparability. All that may be carried out in M_0. The result will be a well-ordering of a higher cardinality than that of X. ⊣ (2·48)

2·49 PROBLEM Might one prove the above by picking out a long well-ordering from the universal extensional well-founded relation, along the following lines ? Let (X, R) be our universal well-founded extensional relation. Extract a long well-ordering from it by putting

$$W = \{x \in X \mid \forall z {:\in} X\ \forall y {:\in} X\ \forall w {:\in} X\ [(zRyRx \implies zRx)\ \&\ (wRzRyRx \implies wRy)]\}.$$

That definition is inspired by a characterisation of the von Neumann ordinals as the hereditarily transitive sets. It would lead to a new proof of the previous proposition if we had the Axiom H property in some kind of Amalgamation form.

2·50 REMARK The constructions of *Slim Models* [E1] show that two different models of Z might lead to the same model of $\mathsf{Z}+\mathsf{H}$.

3: Examination of Axiom H

We have two aims in this section:

1: to deduce Σ_1 Separation from Axiom H in MAC, and thereby complete the proof of Theorems 3 and 4.

The Axiom of Choice plays an important rôle in the above deduction.

2: working in $\mathsf{M}+\mathsf{KP}$, to deduce Axiom H from Σ_1 Separation.

Preparatory to the second we shall show that under AC several statements are equivalent.

One plausible statement that turns out not to be equivalent in general to Axiom H is the assertion that for every von Neumann ordinal κ, a larger initial von Neumann ordinal exists, a statement that we have abbreviated by the formula $\forall\kappa\,\exists\,\kappa^+$.

We shall discuss this statement in §5, where we shall prove in $\mathsf{M}+\mathsf{KPL}$ first that $\forall\kappa\,\exists\,\kappa^+$ and then deduce Axiom H. We shall in §6 in ZF prove that there is a transitive model for $\mathsf{MAC}+\mathsf{KP}$ in which every ordinal is countable, and that there is a transitive model for $\mathsf{MAC}+\mathsf{KP}+\forall\kappa\,\exists\,\kappa^+$ in which Axiom H is false.

Our first step generalises the result we have already seen, that Axiom H implies that $\forall\kappa\ \kappa^+$ exists.

3·0 LEMMA $(\mathsf{M}_0+\mathsf{H})$ *suppose T is u-large as in the statement of Axiom H. Then every extensional well-founded relation of cardinality at most $\overline{\overline{u}}$ is isomorphic to a transitive subset of T.*

Proof : let $(a,\, r)$ be extensional, well-founded, with $\overline{\overline{a}} \leqslant \overline{\overline{u}}$. We define a *$T$-attempt* to be a map f with domain a subset z of a with $r``z \subseteq z$, such that f satisfies the recursion equation:

$$\forall x{:}{\in}\mathrm{Dom}\, f\ f(x) = \{f(y) \mid \langle y,\, x\rangle \in r\}.$$

We note that each such f is injective, since r is extensional, and its image is therefore a transitive set of cardinality $\overline{\overline{\mathrm{Dom}\,(f)}} \leqslant \overline{\overline{u}}$ and thus a subset of T.

So since the image of any attempt is a subset of T, we can form the set of attempts, as it is a Δ_0 (in parameters a, r, etc.) subclass of $\mathcal{P}(T\times a)$, and we can form the union, F, of the set of attempts, which will be a set.

No two attempts disagree, for let f and g be attempts, and consider an r-minimal element $\bar{x}$ of $\{x \in \mathrm{Dom}\,(f)\cap\mathrm{Dom}\,(g) \mid f(x) \neq g(x)\}$. Then $f \restriction r``\bar{x} = g \restriction r``\bar{x}$ and so $f(\bar{x}) = \mathrm{Im}(f \restriction r``\bar{x}) = \mathrm{Im}(g \restriction r``\bar{x}) = g(\bar{x})$, a contradiction.

It follows that F is also an attempt. We assert that its domain is the whole of a: for consider *per impossibile* an r-minimal element $\bar{y}$ of $a \setminus \mathrm{Dom}\, F$. Then $r``\bar{y} \subseteq \mathrm{Dom}\, F$, and hence we might extend $F$ by setting $F(y) = \mathrm{Im}(F \restriction r``\bar{y})$, so $\bar{y}$ is in the domain of F after all. Thus $\mathrm{Dom}\, F = a$ and the image of F is the desired transitive subset of T isomorphic to $\langle a,\, r\rangle$. ⊣ (3·0)

3·1 REMARK No use of choice was made in that argument, nor was it in the formulation of Axiom H.

We have actually established the following equivalence:

3·2 PROPOSITION *Over M_0, the following are equivalent:*

(i) Axiom H;

(ii) the statement that every extensional well-founded relation is isomorphic to a transitive set.

Proof : we have seen that (i) implies (ii). For the reverse, apply (ii) to the "universal" extensional well-founded relations provided by Proposition 2·26. ⊣ (3·2)

The proof of 3·0 shows how recursive definitions succeed in M when a set containing the image of the function can be given in advance. Here is another example:

3·3 PROPOSITION (M_0) *Every subset of a von Neumann ordinal, defined as being a transitive set well-ordered by the ∈-relation, is order-morphic to a von Neumann ordinal that is no greater than the first.*

However, when the containing set has to be created — which is, of course, the purpose of Δ_0 Collection— we get into difficulties:

3·4 EXAMPLE Let $a = \{0, 1, 2, \{2\}\}$, $b = \{1, 2, \{2\}\}$, and $c = \{0, 1, \{1\}\}$. Then b is extensional and isomorphic to c, which is transitive but not a subset of a.

In an early draft of this paper it was asserted that "we can prove in M_0 that an extensional subset of a transitive set is isomorphic to a transitive set", the reason offered being that "the implicit recursion is contained within the power set of the given transitive set".

The last example refutes the reason, and the following proposition and example counter-instance the assertion.

3·5 PROPOSITION (Foundation + Δ_0 Separation) *No two transitive sets are isomorphic.*

Proof : Let a and b be transitive and suppose that $f : a \longrightarrow b$ is such that $\forall x :\in a\ f(x) = \{f(y) \mid y \in x \cap a\}$. Since a is transitive, $x \cap a = x$ for each $x \in a$, so $\forall x :\in a\ f(x) = \{f(y) \mid y \in x\}$. Let $\bar{x}$ be an ∈-minimal element of the set $\{x \in a \mid x \neq f(x)\}$, supposing that to be non-empty. Then $f(\bar{x}) = \{y \mid y \in \bar{x}\} = \bar{x}$, a contradiction. Hence $\forall x :\in a\ f(x) = x$ and so $b = \mathrm{Im} f = a$. ⊣ (3·5)

3·6 REMARK Foundation is necessary in the above argument, since without Foundation we might have $x = \{x\}$, $y = \{y\}$ but $x \neq y$. In such a case, x and y are transitive and isomorphic but not equal. ‡

3·7 EXAMPLE By the remark preceding Lemma 4·1 of *Slim Models* there is a transitive model of $\mathsf{Z} + \mathsf{TCo}$ for which a set, called $Z(\omega)$ in the notation of that paper, is a member of the model, and indeed an extensional subset of the transitive member $T(\omega)$ of the model, but that set is not isomorphic to any transitive set of the model, since $Z(\omega)$ is isomorphic to $HF = Z(0)$ which is not in the model.

In the proof of the next Proposition we shall use a little mild model theory, and therefore must adopt the Axiom of Infinity to ensure that our languages are sets. We defer till §5, where they will be most needed, a statement of our conventions regarding formal (Gödelised) languages. We use φ, ϑ as variables for formulæ of such languages.

3·8 LEMMA ($\mathsf{M}_0 + \omega \in V$) *If A is a set, so is ${}^{<\omega}A$, the set of finite sequences of members of A.*

Proof : the set in question is a subclass of $\mathcal{P}(A \times \omega)$ which is Δ_0 in suitably chosen parameters. ⊣ (3·8)

3·9 REMARK A corresponding result would hold in a set theory with an axiom of infinity but without the assumption that $\omega \in V$.

3·10 PROPOSITION *We can define the model-theoretic satisfaction relation $\models$ in the theory $\mathsf{M}_0 + \omega \in V$, so that for any model $\mathbf{N} = (N, R)$, the class $\{(\varphi, \vec{a}) \mid \vec{a} \in N \ \&\ \mathbf{N} \models \varphi[\vec{a}]\}$ is a set.*

Proof : because in this case the recursions *are* contained. Since there is no reason to expect that V_ω is a set, we must regard the formulæ of formal languages as coded by members of ω. Then for any particular model (N, R) say, $\{(\varphi, \vec{a}) \mid \vec{a} \in {}^{<\omega}N \ \&\ (N, R) \models \varphi[\vec{a}]\}$ will be a subclass of $\omega \times {}^{<\omega}N$, and its characteristic function will be definable by a contained recursion on the well-founded relation "φ is a subformula of ϑ". ⊣ (3·10)

In the proof of the next proposition we shall use the familiar model-theoretic notion of the *Skolem hull* of a subset A of a model. In fact the $\dot{\Delta}_0$ hull $Hull_0$ will suffice. In §5 we shall define the $\dot{\Sigma}_{\mathfrak{k}}$ hull of a subset of the universe with some care, and therefore are content here to leave the reader to complete the details of the following sketch.

To build the $\dot{\Delta}_0$ hull of A, we first define Skolem functions f_φ using a fixed well-ordering of the domain of the model: $f_\varphi(\vec{a})$ is to be the first element of the domain to satisfy a given $\dot{\Delta}_0$ property φ, if such an element exists, and some fixed element of the domain — its empty set will do — otherwise.

‡ See Aczel's non-well-titled tome [C5] for a discussion of possible ways in which the Axiom of Foundation might fail. Note that his Axiom of Infinity as given on page 117 can in his theory be satisfied by a two element set and needs the Axiom of Foundation to produce an infinite set. In his new book *Vicious circles* he corrects this slip.

We then remark that if $A \subseteq X$ and $f : \omega \times {}^{<\omega}X \longrightarrow X$ is a function, then the f-closure of A may be conveniently defined in $\mathsf{M}_0 + \omega \in V$ as

$$\bigcap\{B \in \mathcal{P}(X) \mid A \subseteq B \;\&\; f``{}^{<\omega}B \subseteq B\}.$$

The $\dot\Delta_0$ hull of A will be the closure of A under the function f given by $f(\varphi, \vec{a}) = f_\varphi(\vec{a})$.

3·11 PROPOSITION SCHEME ($\mathsf{MAC} + \mathsf{H}$) *Strong* Δ_0 *Collection.*

Proof : We wish to show that $\forall a \exists b \forall x :\in a\ \big(\exists y \mathfrak{D}(x,y,d) \implies \exists y :\in b\, \mathfrak{D}(x,y,d)\big)$, where $\mathfrak{D}$ is Δ_0 and d is some parameter. Use *TCo* to find and fix a transitive C of which a and d are members. Let T be the transitive set given by Axiom H such that $\forall u(\overline{\overline{u}} \leq \overline{\overline{C}}$ & u transitive $\implies u \subseteq T)$. We shall show that we may take $b = T$.

Let $x \in a$, and suppose that y is such that $\mathfrak{D}(x,y,d)$. Pick a transitive set B containing y, and C. Using a well-ordering of B form $H =_{\mathrm{df}} Hull_0(B, C \cup \{y,\, x\})$. H is extensional and, by the Axiom of Foundation; well-founded, and so we may by Proposition 3·0 build the collapsing map $\varpi_H : H \longrightarrow T$. As $C \subseteq H$ and C is transitive, $\varpi_H \restriction C$ is the identity, and so $\varpi(x) = x$ and $\varpi_H(d) = d$. Let $\bar{y} = \varpi_H(y)$. Then since $\mathfrak{D}$ is Δ_0, its truth is preserved by ϖ: $(\mathfrak{D}(x,y,d))^B$, so $(\mathfrak{D}(x,y,d))^H$ and hence $\mathfrak{D}(x,\bar{y},d)$.

Thus T contains a y such that $\mathfrak{D}(x,y,d)$, for each x in a such that $\exists \mathfrak{D}(x,y,d)$. ⊣ (3·11)

A modest extension of that argument yields

3·12 PROPOSITION SCHEME ($\mathsf{MAC} + \mathsf{H}$) *Strong* Σ_1 *Collection.*

Proof : with the help of pairing functions adapt the previous proof to show, for $\mathfrak{D}$ a Δ_0 formula and d some parameter, that $\forall a \exists b \forall x :\in a\ \big(\exists y \exists z \mathfrak{D}(x,y,z,d) \implies \exists y :\in b\ \exists z :\in b\ \mathfrak{D}(x,y,d)\big)$. ⊣ (3·12)

The following completes the proof of Theorem 4.

3·13 THEOREM SCHEME *In the system* $\mathsf{MAC} + \mathsf{H}$, *all axioms of* KP *plus the scheme of* Π_1 *Separation are provable.*

Proof : Strong Σ_1 Collection implies Π_1 Separation: to form $a \cap \{x \mid \forall y\, \mathfrak{A}\}$, where $\mathfrak{A}$ is Δ_0, use strong Σ_1 Collection to find a b such that $\forall x :\in a\ (\exists y \neg\mathfrak{A} \implies \exists y :\in b\ \neg\mathfrak{A})$, so the desired set is $a \cap \{x \mid \forall y :\in b\ \mathfrak{A}\}$, which is a set by Δ_0 Separation.

Once we have Π_1 Separation, we may deduce Π_1 Foundation from Foundation and Transitive Containment much as in the proof of Proposition 1·21. ⊣ (3·13)

We turn to the deducibility in $\mathsf{MAC} + \mathsf{KP}$ of Axiom H from Σ_1 Separation. That is part of the following Proposition; the eccentric numbering of its clauses is induced by that of a later theorem.

3·14 PROPOSITION *Over* MAC, *the following are equivalent:*

(i') *Axiom H;*

(v') *the scheme of strong* Σ_1 *Collection;*

(vi') *the schemes of* Δ_0 *Collection and* Π_1 *Separation.*

(ii') *the schemes of* Π_1 *Foundation and* Δ_0 *Collection, with the statement that every well-ordering is isomorphic to an ordinal.*

Proof : We have seen in the proof of 3·12 that (i') implies (v') and in that of 3·13 that (v') yields (vi'). We assume (vi') and derive (ii'). Π_1 Foundation follows from Π_1 Separation using set Foundation. To complete the verification of (ii'), we must show that every well-ordering is isomorphic to an ordinal.

Given a well-ordering (X, R), we collapse R by the recursion

$$\varpi_R(x) = \{\varpi_R(y) \mid yRx\};$$

the class $X \setminus \mathrm{Dom}\,(\varpi)$ is Π_1 and therefore a set by Π_1 Separation. Had it a least member under R the definition of ϖ_R could be extended; so it is empty, and ϖ_R is total. Its image is a von Neumann ordinal.

Finally we assume (ii') and derive (i'). Note that MAC together with (ii') includes all axioms of KP. We must derive Axiom H. We know from 3·2 that it will be enough to prove that every well-founded extensional relation is isomorphic to a transitive set.

Let therefore (C,R) be a well-founded extensional relation. By WO, we know that C has a well-ordering, and by our assumption that well-ordering will be isomorphic to some von Neumann ordinal κ. Further, by considering ordinals isomorphic to well-orderings of $\mathcal{P}(C)$, we know that there will be ordinals of cardinality greater than κ; the class of such is Π_1 and therefore by Π_1 foundation has a least member, which we may call κ^+. Our proof divides into two steps.

step 1: considering attempts into κ^+, we see that there is a ranking on (C,R) given by:

$$\varpi_R(x) = \bigcup\{\varpi_R(y)+1 \mid yRx\};$$

step 2: using that ranking, we see that the collapse of R is total.

Here is the proof of step 2. It is a commonplace that KP is not strong enough to prove that every extensional well-founded relation has a Mostowski collapse. We prove that provided the relation is also ranked and the ranks bounded, the collapse does indeed exist:

3·15 LEMMA (KP) *Let R, X, θ and ψ be sets such that R is a relation on X, θ is an ordinal, $\psi : X \longrightarrow \theta$ and*

$$\forall x,y\,{:}{\in}\, X\ xRy \implies \psi(x) < \psi(y).$$

Then the function $\varpi_R : X \longrightarrow V$ defined by

$$\varpi_R(x) = \{\varpi_R(y) \mid y \in X \ \&\ yRx\}$$

is defined on all of X.

Proof : we imitate the usual proof, using *attempts*, of the recursion theorem in KP; our only problem is to finesse the point where Π_1 foundation is applied to find a minimal counterexample to the conclusion.

So consider $\{\nu < \theta \mid \exists u\,{:}{\in}\, X\ u \notin dom(\varpi_R)\ \&\ \psi(u) = \nu\}$: that, by Δ_0 Collection and the fact that ψ is a set, is a Π_1 class, and so if non-empty, has by Π_1 Foundation a least element $\bar\nu$. Let $\bar u \in X \setminus dom(\varpi_R)$ have $\psi(\bar u) = \bar\nu$. Then $\forall y\,{:}{\in}\, X\ (yR\bar u \implies \psi(y) < \bar\nu)$, and so each such y is in $\mathrm{Dom}\,(\varpi_R)$: further, $\{y \in X \mid yR\bar u\}$ is a subclass of $\mathrm{Dom}\,(\varpi_R)$, and therefore a set, B say; we may now proceed to collect into a set v sufficiently many attempts, at least one for each member of B, to define ϖ_R at $\bar u$, a contradiction. ⊣ (3·15)

3·16 COROLLARY (KP) *If $A \subseteq D$, D is transitive and A extensional then A is isomorphic to a transitive set.*

Proof : we may take the usual set-theoretic rank function ϱ for ψ. ⊣ (3·16)

The proof of Proposition 3·14 is complete. ⊣ (3·14)

3·17 REMARK Corollary 3·16 is false for Z, as shown by the model mentioned in 3·7.

The above methods yield the following:

3·18 THEOREM *Over KP + MAC the following are equivalent:*
- *(i) Axiom H;*
- *(ii) every well-ordering is isomorphic to an ordinal;*
- *(iii) every well-founded extensional relation is isomorphic to a transitive set;*
- *(iv) every well-founded relation may be collapsed;*
- *(v) the scheme of strong Δ_0 Collection;*
- *(vi) the scheme of Σ_1 Separation;*
- *(vii) $\forall\kappa\ H_{\leqslant\kappa}$ exists.*

Clause 3·18(iii) is popularly called Mostowski's isomorphism theorem. I am grateful to the anonymous referee of an early draft of this paper for pointing out the following:

3·19 THEOREM *Over KP + WO, the following are equivalent:*
- *(i) Axiom H;*
- *(ii) Power Set + Σ_1 Separation.*

The arguments of §§5 and 6 will prove the following:

3·20 THEOREM *Over* KPL, *the following are equivalent:*

(i) Axiom H;

(ii) $\forall\kappa\exists\kappa^+$

(iii) Power Set.

and will also show that over $\mathsf{KP} + \mathsf{WO}$ alone, without $V = L$, those three statements are not equivalent.

3·21 REMARK KZ_1 proves Axiom H without use of choice, by Proposition 3·2 and the fact that $\mathsf{KP} + \Sigma_1$ Separation proves that every well-founded extensional relation is isomorphic to a transitive set. For the latter, Π_1 Separation enables us to find a minimal counter-example, should any exist, to totality of the recursively defined collapsing function, and thus to prove that no counter-examples exist.

That observation yields the final clause of Theorem 3.

4: Adding KPL to M

The chief problem we encountered in §3 was in proving restricted collection: we may indeed prove directly that (each transitive well-founded relation is isomorphic to a transitive set)^1 and that (Axiom H)^1, but in order to show that Collection holds, we want a version of AC to imitate a downward Löwenheim-Skolem argument to construct sets that are $\dot{\Sigma}_1$-elementary submodels of the universe.

So we apparently need AC in the ground model to get KP in the new one; and once we have KP we can carry out the usual construction of L to get AC. There thus appears to be a circularity in what would otherwise be a natural approach to proving that Consis(M) implies Consis($\mathsf{M} + \mathsf{WO}$).

We avoid that circularity by proving that in the system $\mathsf{M}+\mathsf{H}$ the definition of the constructible hierarchy as given in Gödel's monograph [A3] may be carried out. We have a long, a medium and a short proof of that: the author only found the shorter proofs after working out the details of the long one. Nevertheless the long one is of interest as illuminating the relationship between M and $\mathsf{M} + \mathsf{H}$. So in this section we shall first demonstrate how to simulate the definition of L within the system M, making no use of Axiom H. Then we put on our Axiom H spectacles, by which is meant utilising the translation from $\mathfrak{A}$ to $(\mathfrak{A})^1$ developed in section 2, and show that with the addition of Axiom H, the actual construction of L may be extracted from our simulation. Finally we give a short direct definition of the constructible universe, working in $\mathsf{M} + \mathsf{H}$. For the purpose of proving Theorem 1, that is all that is necessary: we have shown that Consis(M) implies Consis($\mathsf{M} + \mathsf{H}$), and we shall show that Consis($\mathsf{M} + \mathsf{H}$) implies Consis($\mathsf{M} + \mathsf{KPL}$).

The simulation of Gödel's L in the system M by L-strings along well-orderings

Our building bricks in this first approach to L will not be elements of extensional well-founded relations but "constructible" hierarchies along arbitrary well-orderings.

In his monograph Gödel sets up the constructible hierarchy by listing eight functions of two variables, $F_1, \dots F_8$; he establishes an isomorphism $\pi : ON \times ON \times 9 \longleftrightarrow ON$ and decoding functions $\lambda : ON \to ON, \rho : ON \to ON, \phi : ON \to 9$ so that for each ζ,

$$\pi\big(\lambda(\zeta),\ \rho(\zeta),\ \phi(\zeta)\big) = \zeta;$$

the isomorphism is induced from a well-ordering of all triples and is such that if $\phi(\zeta) \neq 0$, $\lambda(\zeta) < \zeta$ and $\rho(\zeta) < \zeta$. For many ζ, $\pi^{-1} \restriction \zeta : \zeta \longleftrightarrow \zeta \times \zeta \times 9$.

Once this has been set up, he then defines a function $\mathcal{F} : ON \to V$ by

$$\mathcal{F}(\zeta) = \begin{cases} F_{\phi(\zeta)}(\mathcal{F}(\lambda(\zeta)), \mathcal{F}(\rho(\zeta))) & \text{if } \phi(\zeta) = 1, \dots, 8; \\ \mathcal{F}``\zeta & \text{if } \phi(\zeta) = 0. \end{cases}$$

We shall mimic this by considering *L-strings* along any well-ordering equipped with decoding functions λ, ρ, ϕ. The difference is that Gödel was making his definitions within an extensional universe; we have to take steps to *ensure* that our L-strings are extensional; and we must do that by recursion.§

§ All Gödel's eight functions are *rudimentary* functions in the sense of [A4] and [A8], the concept termed *basic* in [A5]; the functions they generate coincide with those generated by the first seven of the canonical list of eight rudimentary functions. Closure under Gödel's eight guarantees the truth of Δ_0 separation.

The first step in our construction makes no use of the specific definition of the Gödel functions.

Let $(X, <_X)$ be a well-ordering. We shall develop the concept of an *abstract construction* on X. We shall define, by recursion along the well-ordering $<_X$, two two-place relations E^X, $\equiv^X$ on X, and shall prove that $\equiv^X$ is an equivalence relation that is also a congruence with respect to E^X, and that the induced relation on the equivalence classes is well-founded and extensional.

Let $x \in X$, and suppose that we have achieved such a definition on $\{y \mid y <_X x\}$. We have to extend our definitions to x.

[i] We ordain that $x \equiv^X x$ and $\neg(x\, E^X\, x)$;

[ii] we suppose that we have a function F such that $F(x) \subseteq \{y \mid y <_X x\}$ which is *extensional* in the sense that if $w <_X x$, $y <_X x$, $w \equiv^X y$ and $y \in F(x)$ then $w \in F(x)$. We then set, for $y <_X x$,

$$yE^X x \iff y \in F(x)$$

In the case that concerns us, F will mimic one of the Gödel functions; but for the moment we need not know that. The definition has the consequence that for w and y less than x,

$$w \equiv^X y \implies \big(wE^X x \iff yE^X x\big)$$

[iii] we ensure the continued extensionality of E^X *modulo* $\equiv^X$ by defining, for $y <_X x$, $y \equiv^X x$ and $x \equiv^X y$ to hold if and only if $\forall z :<_X x\, (zE^X x \iff zE^X y)$;

[iv] finally we allow for the possibility that at x we are merely getting a repeat of an earlier object by defining $xE^X y$ for $y <_X x$ thus:

$$xE^X y \iff \exists z{:}<_X y\, (z \equiv^X x \;\&\; z \in F(y)).$$

That completes the inductive step of the definition of $\equiv^X$ and E^X. We must show that these relations continue to obey the laws of identity and extensionality. These laws are

LAW 1:	$a \equiv^X a$
LAW 2:	$a \equiv^X b \implies b \equiv^X a$
LAW 3:	$(a \equiv^X b \;\&\; b \equiv^X c) \implies a \equiv^X c$
LAW 4:	$a \equiv^X b \implies (aE^X c \iff bE^X c)$
LAW 5:	$a \equiv^X b \implies (cE^X a \iff cE^X b)$

We assume that for all a, b, c, less than x, the above laws hold, and must show that the above laws hold for all a, b, c less than or equal to x.

Let us dispose of some easy cases: $x \equiv^X x$ by definition [i]; for $y <_X x$, $y \equiv^X x \iff x \equiv^X y$ by definition clause [iii]; those are the only cases of Laws 1 and 2 needing to be covered. In Law 3, if at least two of a, b and c are equal to x the law is easily verified; if none, then our induction hypothesis applies; so by symmetry we need only show two things:

$$a <_X x \;\&\; c <_X x \;\&\; a \equiv^X x \;\&\; x \equiv^X c \implies a \equiv^X c$$

and

$$b <_X x \;\&\; c <_X x \;\&\; x \equiv^X b \;\&\; b \equiv^X c \implies x \equiv^X c$$

In the first of those, we know that $\forall w <_X x(wE^X a \iff wE^X x)$ and $\forall w <_X x(wE^X x \iff wE^X c)$, then $\forall w < x(wE^X a \iff wE^X c)$; *a fortiori*, $\forall w < \max_X\{a, c\}(wE^X a \iff wE^X c)$, and so $a \equiv^X c$. In the second, we must show from our assumptions that for arbitrary $w < x$, $wE^X x \iff wE^X c$. But we know that $wE^X x \iff wE^X b$, as $x \equiv^X b$; and as b, c, and W are all less than x we know by our inductive hypothesis that $b \equiv^X c \implies (wE^X b \iff wE^X c)$.

Now note that we cannot have, for $y \leq_X x$, $x \equiv^X y$ & $yE^X x$: if $y = x$, that holds by [i]; and for $y <_X x$, $x \equiv^X y$ gives $\forall w < x(wE^X x \iff wE^X y)$; taking $w = y$, $yE^X x$ would by [iii] yield $yE^X y$, which is false.

Note also that we cannot have, for $y \leq x$, $x \equiv^X y$ & $xE^X y$: again by [i] for $x = y$, and for $y < x$, if $xE^X y$, then by [iv] $\exists w < y(x \equiv^X w \,\&\, wE^X y)$; we have seen that $w \equiv^X x \,\&\, x \equiv^X y \implies w \equiv^X y$, so if $x \equiv^X y$, we would have $wE^X y \,\&\, w \equiv^X y$; but both w, y are less than x, so by our inductive assumption we would have $yE^X y$, a contradiction.

These remarks cover the cases of Laws 4 and 5 when c and either a or b are equal to x; in both laws there is nothing to prove when $a = x = b$; so we are left with four things to check:

$$
\begin{aligned}
a <_X x \;\&\; c <_X x \;\&\; a \equiv^X x &\implies (aE^X c \iff xE^X c)\\
a <_X x \;\&\; c <_X x \;\&\; a \equiv^X x &\implies (cE^X a \iff cE^X x)\\
a <_X x \;\&\; b <_X x \;\&\; a \equiv^X b &\implies (aE^X x \iff bE^X x)\\
a <_X x \;\&\; b <_X x \;\&\; a \equiv^X b &\implies (xE^X a \iff xE^X b)
\end{aligned}
$$

The first line: to see that $(a \equiv^X x \;\&\; aE^X c) \implies xE^X c$, fix x and c, and let v be minimal (in the well-ordering) such that $v \equiv^X x \,\&\, vE^X c$. v cannot equal c; I assert that $v < c$, which gives $xE^X z$ by [iv]; were $c < v$, then $\exists w{:}{<_X} c\; v \equiv^X w \;\&\; wE^X c$, but we have seen that $w \equiv^X v \,\&\, v \equiv^X x \implies w \equiv^X x$, so we would have $w \equiv^X x \,\&\, wE^X c$, contradicting the minimality of v. Going the other way, suppose that $a \equiv^X x$ and $xE^X c$: we must show that $aE^X c$. By our supposition, $\exists w < c(x \equiv^X w \,\&\, wE^X c)$; we have seen that $a \equiv^X x \,\&\, x \equiv^X w \implies a \equiv^X w$, so we have $a \equiv^X w \;\&\; wE^X c$; so by our inductive hypothesis, $aE^X c$.

The second line holds by clause [iii]; the third line is covered by the extensionality of F assumed in clause [ii]. We turn to the fourth line.

If $a = b$, there is nothing to prove; the assertion is symmetric, so without loss of generality assume that $a <_X b$. Then $a \equiv^X b$ means that $\forall w{:}{<}b\; (wE^X a \iff wE^X b)$.

If $xE^X a$, then for some $z <_X a$, $(z \equiv^X x \;\&\; zE^X a)$; for this z, $z \equiv^X x \;\&\; zE^X b$, and so $xE^X b$. Going the other way, if $xE^X b$, then for some $z <_X b$, $z \equiv^X x \;\&\; zE^X b$; as $a \equiv^X b$, $z \equiv^X x \;\&\; zE^X a$; so for some $w <_X a$, $w \equiv^X z \;\&\; wE^X a$, so $w \equiv^X x \;\&\; wE^X a$, so finally $xE^X a$.

Our verification of the laws of identity and extensionality is complete.

We may now factor both X and E^X by $\equiv^X$, obtaining (Y, F), say. Y has a well-order $<_Y$ naturally resulting from $<_X$ thus: for distinct equivalence classes A and B, put $A <_Y B$ if the $<_X$-least member of A $<_X$-precedes the $<_X$-least member of B.

4·0 PROPOSITION (Y, F) *is a well-founded extensional relation*

Proof: if Z is a non-empty subset of Y, consider the $<_X$-least element z of X of which the equivalence class belongs to Z. Then for no w can we have $wE^X z$ and the equivalence class of w is in Z. Extensionality holds by our definition [iii].

4·1 REMARK The above recursive definitions are all confined to the power set of $X \times X$, so they may be carried out within the system M.

On elementary submodels

We pause while still in this abstract phase to lay the groundwork for the condensation lemma that will be essential when we come, having defined L, to develop its properties.

4·2 Suppose $(X, <_X)$ is a well-ordering and F^X is an extensional function defined on X with $F(x) \subseteq \{y \mid y <_X x\}$. Let H be a subset of X. We shall follow a convention that letters at the beginning of the alphabet will denote members of H, whilst letters at the end denote arbitrary members of X. We write $<_H$ for the restriction of $<_X$ to H.

We define $F^H(h) = H \cap F^X(h)$, and write $\equiv^H$ and E^H for the relations on H defined using the function F^H as above. We seek mild conditions on H so that F^H is extensional, and for each g, h,

$$(g \equiv^H h \iff g \equiv^X h) \;\;\&\;\; (gE^H h \iff gE^X h).$$

Indeed, suppose that H satisfies the following two “elementarity conditions”:

$$(\alpha) \qquad \text{whenever } c <_H d \;\&\; c \not\equiv^X d,\ \exists b\colon <_H d\, \big(bE^X c \not\Longleftrightarrow b \in F^X(d)\big)$$

$$(\beta) \qquad \text{whenever } d >_H e \;\&\; dE^X e,\ \exists a\colon <_H e\, \big(a \equiv^X d \;\&\; a \in F^X(e)\big)$$

We shall show that those two suffice. We make the following definitions

$$\begin{aligned}
\Psi_{X,H}(c) &\iff_{\rm df} \forall d > c\,(dE^X c \implies dE^H c)\\
\Psi_{H,X}(c) &\iff_{\rm df} \forall d > c\,(dE^H c \implies dE^X c)\\
\Phi_{X,H}(c) &\iff_{\rm df} \forall d > c\,(d \equiv^X c \implies d \equiv^H c)\\
\Phi_{H,X}(c) &\iff_{\rm df} \forall d > c\,(d \equiv^H c \implies d \equiv^X c)\\
\Psi(c) &\iff_{\rm df} \Psi_{X,H}(c) \;\&\; \Psi_{H,X}(c)\\
\Phi(c) &\iff_{\rm df} \Phi_{X,H}(c) \;\&\; \Phi_{H,X}(c)
\end{aligned}$$

We say that F^H is *extensional at h* if $\forall f, g\colon <_H h\ f \equiv^H g \;\&\; g \in F^H(h) \implies f \in F^H(h)$.

4·3 LEMMA *(i)* $cE^X c \iff cE^H c$;
(ii) $w <_H c \implies (wE^X c \iff wE^H c)$;
(iii) if $\Psi_{H,X}(c)$, $\forall b{:}\in H\ (bE^H c \implies bE^X c)$;
(iv) if $\Psi_{X,H}(c)$, $\forall b{:}\in H\ (bE^X c \implies bE^H c)$;
(v) if $\Psi(c)$, $\forall b{:}\in H\ bE^X c \iff bE^H c$;
(vi) $c \equiv^X c \iff c \equiv^H c$;
(vii) $c \equiv^H d \iff d \equiv^H c$.

Proof : (i) because both sides are, by definition, false; (vi) because both sides are true; (vii) by definition; (ii) because both sides are equivalent to $w \in F^X(c) \cap H$; (iii), (iv) and (v) are then immediate. ⊣ (4·3)

4·4 LEMMA *If* $\forall f\colon <_H h\ \Phi_{H,X}(f)$ *then* F^H *is extensional at* h.

Proof : suppose $b \in F^H(h)$ and $b \equiv^H c <_H h$. Let f be the $<_H$-minimum of b and c. Using $\Phi_{H,X}(f)$, $b \equiv^X c$. By the extensionality of F^X, $c \in F^X(h) \cap H = F^H(h)$, as required. ⊣ (4·4)

4·5 LEMMA *If* $\Psi(c)$ *then* $\Phi_{X,H}(c)$.

Proof : Let $c <_H d$ and suppose that $c \equiv^X d$. Then $\forall w : <_X d\ (wE^X c \iff w \in F^X(d))$, so $\forall b : <_H d\ (bE^H c \iff b \in F^H(d))$, using $\Psi(c)$ and Lemma 4·3 (v) , and thus $c \equiv^H d$. ⊣ (4·5)

4·6 LEMMA *If* $\Psi(c)$ *then* $\Phi_{H,X}(c)$.

Proof : suppose that $c \not\equiv^X d$, where $c <_H d$. We must show that $\exists b\colon <_H d\ \big(bE^H c \not\Longleftrightarrow bE^H(d)\big)$. By property (α), $\exists b\colon <_H d\ \big(bE^X c \not\Longleftrightarrow b \in F^X(d)\big)$. Parts (ii) and (i) of Lemma 4·3 forbid respectively $b <_H c$ and $b = c$, so $c <_H b$; from $\Psi(c)$ we know that $bE^H c \iff bE^X c$. As $b \in F^X(d) \iff b \in F^H(d)$, we conclude that $c \not\equiv^H d$, as required. ⊣ (4·6)

4·7 LEMMA *If* $\forall a\colon <_H e\ \Phi_{H,X}(a)$ *then* $\Psi_{H,X}(e)$

Proof : let $d >_H e$, and suppose that $dE^H e$. Then $\exists a\colon <_H e\ a \equiv^H d \;\&\; a \in F^H(e)$. So $\Phi_{H,X}(a)$ holds; as $a < d$, we have $a \equiv^X d$; so as $a \in F^X(e)$, $dE^X e$, as required. ⊣ (4·7)

4·8 LEMMA *If* $\forall a\colon <_H e\ \Phi_{X,H}(a)$ *then* $\Psi_{X,H}(e)$

Proof : let $d >_H e$, and suppose that $dE^X e$. By property (β), $\exists a : <_H e\ \big(a \equiv^X d \;\&\; a \in F^X(e)\big)$. Then $a \in F^H(e)$, and $a <_H d$, so by $\Phi_{X,H}(a)$, $a \equiv^H d$. Hence $dE^H e$. ⊣ (4·8)

From the above,

4·9 PROPOSITION $\Psi(c) \implies \Phi(c)$ *and* $\forall c\colon <_H e\ \Phi(c) \implies \Psi(e)$,

whence by an induction along $<_H$,

4·10 PROPOSITION *If H satisfies properties (α) and (β), then F^H is extensional and for all c, d in H,*

$$(c \equiv^H d \Longleftrightarrow c \equiv^X d) \mathbin{\&} (cE^H d \Longleftrightarrow cE^X d).$$

4·11 REMARK The above proposition is a very weak form of the condensation lemma, which places no reliance on a particular definition of F.

4·12 PROPOSITION *Suppose the first element 0_X of X is in H, and H satisfies (α) and (β) Then for each h in H, if $F^X(h) \neq \varnothing$, $F^H(h) \neq \varnothing$.*

Proof : Given such h, $0_X \not\equiv^X h$, so by (α) $\exists c$: $<_H h\ c \in F^H(h)$, since $\neg(cE^H 0_X)$. ⊣ (4·12)

Gödel's operations

Hitherto we have carried out our definition in abstract form, without specifying the nature of the rule F, because precise knowledge of that rule is not needed for the recursive verification of the laws of identity and extensionality. The time has come to examine Gödel's operations in detail. They are

$$\begin{aligned}
F_1(X,Y) &= \{X,\,Y\}\\
F_2(X,Y) &= X \cap \{\langle u,\,v\rangle \mid u \in v\}\\
F_3(X,Y) &= X \setminus Y\\
F_4(X,Y) &= X \cap \{\langle u,\,v\rangle \mid v \in Y\}\\
F_5(X,Y) &= X \cap \{z \mid \exists y\ \langle y,\,z\rangle \in Y\}\\
F_6(X,Y) &= X \cap \{\langle u,\,v\rangle \mid \langle v,\,u\rangle \in Y\}\\
F_7(X,Y) &= X \cap \{\langle u,v,w,\rangle \mid \langle v,w,u\rangle \in Y\}\\
F_8(X,Y) &= X \cap \{\langle u,v,w,\rangle \mid \langle u,w,v\rangle \in Y\}
\end{aligned}$$

Notice that for $2 \leq i \leq 8$ and for all X, $F_i(X,Y) \subseteq X$.

We shall need to note the actual means of defining the isomorphism between single ordinals and triples:

$$\begin{aligned}
\langle \alpha,\beta,i\rangle < \langle \gamma,\delta,j\rangle \Longleftrightarrow\ & \max\{\alpha,\,\beta\} < \max\{\gamma,\,\delta\} \textbf{ or}\\
& \max\{\alpha,\,\beta\} = \max\{\gamma,\,\delta\} \mathbin{\&} \alpha < \gamma \textbf{ or}\\
& \max\{\alpha,\,\beta\} = \max\{\gamma,\,\delta\} \mathbin{\&} \alpha = \gamma \mathbin{\&} \beta < \delta \textbf{ or}\\
& \max\{\alpha,\,\beta\} = \max\{\gamma,\,\delta\} \mathbin{\&} \alpha = \gamma \mathbin{\&} \beta = \delta \mathbin{\&} i < j
\end{aligned}$$

Thus the first few triples are $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, ... $(0,0,8)$, $(0,1,0)$, $(0,1,1) \ldots (0,1,8)$, $(1,0,0)$, $\ldots (1,0,8)$, $(1,1,0)$, $\ldots (1,1,8)$, $(0,2,0)$ $\ldots (0,2,8)$, $(1,2,0)$, $\ldots (1,2,8)$, $(2,0,0)$, $\ldots (2,1,0)$, $\ldots (2,2,0)$ $\ldots (0,3,0), \ldots$

Examination of the definition shows that if a triple $\langle \alpha,\beta,i\rangle$ has rank ζ in this ordering, then either $\alpha = \beta = i = 0 = \zeta$ or $(\alpha = 0 = i \mathbin{\&} \beta = \zeta)$, or $(\alpha < \zeta \mathbin{\&} \beta < \zeta)$; whence $\max\{\lambda(\zeta), \rho(\zeta), \phi(\zeta)\} \leqslant \zeta$.

Hence if we, following Gödel, define an enumeration of the constructible universe by the equation

$$\mathcal{F}(\zeta) = \begin{cases} \mathcal{F}``\zeta & \text{if } \phi(\zeta) = 0,\\ F_{\phi(\zeta)}(\mathcal{F}(\lambda(\zeta)), \mathcal{F}(\rho(\zeta))) & \text{if } \phi(\zeta) = 1, \ldots, 8; \end{cases}$$

then, for each ζ, $\mathcal{F}(\zeta) \subseteq \{\mathcal{F}(\xi) \mid \xi < \zeta\}$. Hence, provided we are working in a suitable set theory, we may show by recursion on the ordinals that each $\{\mathcal{F}(\eta) \mid \eta < \zeta\}$ is a transitive set. Of course, there are many repetitions; for example $\mathcal{F}(0) = \mathcal{F}(1) = \ldots = \mathcal{F}(8) = \varnothing$, while $\mathcal{F}(9) = \{\varnothing\}$.

But in our present set theory, we do not know that Gödel's recursive definition works. We set out, therefore, to imitate it by an abstract construction of the kind defined above.

Thus, we have a well-ordering $(X, <_X)$, and we equip ourselves with decoding functions λ, ρ, ϕ, where $\lambda : X \longrightarrow X$, $\rho : X \longrightarrow X$ and $\phi : X \longrightarrow \{0,1,2,3,4,5,6,7,8\}$. We use them to define a function $F = F_X^{\text{Gödel}}$

so that the resulting abstract construction along X using F imitates, in its clause [ii], the rules of Gödel defining $\mathcal{F}$. The difference will be that

$$\mathcal{F}(\zeta) \subseteq \{\mathcal{F}(\xi) \mid \xi < \eta\},$$
$$\text{whereas} \quad F(x) \subseteq \{y \mid y <_X x\}.$$

Clause [ii] in our definition of relations $E^X, \equiv^X$ on X by recursion along the well-ordering $<_X$ will read as follows: we are at stage x, and we suppose that the two relations have been defined for all $y, z <_X x$; then
[ii] for $y <_X x$, we determine whether $yE^X x$ according to the value of $\phi(x)$, thus:

$$\phi(x) = 0: \quad y\, E^X\, x$$
$$\phi(x) = 1: \quad y\, E^X\, x \iff y \equiv^X \lambda(x) \text{ or } y \equiv^X \rho(x)$$
$$\phi(x) = 2: \quad y\, E^X\, x \iff yE^X\lambda(x) \;\&\; \exists b{:}{<_X}\, y\, \exists a{:}{<_X}\, b\, \big(aE^X b \,\&\, y \equiv^X \pi(\pi(a,a,1), \pi(a,b,1), 1)\big)$$
$$\phi(x) = 3: \quad y\, E^X\, x \iff y\, E^X\, \lambda(x) \;\&\, \neg(y\, E^X\, \rho(x))$$

and so on for the remaining functions in the basis.

4·13 LEMMA *If $a \equiv^X c$ and $b \equiv^X d$, $\pi(a,b,i) \equiv^X \pi(c,d,i)$.*

Proof: by the extensionality of clause [ii]. ⊣ (4·13)

Some interpretative remarks may be helpful. When $\phi(x) = 0$ we collect together all previously obtained sets. The case $\phi(x) = 1$ does unordered pairs; it follows that we can build ordered pairs and ordered triples, and indeed have explicit formulæ of ordinal arithmetic telling us when the ordered pair of two elements, or the ordered triple of three elements, will be constructed. These will be useful in writing the defining clauses of [ii] for the cases $4 \le \phi(x) \le 8$, which are all defined in terms of ordered pairs and triples, along the lines we have illustrated in writing out the case $\phi(x) = 2$.

The case $\phi(x) = 3$ constructs the difference of two previously obtained sets: this is the only place in the complete definition where negation occurs, and since $z\backslash z = 0$ and $z\backslash 0 = z$ it has the consequence that the empty set occurs repeatedly in the construction, and therefore that every set constructed is constructed repeatedly.

We call abstract constructions where we have followed Gödel's rules, *L-strings*, or, for emphasis, *unfactored* L-strings, to distinguish them from the result of applying the factoring by $\equiv^X$ discussed above in Proposition 4·0, which we shall call *factored* L-strings. Intuitively, the factored L string associated to a well-ordering X is, when transitised, the initial segment $\{\mathcal{F}(\alpha) \mid \alpha < \eta\}$ of the constructible universe; the unfactored L-string along X is isomorphic to the string $\langle \mathcal{F}(\alpha) \mid \alpha < \eta \rangle$, η here being the ordinal that is the length of the well-ordering X.

The above definition, being localised, can be carried out in the system M, and we have established that along any well-ordering there is an L-string. We could now begin our formal development of the proof of Theorem 1 as follows:

4·14 DEFINITION $F_2 =_{\text{df}}$ the class of all L-strings;
$$W_2 =_{\text{df}} \{(\xi, x, e) \mid (x, e) \in F_2 \;\&\; \xi \in x\}.$$

There is a counterpart to the notion of partial isomorphism developed in Section 2, and we may show that maximal partial isomorphisms exist: actually we do slightly better than previously, as we may show that with L-strings every maximal partial isomorphism is at least half-total. We may then use the new notion of maximal partial isomorphism to define relations $\equiv^2$ and E^2 on W_2; finally we define for any formula $\mathfrak{A}$ the formula $(\mathfrak{A})^2$ as that obtained from $\mathfrak{A}$ by first replacing $\in$ and $=$ by E^2 and $\equiv^2$ and then relativising the quantifiers to the class W_2.

4·15 THEOREM SCHEME *For any theorem $\mathfrak{A}$ of* $\mathsf{MAC} + \mathsf{KPL}$, $(\mathfrak{A})^2$ *is provable in* M.

The proof would broadly imitate that of Theorem 2, except that since principles of $\mathcal{P}$-closure and $\bigcup$-closure are not available in full generality, we would resort in many places to Gödel's Condensation Lemma for L-strings, following the lines of [A3] or [A8]. To that we now turn.

Condensation for L-strings

We start with observing that in the case of L-strings, Proposition 4·10 can be considerably strengthened:

4·16 PROPOSITION *Suppose that (X, F^X) is an L-string closed under π^X, and that H is a subset of X containing 0_X, satisfying properties (α) and (β) and closed under π^X, λ^X, ρ^X and ϕ_X. Then H is, in an appropriate interpretation of the phrase, a Δ_0-elementary submodel of X.*

Proof : Let $\Phi(x, a, b)$ be a Δ_0 formula with parameters a and b. We have to show that if a, b and c are in H, and $\exists x :\in F^X(c)\ \Phi^X(x, a, b)$ then $\exists x :\in F^H(c)\ \Phi^X(x, a, b)$, where Φ^X denotes the formula obtained from Φ by replacing $=$ by $\equiv^X$ and $\in$ by E^X, and, as before, $F^H(c) = F^X(c) \cap H$.

Consider $F^X(c) \cap \{x \mid \Phi^X(x, a, b)\}$. We assert that there is a y computable using π, λ, ρ and ϕ from a, b, and c, such that $F^X(y)$ is equal to that set, taking it to be extensionally closed. But then $y \in H$, since H is closed under those functions computing y from a, b and c, and then $F^H(y)$ will be non-empty, by 4·12, proving the proposition.

Our assertion follows from well-known facts about rudimentary functions. Details may be found in Gödel's monograph or in Devlin's exposition of constructibility. ⊣ (4·16)

4·17 REMARK We could remove the vague phrase "in an appropriate interpretation" by factoring the L-string X and regarding H as the submodel consisting of the equivalence classes of elements of H. However, for the moment, we continue to avoid factoring.

We may now state the Condensation Lemma, which in our context takes the following pleasing form.

4·18 THE CONDENSATION LEMMA *Let (X, F) be an L-string closed under π^X. Let $H \subseteq X$ contain 0_X and be closed under the functions π^X, λ^X, ρ^X, ϕ^X, and let it satisfy (α) and (β). Set $F^H(h) = H \cap F(h)$ for $h \in H$. Then (H, F^H) is an L-string.*

Proof : The proposition will give us that counterparts to each of 0, 1, 2, ..., 8 are in H. Notice then that H being closed under λ and the other functions implies that $\lambda^H = \lambda^X \restriction H$, and similarly for ρ, ϕ and π. That follows from the elementary way in which the well-ordering of triples has been defined and the fact that

$$\zeta = \pi(\lambda(\zeta), \rho(\zeta), \phi(\zeta)),$$

so that each element of H is in the image of $\pi \restriction H \times H \times 9$.

Now we have to check that the function F^H as defined above agrees with the definition *via* the Gödel functions. But that readily follows from Proposition 4·16, as all the Gödel functions are defined by Δ_0 formulæ, the apparently unrestricted quantifier $\exists y$ in the definition of F_5 being in its context equivalent to the restricted quantifier $\exists y :\in \bigcup\bigcup Y$. ⊣ (4·18)

We could now, following Gödel, use the Condensation Lemma to develop properties of our version of L. For example, it would now be easy to show that every "subset of ω" in our model is in some countable L-string, and hence that the continuum hypothesis will be true in our model.

The proof of $(\Delta_0\ \mathsf{Separation})^2$ would be straightforward, relying on the fact that our model is closed under the functions F_1, ... F_8, and indeed would have essentially been given in establishing the pivotal assertion of the proof of Proposition 4·16.

The proof of the Power set axiom would use the fact, proved in §2, that given any well-ordering there is one of a greater cardinality. We sketch the argument.

Let $(\xi, x, e) \in W_2$. Let Y be a well-ordering of length $\geq \overline{\overline{x}}^+$. We show that every "subset" of ξ "occurs" in the L-string on Y, by applying Gödel's condensation argument to any π-closed L-string containing an occurrence of a subset of ξ to shrink it to an L-string shorter than that on Y. We may then apply $(\Delta_0\ \mathsf{Separation})^2$ to the string on Y to obtain the "Power set" of ξ.

The proof of $(\mathsf{strong}\ \Sigma_1\ \mathsf{Collection})^2$ would proceed *via* $(\mathsf{Axiom\ H})^2$, as in §2; $(\mathsf{Axiom\ H})^2$ itself would be verified by an application of the Condensation Lemma, imitating the standard proof in $\mathsf{ZF} + V = L$ that if κ is a successor cardinal, $L_\kappa = H_\kappa$.

We would have to show that $(V = L)^2$ is true; that would give us $(\mathsf{AC})^2$, and then Axiom H would yield (strong) Δ_0 Collection. The proof of Theorem 1 would then be complete.

The construction of Gödel's L in the system M + Axiom H by transitised factored L-strings

However, it is conceptually much easier to take a different tack. We have developed the concepts of unfactored and factored L-strings in the system M, and the result is a supply of well-founded equivalence relations. But our model for $\mathsf{M}+\mathsf{H}$, studied in §2, is built from all well-founded equivalence relations. Thus our current version of L may be construed as a sub-model of that, and is a Σ_2 class: something is constructible if there is a well-ordering and an L-string along that well-ordering such that ...

Therefore it is much better, now that we have established the existence of these particular well-founded equivalence relations, to put on our Axiom H spectacles and re-interpret what we have done. First, since, under Axiom H, every well-ordering is isomorphic to a von Neumann ordinal, we save a quantifier in the description of L, for now we may say that something is constructible if there is an ordinal and an L-string along it such that ..., and with the Axiom of Foundation to hand, being an ordinal is a Δ_0 condition.

Secondly, our factored L-strings may, again using Axiom H, be transitised; and thus we are now very close to the KP treatment of of the constructible hierarchy, except that (not having the Axiom of Choice to hand) we have not established that the axioms of KP hold in our current context.

Let ζ be an ordinal. We have an (unfactored) L-string along $(\zeta, < \restriction \zeta)$, defined using $F^\zeta =_{\rm df} F^{\text{Gödel}}_{<\restriction\zeta}$. We form the factored L-string as in Proposition 4·0, and using Proposition 3·2 obtain the transitive set, A_ζ say, isomorphic to it. We factor the object placed at each $\eta < \zeta$, as follows. By definition of L-string, we have for $\xi < \eta < \zeta$,

$$\xi E^\zeta \eta \iff \xi \in F^\zeta(\eta) \subseteq \{\nu \mid \nu < \eta\},$$

so we define by recursion on the ordinals less than ζ, a function G^ζ with domain ζ thus:

$$G^\zeta(\eta) =_{\rm df} \{G^\zeta(\xi) \mid \xi \in F^\zeta(\eta)\}.$$

The recursion may be sustained in $\mathsf{M}+\mathsf{H}$ because each $G^\zeta(\eta)$ will be a member of A_ζ. Then by our careful choice of F^ζ to imitate Gödel's rules, $\langle G^\zeta(\eta) \mid \eta < \zeta\rangle$ can be verified to satisfy Gödel's recursive definition of the sequence $\langle \mathcal{F}(\eta) \mid \eta < \zeta\rangle$. In particular, we may check that for $\eta < \zeta < \theta$, $G^\zeta(\eta) = G^\theta(\eta)$, and may write $\mathcal{F}(\eta)$ for that common value.

We have now established in $\mathsf{M}+\mathsf{H}$ that along every von Neumann ordinal ζ the Gödel sequence $\langle \mathcal{F}(\eta) \mid \eta < \zeta\rangle$ exists. Now it is all plain sailing: we can define L to be the class $\{\mathcal{F}(\eta) \mid \eta \in ON\}$. We know by 2·0 that $\forall\kappa\ \kappa^+$ exists; and therefore there will be in L unboundedly many initial ordinals. We may establish the truth of the axioms of S_0, of Δ_0 Separation and of AC in L, following Gödel's monograph; so successor initial ordinals will in L be regular. Let κ be an initial ordinal in L, and let λ be its successor in L, so that λ is at most the successor of κ computed in V. Then we may prove using the Condensation Lemma that each subset of κ in L is in L_λ, and thus establish the power set axiom. We now know that MAC is true in L. The truth of TCo in L is immediate from its definition, since each member of L is a member of a transitised factored L-string. We use the Condensation Lemma again to show that, in L, $L_\lambda = H_{\kappa^+}$, which is enough to establish the truth of Axiom H in L. Proposition 3·14 then yields the truth of Δ_0 Collection and of Π_1 Separation in L.

We have established the truth in L of all axioms of $\mathsf{MAC}+\mathsf{KPL}+\Sigma_1$ $\mathsf{Separation}$, and the proof of Theorem 1 is complete.

⊣ (Theorem 1)

The construction of Gödel's L in the system M + Axiom H: a direct argument

4·19 PROPOSITION ($\mathsf{M}+\mathsf{H}$) *Let κ be an uncountable von Neumann ordinal. Let $H = H_{\leqslant\kappa}$ be a transitive set, supplied by Axiom H, of which every transitive set of cardinality at most κ is a subset. Then there is a function $f : \kappa^+ \longrightarrow H$ which satisfies Gödel's recursive definition of the sequence $\langle \mathcal{F}(\nu) \mid \nu < \kappa^+\rangle$.*

Proof: essentially by the general principle that contained recursions succeed in M. Let $\Psi(\eta, f)$ assert that f is a function with domain η and values in H that satisfies Gödel's rules and has the properties for all $\delta < \eta$ that $f(\delta) \subseteq \{f(\nu) \mid \nu < \eta\}$ and that $\{f(\nu) \mid \nu < \delta\}$ is transitive. Ψ is Δ_0 in the parameter H.

We have seen in our previous discussion that those properties are indeed maintained by Gödel's rules. It is easily checked that for each η there is at most one such f.

We assert that $\forall\eta{:}{<}\kappa^+\ \exists f\Psi(\eta,f)$. If that fails, then consider the class

$$\{\eta \mid \eta<\kappa^+ \ \&\ \neg\exists f{:}{\in}\mathcal{P}(H\times\kappa^+)\ \Psi(\eta,f)\}:$$

that is Δ_0 in the parameter $\mathcal{P}(H\times\kappa^+)$, which is, provably in $\mathsf{M}+\mathsf{H}$, a set. Hence that class is a set, and by foundation, if non-empty, it will have a least element $\bar\eta$.

Evidently $\bar\eta>0$. The possibility that $\bar\eta$ is a limit ordinal is easily refuted. Suppose that $\bar\eta=\eta+1$.

We then have a function $f:\eta\longrightarrow H$ that satisfies the defining clauses for $\langle\mathcal{F}(\nu)\mid\nu<\eta\rangle$, and for which $\{f(\nu)\mid\nu<\eta\}$ is transitive. Our intended value $\mathcal{F}(\eta)$ is the subset of $\{f(\nu)\mid\nu<\eta\}$ given by Gödel's rules, and thus $\{f(\nu)\mid\nu<\eta\}\cup\{\mathcal{F}(\eta)\}$ is transitive. It is plainly of cardinality $\leqslant\kappa$, since $\eta<\kappa^+$, and so is a subset of H. Thus our intended value $\mathcal{F}(\eta)$ is a member of H and we may extend our definition of f to η, giving a function g with $\Psi(\eta+1,g)$.

Our proof is complete. $\dashv$ (4·19)

A proof of Theorem 1 may now be reached by following the last two paragraphs of our previous proof.

4·20 REMARK In fact $\mathsf{Z}_0+\mathsf{H}$ suffices for the definition of L and proves $(\mathsf{Axiom\ H})^L$ and therefore also $(\mathsf{Z}_1)^L$, for, as we have just seen, $\mathsf{M}+\mathsf{H}$ does, but Z_0^+ is M, $(\mathsf{Axiom\ H})^0$ is provable in $\mathsf{Z}_0+\mathsf{H}$, and the procedure of §1 for getting from Z_0 to Z_0^+ makes no difference to L which is in any case built from transitive sets.

4·21 HISTORICAL NOTE Prior to the publication of his monograph [A3] Gödel published first a brief announcement [A1] and then a sketch [A2] of his proof in the *Proceedings of the National Academy of Sciences* of the U.S.A., in which he proceeds by defining the constructible hierarchy, the stages of which are denoted by M_ν by Gödel and by L_ν by more recent writers.

In the announcement a relative consistency result is stated in this form: if T is consistent it remains so if four propositions are adjoined simultaneously as new axioms, the four propositions being the Axiom of Choice, the generalised continuum hypothesis, the existence of a non-measurable $\boldsymbol{\Delta}^1_2$ set of reals and the existence of an uncountable $\boldsymbol{\Pi}^1_1$ set with no perfect subset. The result is stated to hold for T denoting either von Neumann's system S^*, or the system of *Principia Mathematica* or Fraenkel's system of axioms, leaving AC out in all cases, but including the axiom of infinity in the last two.

Footnote 1 of the sketch begins "This paper gives a sketch of the consistency proof for propositions 1, 2 of *Proc. Nat. Acad. Sci.,* **24**, 556 (1938), if T is Zermelo's system of axioms for set theory (*Math Ann.*, **65** 261) with or without axiom of substitution and if Zermelo's notion of "Definite Eigenschaft" is identified with "propositional function over all sets".

He states in Theorem VII that M_{ω_ω} will be a model of Zermelo's system, and mentions that certain slight modifications will be needed for the corresponding relative consistency proof. Footnote 12 of the sketch reads "In particular for the system without the axiom of substitution we have to consider instead of M_{ω_ω} an isomorphic image of it (with some other relation R instead of the ϵ-relation) because M_{ω_ω} contains sets of infinite type, whose existence cannot be proved without the axiom of subst. The same device is needed for proving the consistency of prop. 3, 4 of the paper quoted in footnote 1."

Though Gödel never published details of his consistency proof for $\mathsf{Z}+V=L$ relative to Z, it seems that he had developed something like the theory of factored L-strings presented in this section.

5: The increasing strength of subsystems of Z

In this section we study systems intermediate in strength between M + KP and $\mathsf{Z} + \mathsf{KP}$, namely $\mathsf{KZ}_{\mathfrak{k}}$, the system KP + $\mathsf{Z}_{\mathfrak{k}}$; $\mathsf{KLZ}_{\mathfrak{k}}$, which is $\mathsf{KZ}_{\mathfrak{k}} + V = L$; for $\mathfrak{k} \geqslant 2$, $\mathsf{KLMZ}_{\mathfrak{k}}$, which is $\mathsf{KLZ}_{\mathfrak{k}}$ with the addition of a "minimality" axiom that we write informally as $ON = \aleph_\omega$; and KLMZ, the union of those systems over all $\mathfrak{k} \geqslant 2$.

In particular, KLZ_0 is the system $\mathsf{M} + \mathsf{KPL}$, of which WO is a theorem, and KZ_0 is $\mathsf{M} + \mathsf{KP}$.

We shall use fraktur letters $\mathfrak{k}$, $\mathfrak{n}$ for indexing the Lévy hierarchy, to remind us that it is very dangerous to quantify over these variables, as we have no truth definition that works for all formulæ at once.

We have two principal aims in this section, for each $\mathfrak{k} \geq 2$.

1: to show that each $\mathsf{KLMZ}_{\mathfrak{k}}$ proves the consistency of $\mathtt{KLMZ}_{\mathfrak{k}-1}$, from which we shall infer that Z is not finitely axiomatisable.

2: to show, reasoning in $\mathsf{KZ}_{\mathfrak{k}}$, that $\mathsf{KZ}_{\mathfrak{k}}$ holds in L.

We shall see shortly that for $\mathfrak{k} = 1$, the first result fails while the second holds. In §6 we shall see that the theory KZ_0 is unable to prove that there is a non-recursive ordinal and hence that the second fails for $\mathfrak{k} = 0$, for $(\mathsf{KZ}_0)^L$ does prove that there is a non-recursive ordinal. We remarked in 4·20 that $\mathsf{Z}_0 + \mathsf{H}$ proves $(\mathsf{Z}_1)^L$; we shall see in Theorem 5·33 that for $\mathfrak{k} \geqslant 2$, $\mathsf{Z}_{\mathfrak{k}} + \mathsf{H}$ proves $(\mathsf{Z}_{\mathfrak{k}})^L$.

Proof of Axiom H in $\mathbf{KLZ_0}$

5·0 Lemma (KLZ_0) *For all κ, κ^+ exists.*

Proof: fix κ, an infinite initial ordinal. Let $S = \mathcal{P}(\kappa)$. S is a set, so there is an ordinal λ such that $S \subseteq L_\lambda$. An easy induction shows that $\overline{\overline{L_\nu}} = \overline{\overline{\nu}}$ for infinite ν; hence such λ must be of cardinality greater than κ, as $\overline{\overline{S}} > \kappa$ by Cantor. By Π_1 Foundation, the class $\{\mu > 0 \mid \neg\exists f : \kappa \xrightarrow{\text{onto}} \mu\}$, being non-empty, has a least element, which will be κ^+. ⊣ (5·0)

5·1 Proposition (KLZ_0) *Axiom H.*

Proof: let u be a set. Since $V = L$, $u \in L_\eta$ for some infinite η, and therefore $\overline{\overline{u}} \leqslant \overline{\overline{\eta}} = \kappa$, say. By the Lemma, κ^+ exists. We take $T = L_{\kappa^+}$ and show that T has the properties promised by Axiom H. That is, we must show that if v is transitive and $\overline{\overline{v}} \leqslant \kappa$, then $v \subseteq T$.

Choose θ with $v \in L_\theta$; as each L_θ is transitive, $v \subseteq L_\theta$. We may, again by the Lemma, suppose that θ is admissible, since if necessary we could replace θ by θ^+. Hence we may without difficulty form $N = Hull(L_\theta, v \cup \{v\})$: again we omit the definition of *Hull*, but the $\dot{\Sigma}_1$ hull, defined in analogy to the $\dot{\Delta}_0$ hull discussed before the statement of Proposition 3·11, will do. By Gödel's Condensation Lemma, which will hold by Corollary 3·16 since it concerns a collapse into L_θ, there is a ζ with

$$\varpi_N : N \cong L_\zeta$$

where ϖ_N is the Mostowski collapsing function.

By a familiar cardinality computation, $\overline{\overline{\zeta}} = \overline{\overline{L_\zeta}} = \overline{\overline{N}} = \overline{\overline{v}} \leqslant \kappa$. Hence $\zeta < \kappa^+$. But $\varpi_N(v) = v$ since v is transitive and $v \subseteq N$; so $v \in L_\zeta \subseteq L_{\kappa^+}$, as required. ⊣ (5·1)

5·2 Metacorollary KLZ_0 *and* KLZ_1 *are the same system.*

Proof: by 5·1 and 3·18. ⊣ (5·2)

5·3 Metacorollary Z_0 *and* Z_1 *are equiconsistent.*

Proof: If Z_0 is consistent, so is Z_0^+, by §1. Z_0^+ is the system M. By §2, the consistency of M implies that of $M + H$, which by the results of §4 proves $(\mathsf{KLZ}_0)^L$, of which $(\mathsf{Z}_1)^L$ is a subsystem by 5·2. ⊣ (5·3)

We saw in 3·21 that Axiom H is provable in KZ_1, whereas the models to be built in §6 will show that it is not provable in $\mathsf{KZ}_0 + \mathsf{WO}$.

5·4 Proposition $\mathsf{KZ}_1 \vdash (\mathsf{KZ}_0)^L$

Proof : KP proves $(\mathsf{KP})^L$, so that our only problem is with the power set axiom. Let $a \in L$. The predicate "$x \in L$" is Σ_1^{KP}, and hence the class $A =_{\rm df} \mathcal{P}(a) \cap L$ is a set. Then using Σ_1 collection we find $\exists\eta\forall x :\in A\ x \in L_\eta$, so that with Δ_0 Separation L may now be proved to satisfy the power set axiom. ⊣ (5·4)

5·5 COROLLARY $\mathsf{KZ}_1 \vdash (\mathsf{KZ}_1)^L$

Proof : since KLZ_0 proves KLZ_1. ⊣ (5·5)

Once the results promised above for section 6 have shown that KZ_0 does not prove $(\mathsf{KZ}_0)^L$, we shall have the

5·6 METACOROLLARY *KZ_0 and KZ_1, though equiconsistent, are not the same system.*

Our plan for the rest of the section is to begin with a discussion of $\dot\Sigma_\mathfrak{k}$ hulls, reasoning in $\mathsf{Z}_\mathfrak{k}$; then to discuss the minimality axiom and use it to establish a weak form of a general fine-structural result; reasoning in $\mathsf{KLMZ}_\mathfrak{k}$ we then build a model of $\mathtt{KLMZ}_{\mathfrak{k}-1}$, thus achieving our first goal; then we show how to strengthen the fine structural result and avoid reliance on the minimality axiom; and finally, reasoning in $\mathsf{KZ}_\mathfrak{k}$, and using that strengthened result, prove that $\dot\Sigma_\mathfrak{k}$ Separation holds in L.

$\dot\Sigma_\mathfrak{k}$ hulls

5·7 We have remarked in 3·10 that the relation $\mathbf{N} \models \varphi$ can be defined satisfactorily in the system M for any structure $\mathbf{N} = (N, R)$ where N is a set. Here we want to define truth in the universe, not in a set; no definition works for all formulæ at once, but we may make in KZ_0 for each $\mathfrak{k}$ a truth definition $\overset{\mathfrak{k}}{\models}$ that works for all $\dot\Sigma_\mathfrak{k}$ formulæ and $\dot\Pi_\mathfrak{k}$ formulæ; however, a formula of the form $\bigwedge x\colon\epsilon\, a\Psi$ where Ψ is $\dot\Sigma_3$ is neither, and need not be equivalent, say in $\mathtt{KZ}_0$, to, a $\dot\Sigma_3$ nor a $\dot\Pi_3$ formula, and therefore *prima facie* our definition of $\overset{3}{\models}$ will not apply.

We start by making this truth definition, $\overset{0}{\models}$, for all $\dot\Delta_0$ formulæ: such a formula is said to be true if it holds in some, or, granted TCo, in every, transitive set containing all its parameters. Thus for $\varphi \in \dot\Delta_0$,

$$\overset{0}{\models} \varphi[a] \iff_{\rm df} \exists u\Big(\bigcup u \subseteq u \ \&\ a \in u \,\&\, \big(u, \{\langle x,\, y\rangle \mid x \in y \in u\}\big) \models \varphi[a]\Big).$$

We know from Remark 1·23 that Z does not prove TCo; but KP does, and indeed the function *tcl* is there available, and of course KPI handles formal languages and the satisfaction relation $\models$ with ease. Thus $\overset{0}{\models}$ is a Δ_1^{KPI} relation.

We then, schematically for each $\mathfrak{k}$, extend the definition to allow for the two strings of length $\mathfrak{k}$ of strictly alternating unrestricted formal quantifiers. For example, for $\mathfrak{k} = 3$ we define for $\varphi \in \dot\Delta_0$ and parameters a,

$$\overset{3}{\models} \bigvee\mathfrak{x}\bigwedge\mathfrak{y}\bigvee\mathfrak{z}\varphi(\mathfrak{x},\mathfrak{y},\mathfrak{z})[a] \iff_{\rm df} \exists x\forall y\exists z \overset{0}{\models} \varphi[a,x,y,z];$$
$$\overset{3}{\models} \bigwedge\mathfrak{x}\bigvee\mathfrak{y}\bigwedge\mathfrak{z}\varphi(\mathfrak{x},\mathfrak{y},\mathfrak{z})[a] \iff_{\rm df} \forall x\exists y\forall z \overset{0}{\models} \varphi[x,y,z,a];$$

More generally, let $\mathcal{Q}_\mathfrak{k}$ denote the unique string of length $\mathfrak{k}$ of of strictly alternating unrestricted quantifiers $\forall$, $\exists$ of the $\in$-language starting with $\exists$, and $\mathcal{R}_\mathfrak{k}$ the dual string starting with $\forall$. Then for $\varphi \in \dot\Delta_0$, we define, in a notation that suppresses details of the binding of variables of φ by quantifiers and their matching to their interpretations,

$$\overset{\mathfrak{k}}{\models} \dot{\mathcal{Q}}_\mathfrak{k}\varphi(\mathfrak{x}_1,\ldots,\mathfrak{x}_\mathfrak{k})[a] \iff_{\rm df} \mathcal{Q}_\mathfrak{k} \overset{0}{\models} \varphi[x_1,\ldots,x_\mathfrak{k},a]$$
$$\overset{\mathfrak{k}}{\models} \dot{\mathcal{R}}_\mathfrak{k}\varphi(\mathfrak{x}_1,\ldots,\mathfrak{x}_\mathfrak{k})[a] \iff_{\rm df} \mathcal{R}_\mathfrak{k} \overset{0}{\models} \varphi[x_1,\ldots,x_\mathfrak{k},a]$$

Thus for $\varphi \in \dot\Delta_0$ and $a \in \mathrm{Fin}_V$, the class of finite sequences of sets, $\overset{\mathfrak{k}}{\models} \dot{\mathcal{Q}}_\mathfrak{k}\varphi[a]$ will be a $\Sigma_\mathfrak{k}^{\mathsf{KPI}}$ predicate of φ and a, and $\overset{\mathfrak{k}}{\models} \dot{\mathcal{R}}_\mathfrak{k}\varphi[a]$ a $\Pi_\mathfrak{k}^{\mathsf{KPI}}$ one.

If $0 \leqslant \mathfrak{n} < \mathfrak{k}$, $\dot\Sigma_\mathfrak{n}$ formulæ and $\dot\Pi_\mathfrak{n}$ formulæ may be construed as $\dot\Sigma_\mathfrak{k}$ formulæ and therefore $\overset{\mathfrak{k}}{\models}$ can be applied to them. This fact will be used without comment.

Now fix $\mathfrak{k}$, and reason in $\mathsf{KZ}_\mathfrak{k}$. We show how to define the $\dot{\Sigma}_\mathfrak{k}$ hull of a subset of the universe. Initially we only consider hulls of the empty set,but we shall allow room for generalisation by speaking of "permitted constants".

We shall use certain manipulations of $\Sigma_\mathfrak{k}$ formulæ that are available using the pairing functions. For example, provably in S_0,

$$\exists a \forall b \exists c \mathfrak{B}(x,a,b,c) \ \&\ \exists d \forall e \exists f \mathfrak{D}(x,d,e,f) \iff \exists p \forall q \exists r \big[\mathfrak{B}(x,(p)_0,(q)_0,(r)_0) \ \&\ \mathfrak{D}(x,(p)_1,(q)_1,(r)_1)\big]$$

and

$$\exists a \forall b \exists c \mathfrak{B}(x,a,b,c) \ \text{ or }\ \exists d \forall e \exists f \mathfrak{D}(x,d,e,f) \iff \exists p \forall q \exists r \big[\mathfrak{B}(x,(p)_0,(q)_0,(r)_0) \ \text{ or }\ \mathfrak{D}(x,(p)_1,(q)_1,(r)_1)\big]$$

Now if $\mathfrak{B}$ and $\mathfrak{D}$ are Δ_0, then both

$$\exists r \big[\mathfrak{B}(x,(p)_0,(q)_0,(r)_0) \,\&\, \mathfrak{D}(x,(p)_1,(q)_1,(r)_1)\big]$$

and

$$\exists r \big[\mathfrak{B}(x,(p)_0,(q)_0,(r)_0) \ \text{ or }\ \mathfrak{D}(x,(p)_1,(q)_1,(r)_1)\big]$$

are $\Sigma_1^{\mathsf{S}_0}$, meaning that they are equivalent in the system S_0 to Σ_1 formulæ. It follows, generalising, that both the conjunction and the disjunction of two $\Sigma_\mathfrak{k}$ formulæ are $\Sigma_\mathfrak{k}^{\mathsf{S}_0}$. Similarly a $\Sigma_\mathfrak{k}$ formula prefaced by an existential quantifier is $\Sigma_\mathfrak{k}^{\mathsf{S}_0}$.

Corresponding manipulations will be available one level down for $\dot{\Sigma}_\mathfrak{k}$ formulæ.

Let $N_0^\mathfrak{k} = \{\varphi \mid \varphi$ is a $\dot{\Sigma}_\mathfrak{k}$ formula with one free variable and no constants$\}$. $N_0^\mathfrak{k} \in V$.

Let $N_1^\mathfrak{k} = N_0^\mathfrak{k} \cap \{\varphi \mid \exists x \overset{\mathfrak{k}}{\models} \varphi[x]\}$. $N_1^\mathfrak{k} \in V$ by $\Sigma_\mathfrak{k}$ separation.

Let $N_2^\mathfrak{k} = N_0^\mathfrak{k} \cap \{\varphi \mid \exists x \exists y \big(x \neq y \ \&\ \overset{\mathfrak{k}}{\models} \varphi[x] \ \&\ \overset{\mathfrak{k}}{\models} \varphi[y]\big)\}$. $N_2^\mathfrak{k} \in V$ by $\Sigma_\mathfrak{k}$ separation.

Finally let $N^\mathfrak{k} = N_1^\mathfrak{k} \setminus N_2^\mathfrak{k}$. $N^\mathfrak{k} \in V$ by the axiom of difference. $N^\mathfrak{k}$ is essentially a nominalist version of the $\Sigma_\mathfrak{k}$ hull of the universe. We may speak of it as the set of $\dot{\Sigma}_\mathfrak{k}$ formulæ in one free variable with unique witnesses.

We define an equivalence relation $\sim_\mathfrak{k}$ on $N^\mathfrak{k}$ by

$$\varphi \sim_\mathfrak{k} \vartheta \iff \exists x \Big(\overset{\mathfrak{k}}{\models} \varphi[x] \ \&\ \overset{\mathfrak{k}}{\models} \vartheta[x]\Big)$$

Plainly for φ, ϑ in $N^\mathfrak{k}$,

$$\varphi \sim_\mathfrak{k} \vartheta \iff \forall x \forall y \Big(\overset{\mathfrak{k}}{\models} \varphi[x] \ \&\ \overset{\mathfrak{k}}{\models} \vartheta[y] \implies x = y\Big);$$

the defining formula for $\sim_\mathfrak{k}$ has form $\Sigma_1(\Sigma_\mathfrak{k} \wedge \Sigma_\mathfrak{k})$, and the second version has form $\Pi_1\Pi_1(\Pi_\mathfrak{k} \vee \Pi_\mathfrak{k} \vee \Delta_0)$, so $\sim_\mathfrak{k}$ is a $\Delta_\mathfrak{k}$ relation on the set $N^\mathfrak{k}$.

We proceed to form the set of equivalence classes of $N^\mathfrak{k}$ induced by $\sim_\mathfrak{k}$. Our oblique approach is caused by the fact that we have no reason to believe that a $\Sigma_\mathfrak{k}$ property prefaced by a varied string of restricted quantifiers remains $\Sigma_\mathfrak{k}$.

For $\varphi \in N^\mathfrak{k}$, $\{\vartheta \in N^\mathfrak{k} \mid \vartheta \sim_\mathfrak{k} \varphi\}$ is a set, it being a $\Delta_\mathfrak{k}$ subclass of the set $N^\mathfrak{k}$; it is therefore a member of the set $\mathcal{P}(N^\mathfrak{k})$.

Let $A^\mathfrak{k} = \mathcal{P}(N^\mathfrak{k}) \cap \{x \mid \exists \varphi \exists \vartheta [\varphi \in x \ \&\ \vartheta \in x \ \&\ \varphi \not\sim_\mathfrak{k} \vartheta]\}$. $A^\mathfrak{k}$ is a set by $\Sigma_\mathfrak{k}$ separation, using the $\Pi_\mathfrak{k}$ definition of $\sim_\mathfrak{k}$. Set $J^\mathfrak{k} =_{\mathrm{df}} \mathcal{P}(N^\mathfrak{k}) \setminus A^\mathfrak{k}$. $J^\mathfrak{k} \in V$. $J^\mathfrak{k} = \{x \subseteq N^\mathfrak{k} \mid x$ is a subset of some $\sim_\mathfrak{k}$ -equivalence class$\}$. Finally put

$$K^\mathfrak{k} =_{\mathrm{df}} J^\mathfrak{k} \cap \big\{x \bigm| x \neq \varnothing \ \&\ \forall y {:\in} J^\mathfrak{k} \ \big(y \subseteq x \ \text{ or }\ y \cap x = \varnothing\big)\big\}$$

$K^\mathfrak{k}$ is a set, by Δ_0 Separation, and is the desired set of $\sim_\mathfrak{k}$-equivalence classes.

$K^\mathfrak{k}$ will form the underlying set of the structure that we shall show, using $V = L + ON = \aleph_\omega$, to model $\dot{\Sigma}_{\mathfrak{k}-1}$ `Separation`. The (membership) relation $E^\mathfrak{k}$ on $K^\mathfrak{k}$ is defined by

$$x E^\mathfrak{k} y \iff_{\mathrm{df}} \exists \varphi {:\in} x \ \exists \vartheta {:\in} y \ \exists a \exists b \big(\overset{\mathfrak{k}}{\models} \varphi[a] \ \&\ \overset{\mathfrak{k}}{\models} \vartheta[b] \ \&\ a \in b.\big)$$

Since

$$xE^{\mathfrak{k}}y \iff_{\mathrm{df}} \forall\varphi{:}{\in}x\ \forall\vartheta{:}{\in}y\ \forall a\forall b\big[\overset{\mathfrak{k}}{\models}\varphi[a]\ \&\ \overset{\mathfrak{k}}{\models}\vartheta[b] \implies a\in b\big],$$

$E^{\mathfrak{k}}$ is actually a $\Delta_{\mathfrak{k}}$ subclass of $K^{\mathfrak{k}}\times K^{\mathfrak{k}}$, and thus is certainly a set.

5·8 Suppose now that $\mathfrak{a}$ is some set: we may form the $\dot{\Sigma}_{\mathfrak{k}}$ hull of $\mathfrak{a}$ by modifying the definition of $N_0^{\mathfrak{k}}$ to permit constants for members of $\mathfrak{a}$ to occur in φ. $N^{\mathfrak{k}}(\mathfrak{a})$ will be the set of $\dot{\Sigma}_{\mathfrak{k}}$ formulæ, in one free variable and the permitted constants, with unique witnesses. We write $K^{\mathfrak{k}}(\mathfrak{a})$ and $E^{\mathfrak{k}}(\mathfrak{a})$ for the outcome of the discussion of 5·7 starting from $\mathfrak{a}$ rather than from $\varnothing$.

We refer to structures of the kind $(K^{\mathfrak{k}}(\mathfrak{a}), E^{\mathfrak{k}}(\mathfrak{a}))$ as *term models*, in view of the natural correspondence between their members and ι-terms of the kind $\iota\mathfrak{x}\varphi$, meaning "the one and only one $\mathfrak{x}$ such that φ".

The minimality axiom

We saw above that the theory KLZ_0 proves that $\forall\kappa\exists\kappa^+$. We wish to consider here a "minimising" addition to that theory, an axiom that we write as $ON=\aleph_\omega$. We outline the background to this axiom.

We wish to define the function $g:\omega\to ON$ by

$$g(p)=\omega_p,$$

It will follow from Lemma 5·9 that given Π_2 Foundation as well, the function g provably has domain ω. Hence we can ask whether $\exists\zeta\forall p{:}{\in}\omega\ g(p)<\zeta$: the least such ζ, if it exists, we shall denote by $\aleph_\omega$, and write the assertion of its existence as $\aleph_\omega<ON$. The denial that such a ζ exists we write as $ON=\aleph_\omega$ and is the minimality axiom, mentioned above.

5·9 LEMMA ($\mathsf{KLZ}_0+\Pi_2$ Foundation) $\forall n\geq 2\,\exists f[f:n\longrightarrow ON\ \&\ \forall m<n-1\ f(m+1)=(f(m))^+]$

Proof: we have seen above that KLZ_0 proves that $\forall\kappa\ \kappa^+$ exists. It follows that a contradiction will result if we assume that there is a least element to the class

$$\Big\{n\ \Big|\ 2\leqslant n\leqslant\omega\ \&\ \neg\exists f\big[\mathrm{Dom}\,(f)=n\ \&\ f(0)=\omega\ \&\ \forall m{:}{<}n\ f(m)\in ON\ \&$$
$$\&\ \forall m{:}{<}n-1\ \forall\zeta{:}{<}f(m+1)\ \exists g\ g:\zeta\xrightarrow{1-1}f(m)\ \&$$
$$\&\ \forall m{:}{<}n-1\ \neg\exists g\ g:f(m+1)\xrightarrow{1-1}f(m)\big]\Big\}$$

That class is Π_2^{KP}; it can have no least member so by Π_2 Foundation it must be empty; from which the theorem follows. ⊣ (5·9)

5·10 COROLLARY ($\mathsf{KLZ}_0+\Pi_2$ Foundation) *The function $p\mapsto\omega_p$ is well-defined and has domain ω.*

5·11 REMARK Π_2 Foundation will hold if we have set Foundation, Transitive Containment and Σ_2 Separation, and thus is provable in KZ_2. Hence the minimality axiom is well-defined in any system extending KZ_2 and we may now define the system $\mathsf{KLMZ}_{\mathfrak{k}}$ for $\mathfrak{k}\geqslant 2$ as the system $\mathsf{KLZ}_{\mathfrak{k}}+ON=\aleph_\omega$.

Continuing to reason in $\mathsf{KLZ}_0+\Pi_2$ Foundation, we distinguish two cases:

Case 1: $\exists\zeta\,\forall p\,\omega_p<\zeta$:

Let ζ_0 be one such ζ. Let $a=\mathcal{P}(\zeta_0\times\omega)$. Then the class $\{\zeta\mid\forall p<\omega\ \omega_p<\zeta\}$ is non-empty, is definable by a formula with the set a as a parameter, of the form $\forall f{:}{\in}a\ (\Pi_1\vee\Sigma_1\vee\Delta_0)$, and so has a least element by Π_2 Foundation.

In this case (remembering that we are assuming that $V=L$) we write $\aleph_\omega$ or ω_ω for the least such ζ. Given the ordinal ω_ω, we may, reasoning in KP, build the corresponding initial segment of the constructible hierarchy.

5·12 PROPOSITION ($\mathsf{KLZ}_0+\Pi_2$ Foundation) *If ω_ω exists, then $L_{\omega_\omega}\models$ KLMZ.*

Proof: it models the Power set axiom by Gödel's famous result, proved using his Condensation Lemma, that each constructible subset of κ is constructed before κ^+, that successor being computed in L. For the same reason it will correctly compute cardinals as far as its own ordinals, and will therefore believe $ON=\aleph_\omega$, it is admissible (indeed a $\dot{\Sigma}_1$ elementary submodel of L), and it will model the full Separation scheme as it is supertransitive in the sense that $x \in L \ \& \ x \subseteq y \in L_{\omega_\omega} \implies x \in L_{\omega_\omega}$. ⊣ (5·12)

So if case 1 holds, we find a model of the theory in which we are reasoning, hence, by Gödel, case 1 cannot be proved to hold, and we must consider case 2. Note that our discussion of case 1 yields the following

5·13 METATHEOREM *For each $\mathfrak{k} \geqslant 2$, the consistency of $\mathsf{KLZ}_{\mathfrak{k}}$ implies, in arithmetic, that of $\mathsf{KLMZ}_{\mathfrak{k}}$.*

Case 2: $\forall \zeta \exists n \ \zeta \leqslant \omega_n$.

This is our minimality axiom, that $ON = \aleph_\omega$, and we turn now to using that to prove a weak fine structural lemma.

A weak fine-structural lemma

Our task is to get a better estimate of the quantifier complexity of $\forall x :\in y \ \mathfrak{A}$ when $\mathfrak{A}$ is $\Sigma_{\mathfrak{n}}$. We shall prove two variants of a general fine-structural lemma of S. Friedman which is proved in Simpson [A6], and which may be seen in further action in Steel [A7]. We use Jensen's J hierarchy, as do Friedman, Simpson and Steel, but the coarser L-hierarchy would suffice for our arguments.

We begin with the first variant in schematic or "undotted" form.

5·14 DEFINITION Let S be any system extending KPI. Write $\mathrm{Fr}_u(\mathfrak{n}, \mathsf{S})$ for the hypothesis that whenever $\mathfrak{A}(a, b, \beta)$ is a $\Sigma_{\mathfrak{n}}$ formula, of which the variables a and b range over sets and the variable β over ordinals, $\forall b :\in J_\beta \ \mathfrak{A}(a, b, \beta)$ is $\Sigma^{\mathsf{S}}_{\mathfrak{n}+1}$.

5·15 LEMMA *(i)* $\mathrm{Fr}_u(0, \mathsf{KPI})$*; indeed* $\forall b :\in J_\beta \ \mathfrak{A}(a, b, \beta)$ *is* Δ^{KPI}_1 *whenever* $\mathfrak{A}$ *is* Δ_0*;*

(ii) $\mathrm{Fr}_u(1, \mathsf{KPI})$*: indeed,* $\forall b :\in J_\beta \ \mathfrak{A}(a, b, \beta)$ *is* Σ^{KPI}_1 *whenever* $\mathfrak{A}$ *is* Σ_1*;*

(iii) for $\mathfrak{n} \geqslant 2$*, if* $\mathrm{Fr}_u(\mathfrak{n}-1, \mathsf{S})$ *and* $\mathsf{S} \vdash \mathsf{KLMZ}_2 \ \& \ \Delta_{\mathfrak{n}}$ Separation*, then* $\mathrm{Fr}_u(\mathfrak{n}, \mathsf{S})$*;*

(iv) for $\mathfrak{n} \leqslant \mathfrak{k}$*,* $\mathrm{Fr}_u(\mathfrak{n}, \mathsf{KLMZ}_{\mathfrak{k}})$.

Proof: part (i) follows from the Δ^{KPI}_1 definability of the constructible hierarchy; (ii) holds by Σ_1 Collection (a consequence in KP of Δ_0 Collection), as the map $\beta \mapsto J_\beta$ is Δ^{KPI}_1; (iv) is immediate from the first three parts. It remains to prove (iii).

Let $\mathfrak{n} \geqslant 2$. The difficulty is that although the easier quantifier manipulations of KP are available to us — we may, for example, amalgamate like quantifiers — $\Sigma_{\mathfrak{n}}$ Collection is not. We shall use the function g defined by $g(p) = \omega_p$, which we have seen to be a $\Sigma^{\mathsf{KZ}_2}_2$ function — this is where we lose the extra point we gained when $\mathfrak{n} = 1$ — and we have the Axiom of Constructibility, our minimality axiom that $\bigcup Im(g) = ON$, and the hypothesis that $\mathrm{Fr}_u(\mathfrak{n}-1, \mathsf{S})$.

Let $\mathfrak{G}(a, \beta)$ be the formula $\beta \in ON \ \& \ \forall b :\in J_\beta \ \mathfrak{A}(a, b, \beta)$. By assumption, $\mathfrak{A}$ is $\Sigma_{\mathfrak{n}}$, and so of the form $\exists c \mathfrak{C}(a, b, c, \beta)$ where $\mathfrak{C}$ is $\Pi_{\mathfrak{n}-1}$.

Suppose that $\mathfrak{G}(a, \beta)$ holds: then

$$\forall b :\in J_\beta \ \exists p :\in \omega \ \exists c :\in J_{g(p)} \ \mathfrak{C}(a, b, c, \beta).$$

Let $\mathfrak{H}(b, p, \beta) \iff_{\mathrm{df}} b \in J_\beta \ \& \ p \in \omega \ \& \ \forall \eta (\eta = g(p) \implies \exists c :\in J_\eta \ \mathfrak{C}(a, b, c, \beta)).$

By our hypothesis that $\mathrm{Fr}_u(\mathfrak{n}-1, \mathsf{S})$, $\exists c :\in J_\eta \ \mathfrak{C}(a, b, c, \beta))$ is $\Pi^{\mathsf{S}}_{\mathfrak{n}}$ and so $\mathfrak{H}$ is $\Pi^{\mathsf{S}}_{\mathfrak{n}}$. But also

$$\mathfrak{H}(b, p, \beta) \iff b \in J_\beta \ \& \ p \in \omega \ \& \ \exists \eta \exists c (\eta = g(p) \ \& \ c \in J_\eta \ \& \ \mathfrak{C}(a, b, c, \beta),$$

so that $\mathfrak{H}$ is $\Sigma^{\mathsf{S}}_{\mathfrak{n}}$. Hence we may apply $\Delta_{\mathfrak{n}}$ Separation to infer that $\{(b, p) \mid \mathfrak{H}(b, p, \beta)\} \subseteq J_\beta \times \omega$ is a set.

So, provably in S,

$$\mathfrak{G}(a, \beta) \iff \beta \in ON \ \& \ \exists H \begin{cases} H \subseteq J_\beta \times \omega \ \& \\ \& \ \forall b :\in J_\beta \ \exists p :\in \omega \ (b, p) \in H \ \& \\ \& \ \forall b :\in J_\beta \ \forall p :\in \omega \ \forall \eta \ \big((b, p) \in H \ \& \ \eta = g(p) \implies \exists c :\in J_\eta \ \mathfrak{C}(a, b, \beta, c)\big) \end{cases}$$

which is $\Sigma^{\mathsf{S}}_{\mathfrak{n}+1}$, as required, since (as we have seen) $\exists c :\in J_\eta \ \mathfrak{C}(a, b, c, \beta))$ is $\Pi^{\mathsf{S}}_{\mathfrak{n}}$. ⊣ (5·15)

We wish, though, for each $\mathfrak{k}$ to be able within our $\in$-language to quantify over all formulæ in a class corresponding to Σ_k. We therefore introduce a concept midway between $\Sigma_{\mathfrak{k}}$ and $\dot{\Sigma}_{\mathfrak{k}}$.

5·16 DEFINITION A $\mathring{\Sigma}_{\mathfrak{k}}$ *formula* of the $\in$-language is one of the form $\mathcal{Q}_{\mathfrak{k}} \models^{0} \vartheta[a]$, where it is intended that $\vartheta \in \dot{\Delta}_0$; similarly, we call $\mathcal{R}_{\mathfrak{k}} \models^{0} \vartheta[a]$ a $\mathring{\Pi}_{\mathfrak{k}}$ *formula*. Such formulæ are chiefly determined by ϑ: so we may introduce one by the simple phrase "Let $\mathfrak{A}$ be $\mathring{\Sigma}_{\mathfrak{k}}(\vartheta)$", thus naming the $\dot{\Delta}_0$ formula concerned.

If Φ denotes the $\dot{\Sigma}_{\mathfrak{k}}$ formula $\dot{\mathcal{Q}}_{\mathfrak{k}}\varphi$, we denote by $\mathring{\Phi}$ the $\mathring{\Sigma}_{\mathfrak{k}}$ formula $\mathcal{Q}_{\mathfrak{k}} \models^{0} \varphi$, the formal variables of φ being changed appropriately to ordinary variables.

Every $\mathring{\Sigma}_{\mathfrak{k}}(\vartheta)$ predicate of a is a $\Sigma_{\mathfrak{k}}^{\mathsf{KPI}}$ predicate of ϑ and a; to indicate the special rôle played by ϑ we purloin Quine's corners and write "$\mathfrak{A}$ is $\Sigma_{\mathfrak{k}}(\ulcorner\vartheta\urcorner, a)$".

Conversely every $\Sigma_{\mathfrak{k}}$ formula $\mathfrak{A}$ is equivalent over KPI to the $\mathring{\Sigma}_{\mathfrak{k}}$ formula where ϑ is $\dot{\mathfrak{A}}$. In particular, over a reasonable base theory such as KPI, the scheme of $\Sigma_{\mathfrak{k}}$ separation is equivalent to a principle of $\mathring{\Sigma}_{\mathfrak{k}}$ separation expressible in a single formula. But if we consider the result of interpreting our set theory with non-standard integers, we see that there might be φ not of the form $\dot{\mathfrak{A}}$; so there is a difference between the two concepts. The important point is that we can practically quantify in the $\in$-language over the $\mathring{\Sigma}_{\mathfrak{k}}$ formulæ, as is illustrated by the following "dotted" definitions, which are of single formulæ of the $\in$-language.

5·17 DEFINITION

(5·17·0) $\mathsf{Fr}_d(\mathfrak{n}) \iff_{\mathrm{df}}$ for every $\dot{\Delta}_0$ formula φ there is a $\dot{\Delta}_0$ formula ϑ such that for all β and for all assignments of values to the relevant free variables of φ and ϑ,

$$\big(\forall x{:}\in J_\beta\ \mathcal{Q}_{\mathfrak{k}} \models^{0} \varphi\big) \iff \mathcal{Q}_{\mathfrak{n}+1} \models^{0} \vartheta.$$

If we paraphrased that by the phrase "whenever $\mathfrak{A}$ is $\mathring{\Sigma}_{\mathfrak{n}}$, $\big(\forall x{:}\in J_\beta\ \mathfrak{A}\big)$ is $\mathring{\Sigma}_{\mathfrak{n}+1}$," we may give two more definitions periphrastically:

(5·17·1) $\mathsf{Fr}'_d(\mathfrak{n})$ is the $\in$-formula similarly paraphrased as "whenever $\mathfrak{A}(a,x,y)$ is $\mathring{\Sigma}_{\mathfrak{n}}$, $\forall x{:}\in y\ \mathfrak{A}(a,x,y)$ is $\mathring{\Sigma}_{\mathfrak{n}+1}$"; and

(5·17·2) $\mathsf{Fr}''_d(\mathfrak{n})$ is the $\in$-formula paraphrased as "whenever $\mathfrak{A}(a,x,y)$ is $\mathring{\Sigma}_{\mathfrak{n}}$, $\forall x{:}<_L y\ \mathfrak{A}(a,x,y)$ is $\mathring{\Sigma}_{\mathfrak{n}+1}$".

5·18 PROPOSITION *(i)* $\mathsf{KPI} \vdash \mathsf{Fr}_d(0)$ & $\mathsf{Fr}'_d(0)$ & $\mathsf{Fr}''_d(0)$; *indeed they hold in the same strengthened sense in which* $\mathrm{Fr}_u(0, \mathsf{KPI})$ *holds;*

(ii) $\mathsf{KPI} \vdash \mathsf{Fr}_d(0)$, $\mathsf{Fr}'_d(1)$ *and* $\mathsf{Fr}''_d(1)$ *hold in the same strengthened sense in which* $\mathrm{Fr}_u(1, \mathsf{KPI})$ *holds;*

(iii) for each $\mathfrak{n} \geqslant 2$, $\mathsf{KPL} \vdash$ *if* $\mathsf{Fr}_d(\mathfrak{n})$ *then both* $\mathsf{Fr}'_d(\mathfrak{n})$ *and* $\mathsf{Fr}''_d(\mathfrak{n})$.

(iv) for $\mathfrak{n} \leqslant \mathfrak{k}$, $\mathsf{KLMZ}_{\mathfrak{k}} \vdash \mathsf{Fr}_d(\mathfrak{n})$ & $\mathsf{Fr}'_d(\mathfrak{n})$ & $\mathsf{Fr}''_d(\mathfrak{n})$

Proof : for parts (i) and (ii), modify the proofs of parts (i) and (ii) of Lemma 5·15. For part (iii), we know that the formula $y \in J_\beta$ and the truth predicate $\models$ are Δ_1^{KPI}. Thus the formulæ $\big(\models_{J_\beta} x \dot{<}_L y\big) \implies \Psi$ and $x \in y \implies \Psi$ are $\mathring{\Sigma}_{\mathfrak{n}}^{\mathsf{KPI}}$ whenever Ψ is $\mathring{\Sigma}_{\mathfrak{n}}$. We may then apply the equivalences

$$\forall x{:}<_L y\ \Psi \iff \exists\beta\Big[y \in J_\beta\ \&\ \forall x{:}\in J_\beta\ \big[(J_\beta \models x \dot{<}_L y) \implies \Psi\big]\Big], \qquad \text{and}$$
$$\forall x{:}\in y\ \Psi \iff \exists\beta\Big[y \in J_\beta\ \&\ \forall x{:}\in J_\beta\ \big[x \in y \implies \Psi\big]\Big].$$

For part (iv), modify the proof of Lemma 5·15, part (iii). ⊣ (5·18)

Consistency proofs for fragments of KLMZ

Let $\mathfrak{k} \geqslant 2$. We recall our discussion of 5·7 and 5·8. The following lemma may be understood informally to imply that the structure $(K^{\mathfrak{k}}, E^{\mathfrak{k}})$ is $\Sigma_{\mathfrak{k}}$-elementarily embeddable in L, the embedding being the map that sends each equivalence class to the unique witness to formulæ in that equivalence class. Remember that the permitted constants are names for members of a set $\mathfrak{a}$. We write $N_0^{\mathfrak{k}}(\mathfrak{a})$ for the set of $\dot{\Sigma}_{\mathfrak{k}}$ formulæ in one free variable and only permitted constants.

5·19 THE WITNESS LEMMA ($\mathsf{KLMZ}_{\mathfrak{k}-1}$) *Let* $\Phi(\mathfrak{x})$ *be a* $\dot{\Sigma}_{\mathfrak{k}}$ *formula with one free variable, the parameters of which are either among the permitted constants or are unique witnesses to formulæ in* $N_0^{\mathfrak{k}}(\mathfrak{a})$. *If* Φ *has a witness, it has one which is itself the unique witness to some formula in* $N_0^{\mathfrak{k}}(\mathfrak{a})$.

Proof : a first attempt would be, given a $\dot{\Sigma}_{\mathfrak{k}}$ formula $\Phi(\mathfrak{x})$, to consider

$$\{x \mid \mathring{\Phi}(x) \; \& \; \forall y :<_L x \, \neg\mathring{\Phi}(y)\},$$

which if $\forall x \neg\mathring{\Phi}(x)$ is empty, and otherwise has one member, that member witnessing Φ. But that class is not $\Sigma_{\mathfrak{k}}$, so we must modify this approach. Further we must consider the presence of parameters.

Suppose then that $\exists x \, \mathring{\Phi}$ holds where Φ is of the form $\bigvee \mathfrak{y} \Psi(x, y, x_1, b)$ where Ψ is $\dot{\Pi}_{\mathfrak{k}-1}$, b is one of the permitted constants (or a finite list thereof), and x_1 is the unique set satisfying a $\dot{\Sigma}_{\mathfrak{k}}$ predicate $\Theta(x_1, c)$, c being a permitted constant (or a finite list of the same). Consider

$$\big\{\langle x, y, z\rangle \bigm| \mathring{\Psi}(x, y, z, b) \; \& \; \mathring{\Theta}(z, c) \; \& \; \forall \langle x', y'\rangle :<_L \langle x, \, y\rangle \, \neg\mathring{\Psi}(x', y', z, b)\big\} :$$

essentially by $\mathsf{Fr}''_d(\mathfrak{k}-1)$, that is $\mathring{\Sigma}_{\mathfrak{k}}$; and contains exactly one member, a triple. The first member of that triple will be a witness to the original Φ, and will be equally definable as the unique witness to a $\dot{\Sigma}_{\mathfrak{k}}$ predicate. ⊣ (5·19)

5·20 THEOREM $(\mathsf{KLMZ}_{\mathfrak{k}})$ $(K^{\mathfrak{k}}, E^{\mathfrak{k}}) \models \mathtt{KLMZ}_{\mathfrak{k}-1}$*; moreover the structure* $(K^{\mathfrak{k}}, E^{\mathfrak{k}})$ *is well-founded and isomorphic to some countable transitive set.*

The proof splits into a series of lemmata, some of which we state in abstract model-theoretic form.

5·21 LEMMA *Let* $\mathbf{M} \preceq_{\Sigma_{\mathfrak{k}}} \mathbf{N}$ *, where* $\mathfrak{k} \geqslant 2$*. If the Power Set axiom is true in* $\mathbf{N}$*, then it is also true in* $\mathbf{M}$*.* ⊣

5·22 REMARK The lemma fails for $\mathfrak{k} = 1$, for let $\delta = \omega_1^L$ and $\kappa = \omega_\omega^L$. Then $L_\delta \preceq_{\Sigma_1} L_\kappa$, L_κ satisfies the power set axiom, and L_δ does not.

Other axioms are easily transported: apply, for example, the fact that the "minimising" axiom, $ON = \aleph_\omega$, is $\Pi_3^{\mathsf{KZ}_2}$. Our problem reduces to proving that $(K^{\mathfrak{k}}, E^{\mathfrak{k}}) \models \dot{\Sigma}_{\mathfrak{k}-1}\text{-}\mathtt{Separation}$.

5·23 LEMMA *If* $\mathbf{M} \preceq_{\Sigma_{\mathfrak{k}}} \mathbf{N}$ *, where* $\mathfrak{k} \geq 2$*, and* $\Sigma_{\mathfrak{k}-1}$ Separation, KPL *and the principle* $\mathsf{Fr}'_d(\mathfrak{k}-1)$ *are true in* $\mathbf{N}$*, then* $\Sigma_{\mathfrak{k}-1}$ Separation *will be true in* $\mathbf{M}$*.*

Proof : let $\Psi(x)$ be $\Sigma_{\mathfrak{k}-1}$. Let $a \in \mathbf{M}$. Then, in $\mathbf{N}$,

$$\exists z \big(z \subseteq a \; \& \; \forall w {:\in} z \; \Psi \; \& \; \forall w (w \notin a \text{ or } \neg\Psi \text{ or } w \in z)\big).$$

The first clause is Δ_0, the second is $\Sigma_{\mathfrak{k}}$ by $\mathsf{Fr}'_d(\mathfrak{k}-1)$, and the third is $\Pi_1(\Pi_{\mathfrak{k}-1})$; so the whole is $\Sigma_{\mathfrak{k}}$. Hence it is also true in $\mathbf{M}$. ⊣ (5·23)

We have used $\Sigma_{\mathfrak{k}}$ Separation to show that $(K^{\mathfrak{k}}, E^{\mathfrak{k}})$ is a set. Recall that its members are equivalence classes of formulæ. The model we would really prefer to consider is $H^{\mathfrak{k}} =_{\text{df}} \{x \mid \exists \varphi {:\in} N^{\mathfrak{k}} \overset{\mathfrak{k}}{\models} \varphi[x]\}$, but that is a $\Delta_0\Sigma_{\mathfrak{k}}$ class; without $\Sigma_{\mathfrak{k}}$ Collection we have no grounds for believing that it is a set. Thus instead we must work with the set $K^{\mathfrak{k}}$, to which $H^{\mathfrak{k}}$ is secretly isomorphic.

For $\varphi \in N^{\mathfrak{k}}$ we write $(\varphi)_{\mathfrak{k}}$ for its equivalence class with respect to $\sim_{\mathfrak{k}}$. The following makes the sense in which $(K^{\mathfrak{k}}, E^{\mathfrak{k}})$ is a $\Sigma_{\mathfrak{k}}$ elementary submodel of L more precise. Note that in part of the statement of Proposition 5·24 the φ's are merely representatives of their equivalence classes which are members of some model, whereas elsewhere they are functioning as formulæ the truth of which is being evaluated.

5·24 PROPOSITION $(\mathsf{KLMZ}_{\mathfrak{k}})$ *Let* $\mathfrak{n} \leqslant \mathfrak{k}$*, let* $\ell \in \omega$ *and let* Φ *be a* $\dot{\Sigma}_{\mathfrak{n}}$ *formula with free variables* $\langle \mathfrak{x}_i \mid i < \ell\rangle$*. For* $\langle \varphi_i \mid i < \ell\rangle \in {}^{\ell}N^{\mathfrak{k}}$,

$$\begin{aligned}(K^{\mathfrak{k}}, E^{\mathfrak{k}}) \models \Phi[\langle(\varphi_i)_{\mathfrak{k}} \mid i < \ell\rangle] &\iff \exists \langle x_i \mid i < \ell\rangle \big((\forall i{:<}\ell \overset{\mathfrak{k}}{\models} \varphi_i[x_i]) \; \& \overset{\mathfrak{k}}{\models} \Phi[\langle x_i \mid i < \ell\rangle]\big)\\ &\iff \forall \langle x_i \mid i < \ell\rangle \big((\forall i{:<}\ell \overset{\mathfrak{k}}{\models} \varphi_i[x_i]) \implies \overset{\mathfrak{k}}{\models} \Phi[\langle x_i \mid i < \ell\rangle]\big)\end{aligned}$$

Proof : the result holds for atomic Φ by definition of the relations $\sim_{\mathfrak{k}}$ and $E^{\mathfrak{k}}$: for φ_1 and φ_2 in $N^{\mathfrak{k}}$,

$$\begin{aligned}(K^{\mathfrak{k}},E^{\mathfrak{k}}) \models (\varphi_1)_{\mathfrak{k}} = (\varphi_2)_{\mathfrak{k}} &\iff \varphi_1 \sim_{\mathfrak{k}} \varphi_2\\ &\iff \exists x_1 \exists x_2 \big(\overset{\mathfrak{k}}{\models} \varphi_1[x] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x] \ \& \ \overset{0}{\models} x_1 = x_2\big)\\ &\iff \forall x_1 \forall x_2 \big((\overset{\mathfrak{k}}{\models} \varphi_1[x] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x]) \implies \overset{0}{\models} x_1 = x_2\big), \quad \text{and}\\ (K^{\mathfrak{k}},E^{\mathfrak{k}}) \models (\varphi_1)_{\mathfrak{k}} \in (\varphi_2)_{\mathfrak{k}} &\iff \varphi_1 E^{\mathfrak{k}} \varphi_2\\ &\iff \exists x_1 \exists x_2 \big(\overset{\mathfrak{k}}{\models} \varphi_1[x] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x] \ \& \ \overset{0}{\models} x_1 \in x_2\big)\\ &\iff \forall x_1 \forall x_2 \big((\overset{\mathfrak{k}}{\models} \varphi_1[x] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x]) \implies \overset{0}{\models} x_1 \in x_2\big).\end{aligned}$$

Then we do an ordinary induction to cover the case of Φ an arbitrary $\dot{\Delta}_0$ wff: for example,

$$\begin{aligned}&(K^{\mathfrak{k}},E^{\mathfrak{k}}) \models \bigvee \mathfrak{x}_3 \colon \in (\varphi_1)_{\mathfrak{k}}\ \vartheta(\mathfrak{x}_3)[(\varphi_1)_{\mathfrak{k}},(\varphi_2)_{\mathfrak{k}}] \iff\\ &\iff (K^{\mathfrak{k}},E^{\mathfrak{k}}) \models \bigvee \mathfrak{x}_3 \big[\mathfrak{x}_3 \in (\varphi_1)_{\mathfrak{k}} \wedge \vartheta(\mathfrak{x}_3)[(\varphi_1)_{\mathfrak{k}},(\varphi_2)_{\mathfrak{k}}]\big]\\ &\iff \exists \varphi_3 \,{:}{\in} N^{\mathfrak{k}}\ (K^{\mathfrak{k}},E^{\mathfrak{k}}) \models (\varphi_3)_{\mathfrak{k}} \in (\varphi_1)_{\mathfrak{k}} \wedge \vartheta[(\varphi_3)_{\mathfrak{k}},(\varphi_1)_{\mathfrak{k}},(\varphi_2)_{\mathfrak{k}}]\\ &\iff \exists \varphi_3 \,{:}{\in} N^{\mathfrak{k}}\ \exists x_1 \exists x_2 \exists x_3 \overset{\mathfrak{k}}{\models} \varphi_1[x_1] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x_2] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_3[x_3] \ \& \ \overset{0}{\models} x_3 \in x_1 \wedge \vartheta[x_3,x_1,x_2]\\ &\iff \exists x_1 \exists x_2 \overset{\mathfrak{k}}{\models} \varphi_1[x_1] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x_2] \ \& \ \overset{0}{\models} \bigvee \mathfrak{x}_3 \colon \in x_1 \vartheta(\mathfrak{x}_3)[x_1,x_2]\\ &\iff \forall x_1 \forall x_2 \big(\overset{\mathfrak{k}}{\models} \varphi_1[x_1] \ \& \ \overset{\mathfrak{k}}{\models} \varphi_2[x_2]\big) \implies \overset{0}{\models} \bigvee \mathfrak{x}_3 \colon \in x_1 \vartheta(\mathfrak{x}_3)[x_1,x_2]\end{aligned}$$

The first equivalence is simply a reformulation of the restricted quantifier, the equivalence of lines 1 and 2 holds by the definition of $\models$, of lines 2 and 3 by the induction hypothesis, of lines 3 and 4 by the Witness Lemma, and of lines 4 and 5 since φ_1, φ_2 determine x_1, x_2 uniquely.

That induction succeeds because we have Σ_1 and Π_1 Foundation available.

Finally, we work our way up from $\mathfrak{n} = 0$ to $\mathfrak{n} = \mathfrak{k}$. For example,

$$\begin{aligned}(K^{\mathfrak{k}},E^{\mathfrak{k}}) \models \bigwedge \mathfrak{y} \Psi(\mathfrak{y})[(\vartheta)_{\mathfrak{k}}] &\iff \forall \varphi \,{:}{\in} N^{\mathfrak{k}}\ (K^{\mathfrak{k}},E^{\mathfrak{k}}) \models \Psi[(\varphi)_{\mathfrak{k}},(\vartheta)_{\mathfrak{k}}]\\ &\iff \forall \varphi \,{:}{\in} N^{\mathfrak{k}}\ \forall y \forall z \Big((\overset{\mathfrak{k}}{\models} \varphi[y] \ \& \ \overset{\mathfrak{k}}{\models} \vartheta[z]) \implies \overset{\mathfrak{k}}{\models} \Psi[y,z]\Big)\\ &\iff \forall z \Big(\overset{\mathfrak{k}}{\models} \vartheta[z] \implies \forall \varphi \,{:}{\in} N^{\mathfrak{k}}\ \forall y \big(\overset{\mathfrak{k}}{\models} \varphi[y] \implies \overset{\mathfrak{k}}{\models} \Psi[y,z]\big)\Big)\\ &\iff \forall z \Big(\overset{\mathfrak{k}}{\models} \vartheta[z] \implies \big(\forall y \overset{\mathfrak{k}}{\models} \Psi[y,z]\big)\Big)\\ &\iff \forall z \big(\overset{\mathfrak{k}}{\models} \vartheta[z] \implies \overset{\mathfrak{k}}{\models} \bigwedge \mathfrak{y} \Psi(\mathfrak{y})[z]\big)\\ &\iff \exists z \big(\overset{\mathfrak{k}}{\models} \vartheta[z] \ \& \ \overset{\mathfrak{k}}{\models} \bigwedge \mathfrak{y} \Psi(\mathfrak{y})[z]\big)\end{aligned}$$

The first equivalence holds by the definition of $\models$, the equivalence of lines 1 and 2 by the induction hypothesis, of lines 2 and 3 by predicate logic, of lines 3 and 4 by the Witness Lemma which tells us that if there is a counter-example there is a $\Sigma_{\mathfrak{k}}$ definable counter-example, of lines 4 and 5 by the definition of $\overset{\mathfrak{k}}{\models}$, and of lines 5 and 6 since ϑ defines z uniquely. ⊣ (5·24)

That $(K^{\mathfrak{k}},E^{\mathfrak{k}}) \models \mathtt{KLMZ}_{\mathfrak{k}-1}$ may now be proved by combining the reasoning of the above lemmata, where Lemmata 5·21 and 5·23 must be adapted to the case that $\mathbf{N}$ is really V, $\mathbf{M}$ the model $(K^{\mathfrak{k}},E^{\mathfrak{k}})$, and "an elementary submodel of" generalised to "elementarily embeddable in".

The well-foundedness of $(K^{\mathfrak{k}},E^{\mathfrak{k}})$ may be proved by the argument given in our proof below that $\mathsf{KZ}_{\mathfrak{k}}$ proves $(\mathsf{KZ}_{\mathfrak{k}})^L$. ⊣ (5·20)

5·25 METACOROLLARY *Zermelo set theory* Z *is not finitely axiomatisable.*

Proof : It has long been known that $\mathsf{KP} + \omega \in V$ is finitely axiomatisable; the above theorem shows by Gödel that $\mathsf{Z} + \mathsf{KP} + V = L + ON = \aleph_\omega$ is not finitely axiomatisable, and it is the result of adding three axioms to Z (which system includes $\omega \in V$.) ⊣ (5·25)

5·26 METACOROLLARY $\mathsf{KLZ}_\mathfrak{k}$ *proves that* $\mathsf{KLZ}_{\mathfrak{k}-1}$ *has a countable transitive model.*

Proof : We have just seen that $\mathsf{KLZ}_\mathfrak{k} + ON = \aleph_\omega$ proves the consistency of $\mathsf{KLMZ}_{\mathfrak{k}-1}$, and therefore the consistency of $\mathsf{KLZ}_{\mathfrak{k}-1}$; and we have also seen that $\mathsf{KLZ}_\mathfrak{k} + \aleph_\omega < ON$ proves the consistency not merely of $\mathsf{KLZ}_{\mathfrak{k}-1}$ but of KLZ. In each case the term model constructed is well-founded, extensional and countable, and we are in a set theory strong enough to transitise it. ⊣ (5·26)

Proving $\Sigma_\mathfrak{k}$ Separation in L

Now we want to adapt those arguments to proving the following scheme:

5·27 THEOREM *Let* $\mathfrak{k} \geqslant 2$. *Then in* $\mathsf{KZ}_\mathfrak{k}$ *we may show that* $\mathsf{KZ}_\mathfrak{k}$ *is true in* L.

We have proved that result in 5·5 for the case $\mathfrak{k} = 1$ and shall see in §6 that it fails for $\mathfrak{k} = 0$.

5·28 COROLLARY *For* $\mathfrak{k} \geqslant 2$, $\mathsf{KZ}_\mathfrak{k}$ *proves the consistency of* $\mathsf{KZ}_{\mathfrak{k}-1}$ *and, further, that* $\mathsf{KZ}_{\mathfrak{k}-1}$ *has a countable transitive model.*

Proof : we have seen that $\mathsf{KLZ}_\mathfrak{k}$ proves $\mathsf{Consis}(\mathsf{KLZ}_{\mathfrak{k}-1})$ and therefore also $\mathsf{Consis}(\mathsf{KZ}_{\mathfrak{k}-1})$; so by the theorem, $\mathsf{KZ}_\mathfrak{k}$ proves $(\mathsf{Consis}(\mathsf{KZ}_{\mathfrak{k}-1}))^L$; but a consistency statement is arithmetical, and $\omega^L = \omega$, and so $\mathsf{KZ}_\mathfrak{k}$ proves $\mathsf{Consis}(\mathsf{KZ}_{\mathfrak{k}-1})$. For the second part, we need only observe that we are in a theory strong enough to transitise a well-founded extensional relation, and that a countable transitive model in L remains that in V. ⊣ (5·28)

To show that $b \cap A^L$ is a set of L, where $b \in L$ and A^L is a $\Sigma_\mathfrak{k}$ class of L, we shall first show that $a \cap A^L \in L$ for a transitive $a \in L$ containing all members of b and all relevant parameters.

To do that, we build a term model not quite as above: we must adjust the definitions of N_1 and N_2 as we are no longer assuming that $V = L$, we have a constant for each member of a, and we obtain a structure which intuitively is $\Sigma_\mathfrak{k}$-elementarily embeddable in L and which is a set containing a copy of the transitive set a as a subset. The structure proves to be well-founded and therefore collapsible to some $L_{\bar\xi}$, and we are then able to build $a \cap A^L$ as a set by using $L_{\bar\xi}$ as an oracle telling us which members of a should go into $a \cap A^L$.

Since we are no longer assuming the minimality axiom, we shall find that we must have as members of a two sets we call the *universal parameters*, the presence of which will guarantee the continued truth of the principles $Fr(\mathfrak{n})$ in a modified form, to the proof of which latter we now turn.

A strong fine-structural lemma

Fix $\mathfrak{k} \geqslant 2$. Let f be a Δ_1 function — for example, $f(x) = J_{\varrho(x)}$, or $f(\bar x) = \{x \mid x <_L \bar x\}$ — which has these three properties:

$$x \in L \implies f(x) \in L$$
$$y \in L \,\&\, z \in L \implies \big(f(y) \subseteq f(z) \text{ or } f(z) \subseteq f(y)\big)$$
$$L = \bigcup\{f(x) \mid x \in L\}$$

5·29 PROPOSITION ($\mathsf{KZ}_\mathfrak{k}$) *There are constructible parameters* $p_\mathfrak{k}$ *and* $a_\mathfrak{k}$, *and* $\vartheta_\mathfrak{k} \in \dot\Delta_0$, *such that for all* $\mathfrak{n} \leqslant \mathfrak{k}$, *whenever* $\Phi(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \dot{\mathcal{Q}}_\mathfrak{n}\varphi$ *is* $\dot\Sigma_\mathfrak{n}$, $\bar a \in L$ *and* $\bar x \in L$, *the formula* $\forall x :\in f(\bar x)\ (\mathring{\Phi})^L(x, \bar x, \bar a)$ *is* $\Sigma_{\mathfrak{n}+1}(\ulcorner\varkappa\urcorner, \bar x, \bar a, p_\mathfrak{k}, a_\mathfrak{k})$ *for some* $\varkappa \in \dot\Delta_0$ *computed uniformly from* φ *and* $\vartheta_\mathfrak{k}$.

We shall refer to $p_\mathfrak{k}$ and $a_\mathfrak{k}$ as the *universal parameters*. Note that they and $\vartheta_\mathfrak{k}$ are independent of φ, $\bar a$ and $\bar x$.

Proof : We prove this by induction on $\mathfrak{n}$, but there may be three phases to the induction. We have a very easy start for $\mathfrak{n} = 0$ and 1, then we have a relatively simple argument that will maintain the induction so long as a certain assumption holds; when we reach an $\mathfrak{n}$ for which the assumption fails, we shall have to change tack.

We begin by determining the point of change. For each $\mathfrak{n} \leqslant \mathfrak{k}$ we ask if for some $\vartheta \in \dot\Delta_0$, some $\bar p \in L$, and some $\bar a \in L$, we have, setting $\Psi = \dot{\mathcal{R}}_{\mathfrak{n}-1}\vartheta$, that

$$\forall x\mathbin{:\in} f(\bar p)\ \exists y\mathbin{:\in} L\ \ (\mathring{\Psi})^L(x,y,\bar p,\bar a)\ \ \text{but}\ \ \neg\exists\bar y\mathbin{:\in} L\ \forall x\mathbin{:\in} f(\bar p)\ \exists y\mathbin{:\in} f(\bar y)\ (\mathring{\Psi})^L(x,y,\bar p,\bar a).$$

If so, take the least such $\mathfrak{n}$, which will be at least 2, and call it $\bar{\mathfrak{n}}$, and choose instances for ϑ, $\bar p$ and $\bar a$, calling them $\vartheta_\mathfrak{k}$, $p_\mathfrak{k}$ and $a_\mathfrak{k}$. If no instances exist, we set $\bar{\mathfrak{n}} = \mathfrak{k}+1$, $p_\mathfrak{k} = a_\mathfrak{k} = \varnothing$, and take $\vartheta_\mathfrak{k}$ to be $\mathfrak{w} = \mathfrak{w}$. Then whatever the value of $\bar{\mathfrak{n}}$ we shall have for all $\mathfrak{n} < \bar{\mathfrak{n}}$ and for all $\dot{\Pi}_{\mathfrak{n}-1}$ predicates $\Psi(\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v})$,

$$\forall\bar x\mathbin{:\in} L\ \forall\bar a\mathbin{:\in} L\ \ \Big[\forall x\mathbin{:\in} f(\bar x)\ \exists y(\mathring{\Psi})^L(x,y,\bar x,\bar a)\iff\exists\bar y\mathbin{:\in} L\ \forall x\mathbin{:\in} f(\bar x)\ \exists y\mathbin{:\in} f(\bar y)\ (\mathring{\Psi})^L(x,y,\bar x,\bar a)\Big] \qquad (**)$$

Finally, set $\Psi_\mathfrak{k}$ to be $\dot{\mathcal{R}}_{\bar{\mathfrak{n}}-1}\vartheta_\mathfrak{k}$.

First phase. For $\mathfrak{n} = 0$ and 1, the result follows simply from admissibility; moreover admissibility shows us that $\bar{\mathfrak{n}} \geqslant 2$.

Second phase. The induction proceeds through all $\mathfrak{n} < \bar{\mathfrak{n}}$, since if Φ is $\Sigma_\mathfrak{n}$, say Φ is $\bigvee\mathfrak{y}\Theta$ where Θ is $\dot{\Pi}_{\mathfrak{n}-1}$, then from $(**)$,

$$\forall x\mathbin{:\in} f(\bar x)\ (\mathring{\Phi})^L \iff \underbrace{\exists\bar y\mathbin{:\in} L\ \underbrace{\forall x\big[\underbrace{x\in f(\bar x)}_{\Delta_1}\implies\underbrace{\exists y\mathbin{:\in} f(\bar y)\ \underbrace{(\mathring{\Theta})^L(x,y,\bar x,\bar a)}_{\Pi_{\mathfrak{n}-1}}}_{\Pi_\mathfrak{n}\text{ by induction}}\big]}_{\Pi_\mathfrak{n}}}_{\Sigma_{\mathfrak{n}+1}}$$

Third phase, for $\mathfrak{n} \geqslant \bar{\mathfrak{n}}$. Again let Φ be $\bigvee\mathfrak{y}\Theta$ where Θ is $\dot{\Pi}_{\mathfrak{n}-1}$, and let

$$\mathcal{H}(\bar x,\bar a) = \Big\{(x,p)\ \Big|\ x\in f(\bar x)\ \&\ p\in f(p_\mathfrak{k})\ \&\ \underbrace{\&\ \forall z\mathbin{:\in} L\ \big(\underbrace{\exists y\mathbin{:\in} f(z)\ (\mathring{\Psi}_\mathfrak{k})^L(p,y,p_\mathfrak{k},a_\mathfrak{k})}_{\Sigma_{\bar{\mathfrak{n}}}}\implies\underbrace{\exists y\mathbin{:\in} f(z)\ (\mathring{\Theta})^L(x,y,\bar x,\bar a)}_{\Pi_\mathfrak{n}\text{ by induction}}\big)}_{\Pi_\mathfrak{n}}\Big\}$$

$\mathcal{H}$ is a $\Pi_\mathfrak{n}(\ulcorner\vartheta_\mathfrak{k}\urcorner,\bar x,\bar a,p_\mathfrak{k},a_\mathfrak{k})$ subclass of $f(\bar x)\times f(p_\mathfrak{k})$, and $\mathfrak{n}\leqslant\mathfrak{k}$, so we may apply $\Sigma_\mathfrak{k}$ separation to conclude that $\mathcal{H}$ is a set for each $\bar x$, $\bar a$ in L.

Hence for $\bar x$ and $\bar a$ in L,

$$\forall x\mathbin{:\in} f(\bar x)\ (\mathring{\Phi})^L\iff\underbrace{\exists H\Big[\overbrace{H\subseteq f(\bar x)\times f(p_\mathfrak{k})\ \&\ \forall x\mathbin{:\in} f(\bar x)\ \exists p\mathbin{:\in} f(p_\mathfrak{k})\ (x,p)\in H}^{\Delta_1^{\mathsf{KP}}}\ \&\ \&\ \forall x\forall p\forall z\mathbin{:\in} L\ \underbrace{\big(\underbrace{(x,p)\in H}_{\Delta_0}\ \&\ \underbrace{\exists y\mathbin{:\in} f(z)\ (\mathring{\Psi}_\mathfrak{k})^L(p,y,p_\mathfrak{k},a_\mathfrak{k})}_{\Sigma_{\bar{\mathfrak{n}}}}\implies\underbrace{\exists y\mathbin{:\in} f(z)\ (\mathring{\Theta})^L(x,y,\bar x,\bar a)}_{\Pi_\mathfrak{n}}\big)}_{\Pi_\mathfrak{n}}\Big]}_{\Pi_\mathfrak{n}}$$

which for some $\varkappa\in\dot{\Delta}_0$ will express the formula $\forall x\mathbin{:\in} f(\bar x)\ (\mathring{\Phi})^L$ in $\Sigma_{\mathfrak{n}+1}(\ulcorner\varkappa\urcorner,\bar x,\bar a,p_\mathfrak{k},a_\mathfrak{k})$ form. It only remains therefore to verify this last equivalence.

Suppose first that $\bar x$ and $\bar a$ are two points in L such that $\forall x\mathbin{:\in} f(\bar x)\ (\mathring{\Phi})^L(x,\bar x,\bar a)$. Take $H = \mathcal{H}(\bar x,\bar a)$. Then $H\subseteq f(\bar x)\times f(p_\mathfrak{k})$. Given $x\in f(\bar x)$, let $y\in L$ be such that $(\mathring{\Theta})^L(x,y,\bar x,\bar a)$ holds. For such x, let $p\in L$ be such that any $f(z)$ (where $z\in L$) containing a witness t with $(\mathring{\Psi}_\mathfrak{k})^L(p,t,p_\mathfrak{k},a_\mathfrak{k})$ is large enough to contain a witness y to $(\mathring{\Theta})^L(x,y,\bar x,\bar a)$. [Thus given x let y witness $(\mathring{\Theta})^L(x,y,\bar x,\bar a)$. Let $y\in f(y_1)$: here we use $L=\bigcup_{w\in L} f(w)$. There is a p in $f(p_\mathfrak{k})$ such that no witness to $(\mathring{\Psi}_\mathfrak{k})^L$ is in $f(y_1)$. For that p, if $f(y_2)$

contains a witness to $\Psi_{\mathfrak{k}}$, then $f(y_2) \not\subseteq f(y_1)$, and so $f(y_1) \subseteq f(y_2)$, and therefore $f(y_2)$ contains a witness to $(\mathring{\Theta})^L$.] Then $\forall x{:\in} f(\bar{x})\ \exists p{:\in} f(p_{\mathfrak{k}})\ (x,p) \in H$.

The third condition on H is trivially seen to be satisfied by $\mathcal{H}(\bar{x},\bar{a})$.

Conversely, if there is an H as above, then we must show that $\forall x{:\in} f(\bar{x})\ \exists y{:\in} L\ (\mathring{\Theta})^L(x,y,\bar{x},\bar{a})$. So let $x \in f(\bar{x})$: there will be a p with $(x,p) \in H$. Since such p will be in $f(\bar{p})$, there will be a $t \in L$ with $\Psi_{\mathfrak{k}}^L(p,t,p_{\mathfrak{k}},a_{\mathfrak{k}})$. Choose $z \in L$ such that $t \in f(z)$. Then $\exists y{:\in} f(z)\ (\mathring{\Theta})^L(x,y,\bar{x},\bar{a})$. ⊣ (5·29)

Armed with that result, we turn to the proof of $\Sigma_{\mathfrak{k}}$ Separation in L. First a convenient reduction:

5·30 LEMMA *Suppose that whenever a is a transitive member of L containing the parameters $p_{\mathfrak{k}}$ and $a_{\mathfrak{k}}$ and Φ is a $\Sigma_{\mathfrak{k}}$ formula with sole free variable x and all its parameters in a, $a \cap \{x \mid \Phi^L(x)\} \in L$. Then the scheme of $\Sigma_{\mathfrak{k}}$ separation is true in L.*

Proof : let $b \in L$, and Φ a $\Sigma_{\mathfrak{k}}$ formula in the sole free variable x, with parameters in L.

Let $a \in L$ be a transitive set of which b, $p_{\mathfrak{k}}$, $a_{\mathfrak{k}}$, and every parameter in Φ are members. By hypothesis $c =_{\mathrm{df}} a \cap \{x \mid \Phi^L(x)\}$ is a member of L. Then $b \cap \{x \mid \Phi^L(x)\} = b \cap c \in L$. ⊣ (5·30)

So now let $\mathfrak{a}$ be a transitive member of L with both universal parameters in $\mathfrak{a}$, and Φ a $\Sigma_{\mathfrak{k}}$ formula with sole free variable x and all its parameters in $\mathfrak{a}$. Write A^L for the class $\{x \mid x \in L \ \&\ \Phi^L(x)\}$. To show that $\mathfrak{a} \cap A^L$ is in L, we build a term model (K,E). However we must proceed slightly differently. In the previous subsection we assumed both $V = L$ and $\Sigma_{\mathfrak{k}}$ Separation; here we have $\Sigma_{\mathfrak{k}}$ Separation but not $V = L$.

Therefore we work with the formal language which has a constant for every member of $\mathfrak{a}$, and we start from the set of $\dot{\Sigma}_{\mathfrak{k}}$ formulæ which have unique witnesses when interpreted in L. We introduce a variant of $\overset{\mathfrak{k}}{\models}$, $\overset{\mathfrak{k}}{\underset{L}{\models}}$ which is defined for $\dot{\Sigma}_{\mathfrak{k}}$ formulæ with all parameters in L, and interprets all unrestricted quantifiers to range over the members of L.

Thus for $\mathfrak{A}$ a Σ_k formula, and $\mathfrak{B}$ a formula Σ_k formula equivalent in KPI to $(\mathfrak{A})^L$ — possible since membership of L is Σ_1^{KPI} — $\overset{\mathfrak{k}}{\models} \dot{\mathfrak{B}}$ will be equivalent over KPI to $\overset{\mathfrak{k}}{\underset{L}{\models}} \dot{\mathfrak{A}}$ when all parameters are members of L. Hence "$a \in L\ \&\ \overset{\mathfrak{k}}{\underset{L}{\models}} \dot{\mathcal{Q}}_{\mathfrak{k}}\varphi[a]$" is $\Sigma_{\mathfrak{k}}^{\mathsf{KPI}}(\ulcorner\varphi\urcorner, a)$.

We proceed to define our term model. For the moment we do not always indicate a subscript L and a superscript $\mathfrak{k}$ in our notation.

5·31 Let $N_0^{\mathfrak{k}}(\mathfrak{a})$ be as before the set of those φ which are a $\dot{\Sigma}_{\mathfrak{k}}$ formula with one free variable and only permitted constants. $N_0^{\mathfrak{k}}(\mathfrak{a}) \in V$.

Let $N_1(\mathfrak{a}) = N_0^{\mathfrak{k}}(\mathfrak{a}) \cap \big\{\varphi \big| \exists x\big(x \in L \ \&\ \overset{\mathfrak{k}}{\underset{L}{\models}}\ \varphi[x]\big)\big\}$. $N_1(\mathfrak{a}) \in V$ by $\Sigma_{\mathfrak{k}}$ separation.

Let $N_2(\mathfrak{a}) = N_0^{\mathfrak{k}}(\mathfrak{a}) \cap \big\{\varphi \mid \exists x \exists y\big(x \in L \ \&\ y \in L \ \&\ x \neq y \ \&\ \overset{\mathfrak{k}}{\underset{L}{\models}}\ \varphi[x] \ \&\ \overset{\mathfrak{k}}{\underset{L}{\models}}\ \varphi[y]\big)\big\}$. $N_2(\mathfrak{a}) \in V$ by $\Sigma_{\mathfrak{k}}$ separation.

Finally let $N_L^{\mathfrak{k}}(\mathfrak{a}) = N_1(\mathfrak{a}) \setminus N_2(\mathfrak{a})$. $N_L^{\mathfrak{k}}(\mathfrak{a}) \in V$ by the axiom of difference.

We proceed as before to define an equivalence relation $\sim_{\mathfrak{k}}$: it is not quite the same as the one we had before, but it scarcely seems worthwhile to introduce a separate notation for it. We factor by that relation, define a membership relation on the resulting set of equivalence classes, and thus obtain the structure $(K_L^{\mathfrak{k}}(\mathfrak{a}), E_L^{\mathfrak{k}}(\mathfrak{a}))$, and prove that it is a set. We have (as yet) no reason to believe that it is a member of L. For simplicity, we call it (K,E).

A new version of the Witness Lemma may now be proved: recall that by choice of $\mathfrak{a}$, constants for the two universal parameters are among those permitted.

5·32 THE WITNESS LEMMA ($\mathsf{KZ}_{\mathfrak{k}-1}$) *Let Φ be a $\dot{\Sigma}_{\mathfrak{k}}$ formula in one free variable whose parameters are either permitted constants or themselves the unique witnesses in L to some member of $N_L^{\mathfrak{k}}(\mathfrak{a})$ when interpreted in L. If Φ has a witness in L, then it has one which is itself the unique witness in L to some member of $N_L^{\mathfrak{k}}(\mathfrak{a})$ when interpreted in L.*

Proof : restriction of the variables to L does not raise the quantifier level, so the previous proof may be followed, the chief modification being the use of the universal parameters in computing quantifier levels.

⊣ (5·32)

Then we may establish, using our strong version of Sy Friedman's principle, a counterpart to 5·24. Our term model will therefore be $\Sigma_\mathfrak{k}$-elementarily embeddable in L. It follows in particular that (K, E) is a model of KPL.

We must show that (K, E) is well-founded. To do so, we use the map that associates to $\varphi \in N_L^\mathfrak{k}$ the set-theoretic rank $\varrho(x)$ of the unique $x \in L$ with $\models^\mathfrak{k}_L \varphi[x]$.

So let Z be a non-empty subset of $N_L^\mathfrak{k}$ closed under $\sim_\mathfrak{k}$. Consider

$$Y =_{\mathrm{df}} \{\eta \mid \exists\varphi{:}{\in} Z\ \exists y\big(y \in L\ \&\ \models^\mathfrak{k}_L \varphi[y]\ \&\ \varrho(y) = \eta\big)\} :$$

Y is a non-empty $\Sigma_\mathfrak{k}$ class, and so if η_0 is some member of it, $A \cap \eta_0$ will be a set, by $\Sigma_\mathfrak{k}$ Separation. If empty, $\eta_0 = \min A$; otherwise $\min(A \cap \eta_0) = \min A$. So $\min A$ exists, call it $\bar\eta$. Let $\varphi \in Z$, y be such that $\varrho(y) = \bar\eta$ and $\models^\mathfrak{k}_L \varphi[y]$. We assert that

$$\vartheta \in Z \implies \neg\big((\vartheta)_\mathfrak{k} E_\mathfrak{k} (\varphi)_\mathfrak{k}\big) :$$

for let z be the unique L-witness to a counterexample ϑ. Then $z \in y$ and so $\varrho(z) < \varrho(y)$, contradicting the minimality of $\bar\eta$.

We now know that (K, E) is a well-founded extensional relation modelling KPL: it is therefore isomorphic to some $L_{\bar\xi}$ which itself models KP. Call the isomorphism ϖ.

To each member x of $\mathfrak{a}$ there corresponds a simple formula φ_x that defines it, namely $\mathfrak{x} = \underline{x}$, where $\underline{x}$ is the (permitted) constant for x.

Since $\mathfrak{a}$ was transitive, we may check by recursion on $\varrho(x)$ that for $x \in \mathfrak{a}$, $\varpi((\varphi_x)_\mathfrak{k}) = x$. Hence for $x \in \mathfrak{a}$,

$$\begin{aligned}\Phi^L(x) &\iff \models^\mathfrak{k}_L \dot\Phi[x]\\ &\iff K_L^\mathfrak{k} \models \dot\Phi[(\varphi_x)_\mathfrak{k}]\\ &\iff L_{\bar\xi} \models \dot\Phi[x],\end{aligned}$$

and so $\mathfrak{a} \cap \{x \mid \Phi^L(x)\} = \mathfrak{a} \cap \{x \mid L_{\bar\xi} \models \dot\Phi[x]\} \in L$, as required. ⊣ (5·27)

⊣ (Theorem 7)

A modification of the above argument yields:

5·33 THEOREM *Let $\mathfrak{k} \geqslant 2$. Then in $\mathsf{Z}_\mathfrak{k} + \mathsf{H}$ we may show that $\mathsf{KZ}_\mathfrak{k}$ is true in L.*

Proof : Remark 4·20 shows that KP will be true in L, and hence we have the admissibility required for the first phase of the fine-structural lemma. At the end of the proof, where we collapse the term model (K, E) to some $L_{\bar\xi}$, we must define the collapsing isomorphism by recursion into L_λ where λ is a cardinal greater than that of (K, E), using the form of the condensation lemma that states that rudimentary functions are preserved under collapse. ⊣ (5·33)

As KZ_1 proves Axiom H, this last result is a slight sharpening of 5·27.

The proof of Theorem 5 is now complete, by 2·46, 5·27 and 5·33.

⊣ (Theorem 5)

6: Two independence results and a consistency proof

We turn to Harvey Friedman's notion of a *power admissible* set, which is a transitive model of the theory $\mathsf{KP}^{\mathcal{P}}$ to be defined below. Roughly, $\mathsf{KP}^{\mathcal{P}}$ is the theory that results from $\mathsf{KP} + \omega \in V$ when "Δ_0" is replaced by "recursive in the power set operation." The theory $\mathsf{KP}^{\mathcal{P}} + \mathsf{AC}$ extends MAC, and our study of $\mathsf{KP}^{\mathcal{P}}$ will yield sharp forms of independence results concerning MAC.

6·0 REMARK We shall see that $L_{(\aleph_\omega)^L}$ is not power-admissible, although it is admissible and satisfies the power-set axiom.

There will be much model theory in this section; in particular we shall be much involved with non-standard models of various set theories. It will be therefore be convenient to distinguish notationally between a variable (x) in a formula and the interpretation $[p]$ of another, possibly invisible, variable, as in $\mathfrak{M} \models \bigvee x \vartheta(x)[p]$, where ϑ is a formal formula with two formal variables x and y.

To give a precise definition of $\mathsf{KP}^{\mathcal{P}}$ we must consider a new class of formulæ, studied first by Takahashi [K1], whose work was later utilised by Forster and Kaye [B3]. We adopt the notation of that latter pair. Thus we shall not follow Friedman's treatment [K2] but shall establish the equivalence of $\mathsf{KP}^{\mathcal{P}}$ to an appropriate variant of Friedman's system PAdm^s.

The Takahashi hierarchy

6·1 Following Forster and Kaye we call a formula $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $Qx\!:\subseteq y$ or $Qx\!:\in y$ where Q is $\forall$ or $\exists$ and x and y are distinct variables. We preserve "restricted" as a description of the quantifiers $Qx\!:\in y$, and speak of the occurrences of y in $Qx\!:\subseteq y$ or $Qx\!:\in y$ as *limiting* the range of the bound variable x.

It is tempting, indeed, to adopt a different presentation of the language by taking there to be three primitive signs, $\in$, $=$ and $\subseteq$, to declare the class of atomic formulæ to consist of every formula of one of the three forms

$$x \in y \qquad\qquad x = y \qquad\qquad x \subseteq y$$

and to have three kinds of quantifiers, $\forall x$, $\forall x\!:\in y$ and $\forall x\!:\subseteq y$ in the language; but we shall not formally adopt this approach here, though we shall indicate some places where it would simplify our treatment.

6·2 We sketch a method of rewriting a $\Delta_0^{\mathcal{P}}$ formula so that all variables are limited by terms constructed from the free variables of the original formula using only $\bigcup$; thus ultimately the terms limiting variables contain no variables that are themselves bound by other quantifiers.

Unlike $\in$, $\subseteq$ is transitive. Hence the following reduction is available:

$$\exists x\!:\subseteq t\ \forall y\!:\subseteq x\ \mathfrak{A} \iff \exists x\!:\subseteq t\ \forall y\!:\subseteq t\ [y \subseteq x \implies \mathfrak{A}].$$

Note here that on the left hand side the x limiting y in the quantifier $\forall y\!:\subseteq x$ is itself bound by the preceding quantifier $\exists x\!:\subseteq t$, whereas on the right hand side the t that limits both quantifiers is itself free. We may speak of t in the above displayed formula or $\bigcup t$ in the next as a *free term*.

Next we should bear in mind in calculating the terms to be used in limiting quantifiers that if $t \in u$ then $t \subseteq \bigcup u$ and that if $u \subseteq v$ then $\bigcup u \subseteq \bigcup v$, so that, for example, a restricted quantifier $\exists x\!:\in t$ may be rewritten thus:

$$\exists x\!:\in t\ \mathfrak{A} \iff \exists x\!:\subseteq \bigcup t\ [x \in t\ \&\ \mathfrak{A}].$$

We thus obtain these reductions:

$$\forall x\!:\in a\ \exists y\!:\in x\ \mathfrak{A} \iff \forall x\!:\in a\ \exists y\!:\in \bigcup a\ [y \in x\ \&\ \mathfrak{A}];$$

$$\forall x\!:\subseteq a\ \exists y\!:\in x\ \mathfrak{A} \iff \forall x\!:\subseteq a\ \exists y\!:\in a\ [y \in x\ \&\ \mathfrak{A}];$$

$$\begin{aligned}\forall x\!:\in a\ \exists y\!:\subseteq x\ \mathfrak{A} &\iff \forall x\!:\in a\ \exists y\!:\subseteq \bigcup a\ [y \subseteq x\ \&\ \mathfrak{A}]\\ &\iff \forall x\!:\in a\ \exists y\!:\subseteq \bigcup a\ [\forall s_1\!:\in \bigcup a\ (s_1 \in y \implies y_1 \in x)\ \&\ \mathfrak{A}];\end{aligned}$$

$$\begin{aligned}\forall x\!:\subseteq a\ \exists y\!:\subseteq x\ \mathfrak{A} &\iff \forall x\!:\subseteq a\ \exists y\!:\subseteq a\ [y \subseteq x\ \&\ \mathfrak{A}]\\ &\iff \forall x\!:\subseteq a\ \exists y\!:\subseteq a\ [\forall s_2\!:\in a\ (s_2 \in y \implies s_2 \in x)\ \&\ \mathfrak{A}].\end{aligned}$$

Those equivalences, which are all valid in S_0, and, where applicable, preserve the stratifiability of the formula under consideration, show that one may progressively rewrite the formula to one in which all limitations are of the form $:\subseteq \bigcup^{\mathfrak{k}} a$ or $:\in \bigcup^{\mathfrak{k}} a$ with a a free variable. We call such a formula one in *free form*. Our expansion of $y \subseteq x$ in the fourth and sixth lines, which would be unnecessary if we treated $y \subseteq x$ as atomic, helps to secure free form. We call the bound variables s_i introduced in those expansions *subsidiary variables*: we shall suppress mention of them in our discussion below.

Given a formula in free form, we replace each limiting free term by a new variable and add a clause expressing the equality of the term and the variable.

We have reached the

6·3 FIRST LIMITED NORMAL FORM *Let Φ be a $\Delta_0^{\mathcal{P}}$ formula with free variables $a_0, \dots a_{\mathfrak{n}}$. Let $\mathfrak{m}+1$ be the number of quantifiers occurring in Φ. Then for $0 \leqslant j \leqslant \mathfrak{m}$, there are numbers $0 \leqslant \mathfrak{k}(j) \leqslant \mathfrak{n}$, $0 \leqslant \mathfrak{l}(j)$, determined by the quantifier structure of Φ, new variables $y_0, \dots y_{\mathfrak{m}}$, and a $\Delta_0^{\mathcal{P}}$ formula Ψ_1 with free variables $a_0, \dots a_{\mathfrak{n}}$, $y_0, \dots y_{\mathfrak{m}}$, in which every quantifier is limited by one of the parameters y_i, such that, abbreviating $\forall y_0, \dots, \forall y_{\mathfrak{m}}$ by $\forall \vec{y}$, we have*

$$\vdash_{\mathsf{S}_0} \forall \vec{a} \forall \vec{y} \Big[\bigwedge_{0 \leqslant j \leqslant \mathfrak{m}} y_j = \bigcup{}^{\mathfrak{l}(j)} a_{\mathfrak{k}(j)} \implies \big[\Phi(\vec{a}) \iff \Psi_1(\vec{a}, \vec{y}) \big] \Big]$$

To take things to a second stage, if we know that we intend using the formula $\Phi(a)$ in a context where a_i will be constrained to be a member of b_i, we may replace the restriction $:\in \bigcup^{\mathfrak{l}} a_i$ by the restriction $:\in \bigcup^{\mathfrak{l}+1} b_i$; and each limitation $:\subseteq \bigcup^{\mathfrak{l}} a_i$ by the limitation $:\subseteq \bigcup^{\mathfrak{l}+1} b_i$, since if $a \in b$, $\bigcup^{\mathfrak{l}} a \subseteq \bigcup^{\mathfrak{l}+1} b$, and make a corresponding adjustment to the matrix.

We could also consider intended limitations $a_i \subseteq b_i$ instead of restrictions $a_i \in b_i$: the replacements to be made then would be $:\in \bigcup^{\mathfrak{l}} a_i$ by $:\in \bigcup^{\mathfrak{l}} b_i$ and $:\subseteq \bigcup^{\mathfrak{l}} a_i$ by $:\subseteq \bigcup^{\mathfrak{l}} b_i$, since if $a \subseteq b$ then $\bigcup^{\mathfrak{l}} a \subseteq \bigcup^{\mathfrak{l}} b$.

Further, we could mix our intentions, and also leave some a_i untouched, which is tantamount to saying $a_i = b_i$. We thus have the

6·4 SECOND LIMITED NORMAL FORM *Continuing the notation of the First Limited Normal Form, let L, R and U be disjoint sets partitioning $[0, \mathfrak{n}]$, and let $b_0, \dots, b_{\mathfrak{n}}$ be variables not occurring in Φ. Then for the same numbers $\mathfrak{k}(j)$, $\mathfrak{l}(j)$, there is a $\Delta_0^{\mathcal{P}}$ formula Ψ_2 with free variables $a_0, \dots a_{\mathfrak{n}}$, $y_0, \dots y_{\mathfrak{m}}$, in which every quantifier is limited to one of the parameters y_i, such that*

$$\vdash_{\mathsf{S}_0} \forall \vec{b} \forall \vec{a} \forall \vec{y} \Bigg[\Big[\bigwedge_{i \text{ in } R} a_i \in b_i \ \& \bigwedge_{i \text{ in } L} a_i \subseteq b_i \ \& \bigwedge_{i \text{ in } U} a_i = b_i \ \& \bigwedge_{\mathfrak{k}(j) \text{ in } R} y_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{\mathfrak{k}(j)} \ \& \bigwedge_{\substack{\mathfrak{k}(j) \text{ in} \\ L \text{ or } U}} y_j = \bigcup{}^{\mathfrak{l}(j)} b_{\mathfrak{k}(j)} \Big] \implies$$
$$\implies \Big[\Phi(\vec{a}) \iff \Psi_2(\vec{a}, \vec{y}) \Big] \Bigg]$$

6·5 Back to free form: when the formula has been rewritten so that all quantifiers are limited by free terms as above, we may, as a last step, replace each limitation $:\subseteq \bigcup^{\mathfrak{k}} a$ by the restriction $:\in \mathcal{P}(\bigcup^{\mathfrak{k}} a)$, but working now in S_1, which is $\mathsf{S}_0 + \mathsf{Power\ Set}$.

6·6 EXAMPLE Let $\mathfrak{A}$ be quantifier-free, with six variables a, b, x, y, z, w. Suppose we want to re-write the formula $\Phi(a, b) \iff_{\text{df}} \exists x :\in a \ \forall y :\subseteq x \ \exists z :\in x \ \forall w :\subseteq z \ \mathfrak{A}(a, b, x, y, z, w)$.

Let

$$\mathfrak{B}(a, b, x, y, z, w) \iff_{\text{df}} \big(y \subseteq x \implies [z \in x \ \& \ (w \subseteq z \implies \mathfrak{A}(a, b, x, y, z, w))] \big).$$

Notice that $\mathfrak{B}$ is Δ_0, or indeed quantifier-free if we count $s \subseteq t$ as atomic. Then

$$\begin{aligned} &\exists x :\in a \ \forall y :\subseteq x \ \exists z :\in x \ \forall w :\subseteq z \ \mathfrak{A}(a, b) \iff \\ &\quad \iff \exists x :\in a \ \forall y :\subseteq \bigcup a \ \exists z :\in \bigcup a \ \forall w :\subseteq \bigcup \bigcup a \ \big[\mathfrak{B}(a, b, x, y, z, w) \big] \\ &\quad \iff \exists x :\in a \ \forall y :\in \mathcal{P}(\bigcup a) \ \exists z :\in \bigcup a \ \forall w :\in \mathcal{P}(\bigcup \bigcup a) \ \big[\mathfrak{B}(a, b, x, y, z, w) \big] \end{aligned}$$

In order not to use $\mathcal{P}$ applied to a term that is not a variable, we introduce further variables z_j.

6·7 First Restricted Normal Form *Continuing the notation of the First Limited Normal Form, for the same numbers* $\mathfrak{k}(j)$, $\mathfrak{l}(j)$, *there is a partition of* $\{j \mid 0 \leqslant j \leqslant \mathfrak{m}\}$ *into disjoint sets* L_Φ, R_Φ; *there are new variables* y_j, z_j *for* $0 \leqslant j \leqslant \mathfrak{m}$; *and there is a* Δ_0 *formula* Ψ_3, *with free variables the a's and the y's; such that every quantifier in* Ψ_3 *is restricted to one of the parameters* y_i, *and*

$$\vdash_{\mathsf{S}_1} \forall\vec{a}\forall\vec{y}\forall\vec{z}\Big[\Big[\bigwedge_{j \text{ in } R_\Phi}\big(y_j = z_j \,\&\, z_j = \bigcup{}^{\mathfrak{l}(j)} a_{\mathfrak{k}(j)}\big) \,\&\, \bigwedge_{j \text{ in } L_\Phi}\big(y_j = \mathcal{P}(z_j) \,\&\, z_j = \bigcup{}^{\mathfrak{l}(j)} a_{\mathfrak{k}(j)}\big)\Big] \Longrightarrow \Big[\Phi(\vec{a}) \Longleftrightarrow \Psi_3(\vec{a},\vec{y})\Big]\Big]$$

Taking that to the corresponding second stage, and noting that if $a \subseteq b$ then $\mathcal{P}(\bigcup{}^{\mathfrak{l}} a) \subseteq \mathcal{P}(\bigcup{}^{\mathfrak{l}} b)$, whereas if $a \in b$, $\mathcal{P}(\bigcup{}^{\mathfrak{l}} a) \subseteq \mathcal{P}(\bigcup{}^{\mathfrak{l}+1} b)$, we reach the

6·8 Second Restricted Normal Form *Let* Φ *be a* $\Delta_0^{\mathcal{P}}$ *formula with free variables* $a_0, \dots a_{\mathfrak{n}}$. *Let* L, R *and* U *be disjoint sets partitioning* $[0,\mathfrak{n}]$, *and let* $b_0, \dots, b_{\mathfrak{n}}$ *be variables not occurring in* Φ. *Let* $\mathfrak{m}+1$ *be the number of quantifiers occurring in* Φ. *Then there is a partition of* $\{j \mid 0 \leqslant j \leqslant \mathfrak{m}\}$ *into disjoint sets* L_Φ, R_Φ; *for* $0 \leqslant j \leqslant \mathfrak{m}$, *there are numbers* $0 \leqslant \mathfrak{k}(j) \leqslant \mathfrak{n}$, $0 \leqslant \mathfrak{l}(j)$, *determined by the quantifier structure of* Φ, *there are new variables* y_j, z_j *for* $0 \leqslant j \leqslant \mathfrak{m}$; *and there is a* Δ_0 *formula* Ψ_4 *with free variables the a's and the y's, in which every quantifier is restricted to one of the parameters* y_i; *such that,*

$$\begin{aligned}\vdash_{\mathsf{S}_1} \forall\vec{b}\forall\vec{a}\forall\vec{y}\forall\vec{z}\Big[\Big[&\bigwedge_{i \text{ in } R} a_i \in b_i \;\&\; \bigwedge_{i \text{ in } L} a_i \subseteq b_i \;\&\; \bigwedge_{i \text{ in } U} a_i = b_i \;\&\\
&\& \bigwedge_{\substack{j \text{ in } R_\Phi,\\ \mathfrak{k}(j) \text{ in } R}} \big(y_j = z_j \;\&\; z_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{\mathfrak{k}(j)}\big) \;\&\; \bigwedge_{\substack{j \text{ in } R_\Phi,\\ \mathfrak{k}(j) \text{ in } R}} \big(y_j = \mathcal{P}(z_j) \;\&\; z_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{\mathfrak{k}(j)}\big) \;\&\\
&\& \bigwedge_{\substack{j \text{ in } R_\Phi,\\ \mathfrak{k}(j) \text{ in } L \text{ or } U}} \big(y_j = z_j \;\&\; z_j = \bigcup{}^{\mathfrak{l}(j)} b_{\mathfrak{k}(j)}\big) \;\&\; \bigwedge_{\substack{j \text{ in } L_\Phi,\\ \mathfrak{k}(j) \text{ in } L \text{ or } U}} \big(y_j = \mathcal{P}(z_j) \;\&\; z_j = \bigcup{}^{\mathfrak{l}(j)} b_{\mathfrak{k}(j)}\big)\Big] \Longrightarrow\\
&\Longrightarrow \Big[\Phi(\vec{a}) \Longleftrightarrow \Psi_4(\vec{a},\vec{y})\Big]\Big]\end{aligned}$$

From the Second Restricted Normal Form we deduce the

6·9 Theorem scheme *(i)* M_0 *proves all instances of the scheme of* $\Delta_0^{\mathcal{P}}$ *separation.*

(ii) KF *proves all instances of the scheme of stratifiable* $\Delta_0^{\mathcal{P}}$ *separation.*

Proof : given $\Phi(\vec{a})$, form Ψ_4 as above; then to show in the system M_0 that $b_1 \cap \{a_1 \mid \Phi(\vec{y})\} \in V$, note that $\mathsf{M}_0 \vdash \forall\vec{y}\; b_1 \cap \{a_1 \mid \Psi_4(\vec{a},\vec{y})\} \in V$, and that $\mathsf{S}_1 \vdash \exists\vec{y}\exists\vec{z} \bigwedge_j \Big(y_j = \mathcal{P}(z_j) \,\&\, z_j = \bigcup{}^{\mathfrak{l}(j)+1} b_{k(j)}\Big)$. Part (i) follows by predicate logic.

For Part (ii), note that if Φ is stratifiable, so is Ψ_4, and hence $\mathsf{KF} \vdash \forall\vec{y}\; b_1 \cap \{a_1 \mid \Psi_4(\vec{a},\vec{y})\} \in V$. ⊣ (6·9)

6·10 We may continue the Takahashi hierarchy by defining a $\Pi_1^{\mathcal{P}}$ *formula* to be the result of prefixing a single unlimited universal quantifier, a $\Sigma_1^{\mathcal{P}}$ *formula* a single unlimited existential quantifier, to a $\Delta_0^{\mathcal{P}}$ formula.

We are going to study the system $\mathsf{KP}^{\mathcal{P}}$ which we specify as $\mathsf{KP} + \omega \in V + \mathsf{Power\ Set} + \Pi_1^{\mathcal{P}}$ Foundation + $\Delta_0^{\mathcal{P}}$ Collection, of which theory M is a subtheory.

Transitive models of the system $\mathsf{KP}^{\mathcal{P}}$, which in some ways seems as strong as ZF but in others is no stronger than KP, are termed by H. Friedman [K2] *power-admissible*: we shall in due course use his results relating well-founded and ill-founded models of $\mathsf{KP}^{\mathcal{P}}$.

6·11 Remark Apart from his different axiomatisation of the class $\Delta_0^{\mathcal{P}}$, which we shall treat in a moment, Friedman includes only set Foundation in his formulation PAdm^s of $\mathsf{KP}^{\mathcal{P}}$, but remarks that it might be better to include class Foundation. We choose $\Pi_1^{\mathcal{P}}$ Foundation in analogy to our formalisation of KP with Π_1 Foundation.

Takahashi's reductions

We review some comparisons between the hierarchies of Lévy and of Moto-o Takahashi established by the latter, who, however wrote $\tilde{\Delta}_0$, $\tilde{\Sigma}_1$, ... instead of $\Delta_0^{\mathcal{P}}$, $\Sigma_1^{\mathcal{P}}$ ... He worked in ZFC, but suggested in [K1] that it would be interesting to work with only $\Delta_1^{\mathcal{P}}$ replacement. Indeed even less than that suffices for his results:

6·12 PROPOSITION SCHEME (Takahashi) $\Sigma_1 \subseteq (\Delta_1^{\mathcal{P}})^{\mathsf{MOST}}$; $\Delta_0^{\mathcal{P}} \subseteq \Delta_2^{\mathsf{S}_1}$.

Proof : We define a predicate expressing with limited quantifiers the fact that b is a transitive set satisfying the conclusion of H for $u = tcl(\{a\} \cup \omega)$, a transitive infinite set of which a is a member:

$$T(b,a,c,u) \iff_{\rm df} u = tcl(\{a\}\cup\omega) \,\&\, c = u\times u \,\&\, \bigcup b \subseteq b \,\&\, \forall u'\!:\subseteq u\; \forall r\!:\subseteq c\; \Big(\text{if } r \subseteq u'\times u' \text{ and } (a',r) \text{ is well-founded and extensional then } \exists d\!:\subseteq b\; \big(\bigcup d \subseteq d \;\&\; \exists f\!:\subseteq b\times a\;\, f : (u',r) \cong (d, \in \restriction d)\big)\Big)$$

Note, with Proposition 1·29 in mind, that the predicate T is $\Delta_0^{\mathcal{P}}$, and that if $T(b,a,c,u)$ then $a \in b$; so if $\Phi(x,a)$ is Δ_0, then the following equivalences, establishable using $\omega \in V$, H and AC, much as in the proof of 3·11,

$$\begin{aligned}\exists x\Phi(x,a) &\iff \forall b\forall c\forall u\;\big(T(b,a,c) \implies \exists x\!:\in b\; (\Phi)^b\big)\\ &\iff \exists b\exists c\exists u\;\big(T(b,a,c) \;\&\; \exists x\!:\in b\; (\Phi)^b\big)\end{aligned}$$

express $\exists x\Phi$ in $\Delta_1^{\mathcal{P}}$ form.

On the other hand, if $\Psi(\vec{a})$ is $\Delta_0^{\mathcal{P}}$ we may by expressing it in First Restricted Normal Form, find a Δ_0 formula $\Psi_3(\vec{a},\vec{y})$ and a conjunction $\Theta(\vec{a},\vec{y},\vec{z})$ of finitely many equations, each of one of the forms $y = \mathcal{P}(z)$, $y = z$ or $z = \bigcup{}^{\iota} a$, so that, provably in S_1,

$$\begin{aligned}\Psi &\iff \exists\vec{y}\exists\vec{z}\big(\Theta(\vec{a},\vec{y},\vec{z}) \;\&\; \Psi_3(\vec{a},\vec{y})\big)\\ &\iff \forall a\forall c\big(\Theta(\vec{a},\vec{y},\vec{z}) \implies \Psi_3(\vec{a},\vec{y})\big)\end{aligned}$$

so that Ψ is $\Delta_2^{\mathsf{S}_1}$. ⊣ (6·12)

6·13 REMARK The predicate T, involving the concept of a transitive set, is not apparently stratifiable, and hence we cannot refine the first inclusion above to showing that stratifiable Σ_1 formulæ are equivalent to stratifiable formulæ in $(\Sigma_1^{\mathcal{P}})^{\mathsf{MOST}}$ and in $(\Pi_1^{\mathcal{P}})^{\mathsf{MOST}}$.

However, if Ψ is stratifiable $\Delta_0^{\mathcal{P}}$, then the Ψ_3 obtained above is stratifiable Δ_0, and thus, easily, Ψ is stratifiable Δ_2^{KF}, giving

$$\text{strat-}\Delta_0^{\mathcal{P}} \subseteq \text{strat-}\Delta_2^{\mathsf{KF}}.$$

We shall see during this section that $\mathsf{KP}^{\mathcal{P}} + \mathsf{AC}$ cannot prove Σ_1 separation whereas $\mathsf{KP}^{\mathcal{P}}$ does prove $(\Delta_1^{\mathcal{P}})^{\mathsf{KP}^{\mathcal{P}}}$ separation, so that there are Σ_1 wffs not in $(\Delta_1^{\mathcal{P}})^{\mathsf{KP}^{\mathcal{P}}+\mathsf{AC}}$, still less in $(\Delta_1^{\mathcal{P}})^{\mathsf{MAC}}$: thus the use of Axiom H in the proof of 6·12 is indispensable. We pause to note a consequence for various principles of collection.

6·14 PROPOSITION *Let Γ be any of the following classes of formulæ: Π_1; $\Delta_0^{\mathcal{P}}$; strat-Π_1; strat-$\Delta_0^{\mathcal{P}}$. Then it is provable in* KF *that*

(i) Γ Collection implies $\exists\Gamma$ Collection;

(ii) strong Γ Collection implies strong $\exists\Gamma$ Collection.

Thus $\Delta_0^{\mathcal{P}}$ Collection implies $\Sigma_1^{\mathcal{P}}$ Collection; and similar results with the words "strong" or "stratifiable" or both added to each hypothesis and each conclusion.

Proof : similar to that of 3·12, though we must take care if we are to preserve stratifiability. The general form of the argument without concern for stratification is this. Let $\mathfrak{A}$ be a formula in Γ.

$$\forall x\exists y\exists z\mathfrak{A}(x,y,z) \implies \forall x\exists w[w \text{ is an ordered pair and } \mathfrak{A}(x,(w)_0,(w)_1)]$$
$$\implies \forall u\exists v\forall x\,{:}{\in}\, u\ \exists w\,{:}{\in}\, v\ [w \text{ is an ordered pair and } \mathfrak{A}(x,(w)_0,(w)_1)].$$

In a system which includes TCo, we may replace v by some transitive set v' including it, and infer that

$$\forall u\exists v'\forall x\,{:}{\in}\, u\ \exists y\,{:}{\in}\, v'\ \exists z\,{:}{\in}\, v'\ \mathfrak{A}(x,y,z)$$

as required. Without TCo, we remark that if v contains the ordered pair w then $(w)_0$ and $(w)_1$ are both members of $v'' = \bigcup\bigcup v$, and apply the Axiom of Sumset twice to infer that

$$\forall u\exists v''\forall x\,{:}{\in}\, u\ \exists y\,{:}{\in}\, v''\ \exists z\,{:}{\in}\, v''\ \mathfrak{A}(x,y,z).$$

For strong Collection, we wish to prove that

$$\forall u\exists v\forall x\,{:}{\in}\, u\ (\exists y\exists z\mathfrak{A} \implies \exists y\,{:}{\in}\, v\ \exists z\mathfrak{A})$$

so we begin by a similar rephrasing:

$$\forall u\exists v\forall x\,{:}{\in}\, u\ \big(\exists w[w \text{ is an ordered pair and } \mathfrak{A}(x,(w)_0,(w)_1)] \implies$$
$$\implies \exists w\,{:}{\in}\, v\ [w \text{ is an ordered pair and } \mathfrak{A}(x,(w)_0,(w)_1)]\big)$$

and reason that since the formula "w is an ordered pair " is in Δ_0, "w is an ordered pair and $\mathfrak{A}$" is in Γ.

To stay within a class Γ of stratifiable formulæ we must be more devious. Suppose, for example, that we have stratified $\forall x\exists y\exists z\mathfrak{A}$ giving y type 3 and z type 5; we cannot take w to be $\langle x,\, y\rangle$ and preserve the stratification. Instead we raise the lower type, in our example that of y, to that of the higher type, that of z, by applying the singleton function the appropriate number of times. In our example, we would have this equivalence, for any x,

$$\exists y_3\exists z_5\mathfrak{A}(x,y_3,z_5) \iff \exists w_7\big[w_7 \text{ is an ordered pair } \&$$
$$\&\ \exists y_3\,{:}{\in}\bigcup{}^3 w_7\ (w_7)_0 = \{\{y_3\}\}\ \&\ \exists z_5\,{:}{\in}\bigcup w_7\ (w_7)_1 = z_5\ \&\ \mathfrak{A}(x,y_3,z_5)\big]$$

and we have successfully absorbed the extra existential quantifier whilst preserving the stratification and staying within Γ. Applying Γ Collection, we have

$$\forall u\exists v_8\forall x\,{:}{\in}\, u\ \exists w_7\,{:}{\in}\, v_8\ \big[w_7 \text{ is an ordered pair } \&$$
$$\&\ \exists y_3\,{:}{\in}\bigcup{}^3 w_7\ (w_7)_0 = \{\{y_3\}\}\ \&\ \exists z_5\,{:}{\in}\bigcup w_7\ (w_7)_1 = z_5\ \&\ \mathfrak{A}(x,y_3,z_5)\big]$$

whence

$$\forall u\exists v_8\forall x\,{:}{\in}\, u\ \exists y_3\,{:}{\in}\bigcup{}^4 v_8\ \exists z_5\,{:}{\in}\bigcup{}^2 v_8\ \mathfrak{A}(x,y_3,z_5)\big]$$

which implies $\forall u\exists t\forall x\,{:}{\in}\, u\ \exists y\,{:}{\in}\, t\ \exists z\ \mathfrak{A}(x,y,z)$, yielding $\exists\Gamma$ Collection as desired.

Essentially the same device will work for the case of strong stratifiable Collection. ⊣ (6·14)

6·15 COROLLARY *Over* KF*, the scheme of* Π_1 *Collection implies that of* $\Delta_0^{\mathcal{P}}$ *Collection; over* MOST*, the two schemes are equivalent; over* KF*, the scheme of stratifiable* Π_1 *Collection implies that of stratifiable* $\Delta_0^{\mathcal{P}}$ *Collection. Similar results hold for strong Collection.*

Proof : by 6·12 and 6·14. ⊣ (6·15)

6·16 REMARK Note that it is not being alleged that the classes of formulæ that are respectively $(\Delta_0^{\mathcal{P}})^{\mathsf{MOST}}$ and $(\Pi_1)^{\mathsf{MOST}}$ are identical: a plainly suspect allegation, since the first class is closed under negation and the second is not. For a specific counter-example, note that over MOST, the statement that every real is constructible is, with the help of recursive pairing functions, a $\Delta_0^{\mathcal{P}}$ statement about ω; but not Σ_1 since it is true in L but liable to be false in extensions of L; so "there is a non-constructible real" is $\Delta_0^{\mathcal{P}}$ but not Π_1.

In the other direction, the allegation is close to being true: MOST proves that if a is transitive, infinite and closed under pairing, and $\Phi(x, a)$ is Δ_0, then

$$\exists x\Phi(a,x) \iff \exists u \bigcup u \subseteq u \;\&\; a \in u \;\&\; \overline{\overline{u}} = \overline{\overline{a}} \;\&\; u \models \bigvee x\dot{\Phi}(x)[a]$$
$$\iff \exists R{:}{\subseteq}\, a \times a\ (a, R) \text{ is a well-founded extensional relation } \&$$
$$\&\ \exists k{:}{\in}\, a\ k \text{ represents } a \;\&\; (a, R) \models \bigvee x\dot{\Phi}(x)[k].$$

Here "k represents a" means that $(a, R) \models [k]$ `is transitive` and there is an isomorphism between $(k, R \restriction k)$ and $(a, \in\restriction a)$. When a is closed under pairing, we may replace $:\subseteq a \ \times a$ by $:\subseteq a$, and we have therefore reached a $\Delta_0^{\mathcal{P}}$ formulation of $\exists x\Phi(x, a)$. These remarks lead to the following

6·17 PROPOSITION *If* $\mathbf{M} \models$ MOST *and* $\mathbf{N} \models$ MAC *and* $\mathbf{M} \subseteq_e^{\mathcal{P}} \mathbf{N}$ *then* $\mathbf{M} \preceq_{\Sigma_1} \mathbf{N}$.

The relation $\mathbf{M} \subseteq_e^{\mathcal{P}} \mathbf{N}$ is that studied in the paper [B3] of Forster and Kaye on end-extensions preserving power set. It implies the relation $\mathbf{M} \preceq_{\Delta_0^{\mathcal{P}}} \mathbf{N}$ of being an elementary submodel with respect to $\Delta_0^{\mathcal{P}}$ formulæ.

Proof : Let $a \in \mathbf{M}$. Reasoning in $\mathbf{M}$, find an infinite transitive u containing a and closed under pairing functions. Let Φ be Σ_1 and suppose that $\mathbf{N} \models \bigvee x\Phi(x)[a]$. Reasoning in $\mathbf{N}$ and using AC there, take Skolem hulls inside an appropriate transitive set containing u, therefore a, and a witness x, and deduce that there is a v, not necessarily transitive, with $\overline{\overline{u}} = \overline{\overline{v}}$, $u \in v$, $u \subseteq v$, $x \in v$, with $v \models \Phi[x, a]$.

Thus (still reasoning in $\mathbf{N}$) there is a well-founded extensional relation $R \subseteq u \times u$, there is a $k \in u$ representing the set u in (u, R), there is an $\ell \in u$ representing the set a in (u, R) — which means as in §2 that there is a function $f \subseteq u \times u$ giving an isomorphism between $(a, u, \in)$ and (ℓ, k, R), such that $(u, R) \models \bigvee x\, \Phi(x)[\ell]$. But that is a $\Delta_0^{\mathcal{P}}$ statement about u and ℓ — it is here that we need all members of u which are in $\mathbf{N}$ to be in $\mathbf{M}$ — and therefore pulls down to being true in $\mathbf{M}$. Using the truth of Axiom H in $\mathbf{M}$ we can turn the relation (u, R) back into a transitive set and recover a witness to Φ. ⊣ (6·17)

6·18 REMARK The Proposition becomes false if we weaken the hypothesis $\mathbf{M} \models$ MOST to $\mathbf{M} \models$ MAC: $\mathbf{N}$ might see a non-recursive von Neumann ordinal invisible to $\mathbf{M}$.

Equivalence of $\mathbf{KP}^{\mathcal{P}}$ and $\mathbf{KP}^s(\mathcal{P})$

6·19 HISTORICAL NOTE Friedman in [K2] attacked the problem mentioned but not addressed by Takahashi in [K1], that of finding a formal system appropriate to the concept of recursion in the power set operation $\mathcal{P}$. Working later than but independently of Takahashi, he defined a possibly smaller class of formulæ than $\Delta_0^{\mathcal{P}}$, and proposed a system PAdm^s, defined below, besides treating other classes and systems with which we are not concerned.

As we wish to apply the model-existence results of [K2], we shall review his treatment and show that the transitive well-founded models of PAdm^s coïncide with those of our sytem $\mathsf{KP}^{\mathcal{P}}$.

We mention that in the syntax of Friedman's presentation, certain symbols, which he calls parameters, are used only as free variables, and others, which he calls variables, are used only as bound variables. If it is wished to bind the occurrences of a parameter by a quantifier or class-forming operator, the parameter in all its occurrences must first be changed to a variable.

Friedman introduces a class of formulæ that he calls pseudo-$\Delta_0^s(\mathcal{P})$ wffs, which are those built up from $\mathcal{P}$-terms which are terms of the form $\mathcal{P}^{\mathsf{n}}(a)$: atomic $\mathcal{P}$-wffs are $s = t$ and $s \in t$ where s and t are $\mathcal{P}$-terms, and then other formulæ are built from atomic ones using propositional connectives and restricted quantifiers $Qx{:}{\in}\, a$.

He then defines the class of $\Delta_0^s(\mathcal{P})$ wffs as the result of eliminating all the $\mathcal{P}$-terms from the pseudo-$\Delta_0^s(\mathcal{P})$ wffs; for that these reductions, all provable in S_1, suffice:

$$\begin{aligned}
\mathcal{P}(x) \subseteq \mathcal{P}(y) &\iff x \subseteq y \\
x \subseteq \mathcal{P}(y) &\iff \forall z{:}{\in} x\ z \subseteq y \\
x \subseteq \mathcal{P}^{\mathfrak{k}+1}(y) &\iff \forall z{:}{\in} x\ z \subseteq \mathcal{P}^{\mathfrak{k}} y \\
\mathcal{P}(x) = \mathcal{P}(y) &\iff x = y \\
x = \mathcal{P}(z) &\iff \forall y{:}{\in} x\ y \subseteq z \ \&\ \forall y{:}{\subseteq} z\ y \in x \\
x = \mathcal{P}^{\mathfrak{k}+1}(y) &\iff \exists z{:}{\in} x\ [x = \mathcal{P}(z)\ \&\ z = \mathcal{P}^{\mathfrak{k}}(y)] \\
\mathcal{P}^{\mathfrak{k}}(x) \subseteq y &\iff \exists z{:}{\subseteq} y\ z = \mathcal{P}^{\mathfrak{k}}(x) \\
x \in \mathcal{P}(y) &\iff x \subseteq y \\
x \in \mathcal{P}^{\mathfrak{k}+1}(y) &\iff x \subseteq \mathcal{P}^{\mathfrak{k}}(y) \\
\mathcal{P}^{\mathfrak{k}}(x) \in y &\iff \exists w{:}{\in} y\ w = \mathcal{P}^{\mathfrak{k}}(x) \\
\mathcal{P}^{\mathfrak{k}}(x) \in \mathcal{P}^{\mathfrak{l}+1}(y) &\iff \mathcal{P}^{\mathfrak{k}}(x) \subseteq \mathcal{P}^{\mathfrak{l}}(y)
\end{aligned}$$

Those show that Friedman's atomic formulæ can be written in $\Delta_0^{\mathcal{P}}$ form; and therefore so can his pseudo-$\Delta_0^s(\mathcal{P})$ formulæ and $\Delta_0^s(\mathcal{P})$ formulæ.

As for a possible converse to the above discussion, the closest we come is to observe that Takahashi's proof that every $\Delta_0^{\mathcal{P}}$ formula is $\Delta_2^{\mathsf{S}_1}$ shows too that every such formula is also $(\Delta_1^s(\mathcal{P}))^{\mathsf{S}_1}$. Hence every Forster–Kaye $\Pi_1^{\mathcal{P}}$ formula is also $\Pi_1^s(\mathcal{P})$: here is an illustrative equivalence. Let $\Phi(a,w)$ be $\Delta_0^{\mathcal{P}}$: then using the First Restricted Normal Form,

$$\forall w \Phi(a,w) \iff \forall w \forall y_1 \forall d_1 \forall d_2 \Big[\big[y_1 = \bigcup{}^{l}(a)\ \&\ z_2 = \bigcup{}^{k}(a)\ \&\ y_2 = \mathcal{P}(z_2)\big] \implies \Psi_3\Big],$$

which once we amalgamate like quantifiers, will evidently be in $\Pi_1^s(\mathcal{P})$ form. The unlimited quantifiers have absorbed the extra variables introduced in Restricted Normal Form. Thus over S_1 the formula classes $\Pi_1^{\mathcal{P}}$ and $\Pi_1^s(\mathcal{P})$ are equivalent, as are the classes $\Sigma_1^{\mathcal{P}}$ and $\Sigma_1^s(\mathcal{P})$.

It is not clear that every $\Delta_0^{\mathcal{P}}$ formula is $\Delta_0^s(\mathcal{P})$; but nevertheless if we denote by $\mathsf{KP}^s(\mathcal{P})$ the system $\mathsf{KP} + \Delta_0^s(\mathcal{P})$ Separation $+\ \omega \in V +$ Power Set $+\ \Pi_1^s(\mathcal{P})$ Foundation $+\ \Delta_0^s(\mathcal{P})$ Collection, we have the following

6·20 METATHEOREM *$\mathsf{KP}^{\mathcal{P}}$ and $\mathsf{KP}^s(\mathcal{P})$ are the same system.*

Friedman's system PAdm^s is $\mathsf{KP}^s(\mathcal{P})$ without the axiom of infinity and with only set Foundation.

Proof : Over S_1, $\Delta_0^s(\mathcal{P})$ Separation is included in $\Delta_0^{\mathcal{P}}$ Separation; that, over M_1, follows from Δ_0 Separation; that is included in $\Delta_0^s(\mathcal{P})$; so over M_1 the three schemata are equivalent.

We have just seen why the schemata of $\Pi_1^{\mathcal{P}}$ Foundation and $\Pi_1^s(\mathcal{P})$ Foundation are equivalent. We have only to prove the equivalence of the two versions of Δ_0 Collection. Every $\Delta_0^s(\mathcal{P})$ formula is $(\Delta_0^{\mathcal{P}})^{\mathsf{S}_1}$. In the other direction, we know in analogy to previous arguments that $\Delta_0^s(\mathcal{P})$ Collection will yield $\Sigma_1^s(\mathcal{P})$ Collection; but every $\Delta_0^{\mathcal{P}}$ formula is $\Sigma_1^s(\mathcal{P})$. Here is some further detail:

6·21 LEMMA *(i) $\{\bigcup^{\ell} a \mid a \in A\}$ is a stratifiable Δ_0 subclass of $\mathcal{P}(\bigcup^{\ell+1} A)$;*

(ii) $\{\mathcal{P}(x) \mid x \in X\}$ is a stratifiable Δ_0 subclass of $\mathcal{P}\mathcal{P}\bigcup X$.

Proof of (ii): The class in question equals

$$\mathcal{P}\mathcal{P}\bigcup X \cap \{y \mid \exists x{:}{\in} X\ \big[[\forall z{:}{\in}\mathcal{P}\bigcup X\ (z \subseteq x \implies z \in y)]\ \&\ [\forall z{:}{\in} y\ z \subseteq x]\big]\}. \qquad \dashv (6{\cdot}21)$$

We wish to show that every instance of $\Delta_0^{\mathcal{P}}$ Collection is derivable from an instance of pseudo-$\Delta_0^s(\mathcal{P})$ Collection.

Let $\Phi(a_1,a_2)$ be $\Delta_0^{\mathcal{P}}$. We assume that $\forall a_1 \exists a_2 \Phi$. We use the First Restricted Normal Form to express that as:

$$\forall a_1 \forall y_1 \forall z_1 \exists a_2 \exists y_2 \exists z_2 \Big[\underbrace{y_1 = \mathcal{P}(z_1)\ \&\ z_1 = \bigcup{}^{\mathfrak{l}_1} a_1 \implies [y_2 = \mathcal{P}(z_2)\ \&\ z_2 = \bigcup{}^{\mathfrak{l}_2} a_2\ \&\ \Phi_3(a_1,a_2)]}_{\Delta_0^s(\mathcal{P})}\Big]$$

We wish to deduce that $\forall A_1 \exists A_2 \forall a_1 :\in A_1\ \exists a_2 :\in A_2\ \Phi$. So for given A_1, form the class

$$C_1 = A_1 \cup \{\bigcup^{\ulcorner_1} a_1 \mid a_1 \in A_1\} \cup \{\mathcal{P}(\bigcup^{\ulcorner_1} a_1) \mid a_1 \in A_1\}$$

which by the Lemma will be a set.

Applying $\Delta_0^s(\mathcal{P})$ Collection, we find that

$$\exists C_2 \forall a_1, y_1, z_1 :\in C_1\ \exists a_2, y_2, z_2 :\in C_2 \\ \Big[y_1 = \mathcal{P}(z_1)\ \&\ z_1 = \bigcup^{\ulcorner_1} a_1 \implies \big[y_2 = \mathcal{P}(z_2)\ \&\ z_2 = \bigcup^{\ulcorner_2} a_2\ \&\ \Phi_3(a_1, a_2)\big]\Big]$$

whence elementary set theory yields the desired conclusion that $\forall a_1 :\in A_1\ \exists a_2 :\in C_2\ \Phi(a_1, a_2)$.

6·22 REMARK The same method will derive strong $\Delta_0^{\mathcal{P}}$ Collection from strong $\Delta_0^s(\mathcal{P})$ Collection.

Henceforth we shall work solely with the version $\mathsf{KP}^{\mathcal{P}}$.

Development of $\mathbf{KP}^{\mathcal{P}}$ and KPR

$\Delta_0^{\mathcal{P}}$ Separation, as we have seen, is provable in M_0 and therefore in $\mathsf{KP}^{\mathcal{P}}$. We therefore seek to develop $\mathsf{KP}^{\mathcal{P}}$ in analogy to the development of KP: our treatment broadly parallels that of Takahashi in [K1] of a calculus of $\Delta_1^{\mathcal{P}}$ functions.

Our next aim is the $\Sigma_1^{\mathcal{P}}$ recursion theorem.

6·23 LEMMA $\mathsf{KP}^{\mathcal{P}}$ *proves* $\Sigma_1^{\mathcal{P}}$ *Collection.*

Proof : as in the KP case. If $\forall a \exists y \exists z \Phi(a, y, z)$ where Φ is $\Delta_0^{\mathcal{P}}$, collect pairs (y, z), so that

$$\forall u \exists v \forall a :\in u\ \exists w :\in v\ [w \text{ is an ordered pair and } \Phi(a, (w)_0, (w)_1)]. \qquad \dashv (6{\cdot}23)$$

6·24 LEMMA $\mathsf{KP}^{\mathcal{P}}$ *proves* $\Delta_1^{\mathcal{P}}$ *Separation.*

Proof : if $\forall x :\in a$ either $\exists y \Phi$ or $\exists z \Psi$ but not both, where Φ and Ψ are $\Delta_0^{\mathcal{P}}$, apply $\Delta_0^{\mathcal{P}}$ Collection to find a b such that $\forall x :\in a$ either $\exists y :\in b\ \Phi$ or $\exists z :\in b\ \Psi$. Then

$$a \cap \{x \mid \exists y \Phi\} = a \cap \{x \mid \exists y :\in b\ \Phi\},$$

and the right hand side is a set by $\Delta_0^{\mathcal{P}}$ Separation. $\dashv$ (6·24)

6·25 LEMMA *If* G *is total and* $\Sigma_1^{\mathcal{P}}$ *then* $x = G(y)$ *is* $\Delta_1^{\mathcal{P}}$

Proof : $x = G(y) \iff (x, y) \in G$; $x \neq G(y) \iff \exists z\ z \neq x\ \&\ (z, y) \in G$. $\dashv$ (6·25)

6·26 THE $\Sigma_1^{\mathcal{P}}$ RECURSION THEOREM. *Let* G *be a* $\Sigma_1^{\mathcal{P}}$ *class. Then there is a* $\Sigma_1^{\mathcal{P}}$ *class* F *such that*

$$\mathsf{KP}^{\mathcal{P}} \vdash \mathrm{Fn}(G)\ \&\ \mathrm{Dom}\,(G) = V \implies \mathrm{Fn}(F)\ \&\ \mathrm{Dom}\,(F) = V\ \&\ \forall x\ F(x) = G(F \restriction x).$$

Proof : We define an *attempt* to be a set f such that

$$\mathrm{Fn}(f)\ \&\ \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f)\ \&\ \forall x :\in \mathrm{Dom}\,(f)\ f(x) = G(f \restriction x).$$

By the lemma, that is in this context a $\Delta_1^{\mathcal{P}}$ formula. We take F to be the union of all attempts. Then F is $\Sigma_1^{\mathcal{P}}$. We have to check that it is a function with domain V.

1) if f and g are attempts, we show $\forall x :\in \mathrm{Dom}\,(f) \cap \mathrm{Dom}\,(g)\ f(x) = g(x)$.

Deny, consider $\{x \mid x \in \mathrm{Dom}\,(f) \cap \mathrm{Dom}\,(g)\ \&\ f(x) \neq g(x)\}$. That is Δ_0^{KP}, and so is a set which if not empty will have a least element, $\bar{x}$ say. Then $f(\bar{x}) = G(f \restriction \bar{x}) = G(g \restriction \bar{x}) = g(\bar{x})$, a contradiction.

Thus F is a function.

2) $\mathrm{Dom}\,(F) = V$.

Certainly $\mathrm{Dom}\,(F) = \bigcup\{\mathrm{Dom}\,(f) \mid f \text{ is in attempt}\}$. That is a $\Sigma_1^{\mathcal{P}}$ class. By $\Pi_1^{\mathcal{P}}$ Foundation its complement, if non-empty has a minimal element, $\bar{x}$ say. By minimality, $\bar{x} \subseteq \mathrm{Dom}\,(F)$. Hence

$$\forall a \mathpunct{:}\in \bar{x}\ \exists f\ \bigl(f \text{ is an attempt and } a \in \mathrm{Dom}\,(f)\bigr).$$

Apply $\Sigma_1^{\mathcal{P}}$ Collection to find a y such that

$$\forall a \mathpunct{:}\in \bar{x}\ \exists f \mathpunct{:}\in y\ \ \bigl(f \text{ is an attempt and } a \in \mathrm{Dom}\,(f)\bigr).$$

Using $\Delta_1^{\mathcal{P}}$ Separation, form $y \cap \{f \mid f \text{ is an attempt }\}$. Its union is an attempt, g say, with domain a transitive set $\supseteq \bar{x}$, and so can be extended to an attempt h with $\bar{x} \in \mathrm{Dom}\,(h)$ by setting $h = g \cup \{(G(g \restriction \bar{x}), \bar{x})\}$ — a contradiction.

3) $\forall x\ F(x) = G(F \restriction x)$.

For any x $F(x) = f(x)$ where f is any attempt with x in its domain. Then $f(x) = G(f \restriction x)$, but $f \restriction x = F \restriction x$, for the domain of f is transitive and so $x \subseteq \mathrm{Dom}\,(f)$. ⊣ (6·26)

6·27 Remark We have treated the easy case of $\in$-recursion: definitions by recursion on other well-founded relations are formally possible but the resulting functions might not be provably total.

In a set theory such as KP in which the set-theoretic rank ϱ may be defined, the following hypothesis makes sense.

$$\forall\zeta\exists v\ v = \{x \mid \varrho(x) < \zeta\}.$$

We shall refer to that hypothesis by the English phrase "ranks are sets". When the hypothesis holds, we may write V_ζ for the set $\{x \mid \varrho(x) < \zeta\}$. The following equations will then hold:

$$V_0 = \varnothing; \quad V_{\zeta+1} = \mathcal{P}(V_\zeta); \quad V_\lambda = \bigcup_{\nu<\lambda} V_\nu \quad \text{for limit } \lambda.$$

Those equations show that, by the $\Sigma_1^{\mathcal{P}}$ recursion theorem,

6·28 Proposition ($\mathsf{KP}^{\mathcal{P}}$) *ranks are sets.*

6·29 Remark KLZ_1 does not prove that ranks are sets: a natural model for KLZ_1 is, assuming $V = L$, $L_{\aleph_\omega}$. Then $V_{\omega+\omega}$ is a subclass but not a set of that model.

Let KPR be the theory $\mathsf{KP} + \omega \in V$ + RAS.

6·30 Lemma (KPR) *The Power Set axiom.*

Proof : let a be a set, η its rank. Then each $b \subseteq a$ has $\varrho(b) \leqslant \eta$. So $\mathcal{P}(a)$ is a Δ_0 subclass of $V_{\eta+1}$, which is a set. ⊣ (6·30)

6·31 Lemma (KPR) *Z is consistent, and indeed has a transitive model.*

Proof : $\omega + \omega$ exists, and therefore $V_{\omega+\omega}$ is a set. It models Z, for if a and p are in $V_{\omega+\omega}$, the class $\{x \mid V_{\omega+\omega} \models \varphi[x, p]\}$ is Δ_1^{KP} in the set $V_{\omega+\omega}$ as parameter. Hence $a \cap \{x \mid V_{\omega+\omega} \models \varphi[x, p]\}$ is a set; it is of rank less than $\omega + \omega$, and therefore is in $V_{\omega+\omega}$. ⊣ (6·31)

The following simplified restricted normal form becomes available once we assume that ranks are sets. To return to our previous example with $\mathfrak{A}$ and $\mathfrak{B}$: let η be an ordinal. Then for all a and b in V_η,

$$\exists x \mathpunct{:}\in a\ \forall y \mathpunct{:}\subseteq x\ \exists z \mathpunct{:}\in x\ \forall w \mathpunct{:}\subseteq z\ \mathfrak{A}(a, b) \Longleftrightarrow \exists x \mathpunct{:}\in V_\eta\ \forall y \mathpunct{:}\in V_\eta\ \exists z \mathpunct{:}\in V_\eta\ \forall w \mathpunct{:}\in V_\eta\ \mathfrak{B}(a, b, x, y, z, w)$$

and if V_η is a set the right-hand side is a Δ_0 formula, using the set (not the term) V_η as a parameter. That equivalence uses only the simple facts that $u \subseteq v \implies \varrho(u) \leqslant \varrho(v)$ and $u \in v \implies \varrho(u) < \varrho(v)$. Further, the left-hand side is independent of η, subject only to the condition that V_η contains all the parameters of the formula, in this case a and b.

6·32 LEMMA *Let* $\Phi(\vec{x})$ *be* $\Delta_0^{\mathcal{P}}$, *with free variables in the list* $\vec{x}$. *Let* w *be a new variable. Then there is a* Δ_0 *formula* $\Theta(\vec{x}, w)$ *in which all quantifiers are restricted by* w *such that*

$$\mathsf{KPR} \vdash \forall\eta\forall\vec{x}\Big[\big[\eta \in ON \;\&\; \bigwedge_i x_i \in V_\eta\big] \implies \big(\Phi(\vec{x}) \iff \Theta(\vec{x})[V_\eta]\big)\Big].$$

6·33 LEMMA $(\mathsf{KP}^{\mathcal{P}} + \forall\kappa\exists\kappa^+)$ $\forall\eta\exists\omega_\eta$.

Proof: We must apply the $\Sigma_1^{\mathcal{P}}$ recursion theorem to a suitable G. Start from the observation that

$$\beta = \alpha^+ \iff \neg\exists f :\subseteq \alpha\times\beta\; f : \beta \xrightarrow{1-1} \alpha \;\&\; \forall\gamma :< \beta\; \exists f :\subseteq \alpha\times\nu\; f : \nu \xrightarrow{1-1} \alpha.$$

In that formula the quantifiers are limited not by a variable but by a term of the form $b\times c$. Now apply the idea underlying Remark 6·16: in KPI it is easy to prove that each set is a member of a transitive set a such that $\forall x, y :\in a\; \{x, y\} \in a$, which is thus closed under pairing; so for ordinals α and β, and writing $\Phi(a, \alpha, \beta)$ for the Δ_0^{KP} formula $a \times a \subseteq a \;\&\; \bigcup a \subseteq a \;\&\; \beta \in a \;\&\; \alpha \in a$, we

$$\begin{aligned}\beta = \alpha^+ &\iff \exists a\big(\Phi(a,\alpha,\beta) \;\&\; \neg\exists f :\subseteq a\; f : \beta \xrightarrow{1-1} \alpha \;\&\; \forall\gamma :< \beta\; \exists f :\subseteq \alpha\times\nu\; f : \nu \xrightarrow{1-1} \alpha\big)\\ &\iff \forall a\big(\Phi(a,\alpha,\beta) \implies \neg\exists f :\subseteq a\; f : \beta \xrightarrow{1-1} \alpha \;\&\; \forall\gamma :< \beta\; \exists f :\subseteq \alpha\times\nu\; f : \nu \xrightarrow{1-1} \alpha\big)\end{aligned}$$

which is $\Delta_1^{\mathcal{P}}$ in our system. An appropriate $\Sigma_1^{\mathcal{P}}$ G may now be constructed. ⊣ (6·33)

6·34 PROPOSITION $(\mathsf{KP}^{\mathcal{P}} + \forall\kappa\exists\kappa^+)$ $\forall\zeta\,\exists\nu\,(\zeta < \nu = \aleph_\nu)$.

Proof: We have just found a $\Sigma_1^{\mathcal{P}}$ function F defined on the class of ordinals such that $\forall\nu\; F(\nu) = \omega_\nu$. G is continuous at limits. For any ζ define by a further recursion on ω,

$$f(0) = \zeta + 1; \quad f(n+1) = F(f(n))$$

Then $\alpha_0 =_{\rm df} \bigcup_{n<\omega} f(n)$ will exist by $\Sigma_1^{\mathcal{P}}$ collection and will satisfy $F(\alpha) = \alpha > \zeta$. ⊣ (6·34)

We saw in §3 that KLZ_0 proves that $\forall\kappa\exists\kappa^+$, and therefore so does $\mathsf{KPR} + V = L$.

6·35 LEMMA $(\mathsf{KP}^{\mathcal{P}} + V = L)$ $\forall\nu\big[V_{\omega+\nu} \subseteq L_{\omega_\nu}\big]$.

Proof: an induction on ν. $V_\omega = L_\omega$. $\mathcal{P}(L_{\omega_\nu}) \subseteq L_{\omega_{\nu+1}}$. The induction easily continues at limits. Hence we need only show that if there is a counterexample, there is a minimal one. But consider $\{x \mid \varrho(x) \geqslant \omega \;\&\; x \notin L_{\omega_{\varrho(x)-\omega}}\}$: that is a $\Delta_1^{\mathcal{P}}$ class, and so if non-empty has a minimal element, the rank of which will be a minimal ν for which the theorem fails. ⊣ (6·35)

6·36 PROPOSITION *The theory* $\mathsf{KP}^{\mathcal{P}} + V = L$ *proves the existence of arbitrarily large transitive models of* `KPR`.

Proof: for any $\eta = \omega_\eta$, L_η will model `KLZ` + `RAS`. ⊣ (6·36)

Well-founded parts of ill-founded models

We shall show that the standard part of any non-standard ω-model of `KPR` is a model of $\mathtt{KP}^{\mathcal{P}}$ plus `Class Foundation`, and use that fact to show that $\mathsf{KP}^{\mathcal{P}} + V = L$ proves the consistency of $\mathsf{KP}^{\mathcal{P}}$.

6·37 An *ω-model* of a set theory including $\mathtt{KP} + \dot\omega \,\epsilon\, \dot V$ is one of which the natural numbers are well founded. Let $\mathbf{N}$ be such a model. Let $\varpi_o(\mathbf{N})$, the *standard ordinal* of $\mathbf{N}$, be the supremum of von Neumann ordinals isomorphic to (well-founded) ordinals of $\mathbf{N}$. $\omega < \varpi_o(\mathbf{N})$ as $\mathbf{N}$ is an ω-model. $\mathbf{N}$ can compute the rank of a set, being a model of KP. Let

$$\mathbf{M} = \{x \in \mathbf{N} \mid \exists\zeta :< \varpi_o(\mathbf{N})\; \varrho^{\mathbf{N}}(x) \cong \zeta\}.$$

$\mathbf{M}$ is the *well-founded* or *standard* part of $\mathbf{N}$, comprising those elements x of M whose rank $\varrho^{\mathbf{N}}(x)$, as computed in $\mathbf{N}$, is a well-founded ordinal; we may denote it by $\varpi_p(\mathbf{N})$.

6·38 Remark We use Σ_1 Separation, to form the standard part as a set, and we use the truth predicates $\mathbf{M}\models$, $\mathbf{N}\models$, so that $\mathsf{M}+\mathsf{H}$ suffices for the construction of $\mathbf{M}$ from $\mathbf{N}$.

In the next two proofs, we shall need Σ_1 Separation to ensure that the maximal well-founded initial segment of a linear ordering is a set. KP of course enables us to form the truth predicate $\models$ for $\mathbf{N}$ and $\mathbf{M}$.

6·39 Proposition (KZ_1) *The standard part* $\mathbf{M} = \varpi_p(\mathbf{N})$ *of an ill-founded* ω*-model* $\mathbf{N}$ *of* $\mathsf{KP}+\mathsf{M}+\dot\Sigma_1$ `Foundation` *is a model of* $\mathsf{KP}+\mathsf{M}+$ `Class Foundation`*; further,* WO *will hold in* $\varpi_p(\mathbf{N})$ *if it held in* $\mathbf{N}$.

Proof : (1) That KP might hold in the standard part of a non-standard model is a result that goes back to M[lle] Ville. The argument we give is from Barwise's book [K3].

That $\mathbf{M}\models\dot\Delta_0$ `Separation` is readily checked, because $\dot\Delta_0$ formulæ are absolute between $\mathbf{M}$ and $\mathbf{N}$. $\mathbf{M}\models$ `Infinity` because $\mathbf{N}$ is an ω-model. Union is easily checked, and pairing will hold because $\varpi_o(\mathbf{N})$ is a limit ordinal. Π_1 Foundation, indeed full Class Foundation, will hold because $\mathbf{M}$ is a set and its membership relation is well-founded.

The hardest part will be the verification of $\dot\Delta_0$ Collection. So let p and b be in $\mathbf{M}$, φ be $\dot\Delta_0$ and suppose that $\mathbf{M}\models\bigwedge x\colon\epsilon\, b\bigvee y\varphi(x,y)[p]$: here p is just a parameter which is hardly important. Let d be an ill-founded ordinal of $\mathbf{N}$. $\mathbf{N}\models\bigwedge x\colon\epsilon\, b\bigvee\zeta\colon<d\ \bigvee y\big[\dot\varrho(y)=\zeta\wedge\varphi(x,y)[p]\big]$. Form the class

$$A=\big\{c\ \big|\ \mathbf{N}\models c\,\epsilon\,\dot{ON}\wedge\bigwedge x\colon\epsilon\, b\bigvee\zeta\colon<c\ \bigvee y\big[\dot\varrho(y)=\zeta\wedge\varphi(x,y)[p]\big]\big\}.$$

That is non-empty, as it contains all ill-founded ordinals of $\mathbf{N}$. Its defining formula is $\dot\Sigma_1^{\mathsf{KP}}$, and hence as $\mathbf{N}$ models $\dot\Sigma_1^{\mathsf{KP}}$ `Foundation`, there is a least element. That must be a well-founded ordinal, as there is no least ill-founded ordinal. Call it $\bar\eta$; so $\bar\eta<\varpi_o(\mathbf{N})$ and

$$\mathbf{N}\models\bigwedge x\colon\epsilon\, b\bigvee y\big[\dot\varrho(y)<\bar\eta\wedge\varphi(x,y,p)\big].$$

Applying $\dot\Delta_0$ Collection in $\mathbf{N}$,

$$\mathbf{N}\models\bigvee z\bigwedge x\colon\epsilon\, b\bigvee y\colon\epsilon\, z\big[\dot\varrho(y)<\bar\eta\wedge\varphi(x,y,p)\big].$$

Let z witness that last statement, and form $z\cap\{y\mid\mathbf{N}\models\dot\varrho(y)<\bar\eta\}$: that is a member of $\mathbf{N}$ by $\dot\Delta_1^{\mathsf{KP}}$ Separation, and is a set of rank at most $\bar\eta$; hence it is in $\mathbf{M}$, and we have established $\dot\Delta_0$ Collection for $\mathbf{M}$.

(2) The power set of a object of well-founded rank is also of well-founded rank. More exactly,

Let $x\in\mathbf{M}$*, and let* $y\in\mathbf{N}$ *be the object such that* $\mathbf{N}\models y=\dot{\mathcal{P}}(x)$*. Then* $y\in\mathbf{M}$ *and* $\mathbf{M}\models y=\dot{\mathcal{P}}(x)$.

(3) Evidently a well-ordering of an object of well-founded rank is also an object of well-founded rank. ⊣ (6·39)

The following is a slightly sharpened version of Harvey Friedman's extension of the above, in [K2], to models of $\mathsf{KP}^{\mathcal{P}}$.

6·40 Proposition (KZ_1) *Let* $\mathbf{N}$ *be a non-standard* ω*-model of* KPR*. Let* $\mathbf{M}$ *be the standard part of* $\mathbf{N}$*. Then* $\mathbf{M}$ *is a model of* $\mathsf{KP}^{\mathcal{P}}+$ `Class Foundation`*. Further,* WO *will hold in* $\mathbf{M}$ *if it did in* $\mathbf{N}$.

Proof : We shall use repeatedly the fact that for φ a $\dot\Delta_0^{\mathcal{P}}$ formula with all parameters in $\mathbf{M}$, $\mathbf{N}\models\varphi\iff\mathbf{M}\models\varphi$. The truth of $\dot\Delta_0^{\mathcal{P}}$ Separation in $\mathbf{M}$ thus follows from its truth in $\mathbf{N}$. The axiom of power set holds in $\mathbf{N}$ by Lemma 6·30, and therefore also in $\mathbf{M}$ by our previous reasoning. $\mathbf{M}$ will satisfy full Foundation because the standard part is a well-founded set. Our sole problem therefore will be proving $\dot\Delta_0^{\mathcal{P}}$ Collection.

Suppose that $\mathbf{M}\models\bigwedge x\colon\epsilon\, a\bigvee y\varphi(a,b,x,y)$ where $\varphi(a,b,x,y)$ is $\dot\Delta_0^{\mathcal{P}}$, and a, b, are parameters in $\mathbf{M}$.

Let $\bar e$ be any non-standard ordinal. Then

$$\mathbf{N}\models\bigwedge x\colon\epsilon\, a\bigvee y\colon\epsilon\, V_{\bar e}\ \varphi.$$

By the formal counterpart to Lemma 6·32, there is a $\dot\Delta_0$ formula $\vartheta(\mathfrak{a},\mathfrak{b},x,y,w)$ with all quantifiers restricted to w such that $\mathbf{N}\models\varphi\iff\vartheta[V_{\bar e}]$. Therefore let us consider

$$\{e\mid\mathbf{N}\models e\,\epsilon\,\dot{ON}\wedge\bigwedge x\colon\epsilon\, a\bigvee y\colon\epsilon\,\dot V_{\bar e}(\dot\varrho(y)<e\wedge\vartheta[V_{\bar e}]).\}^{\mathbf{N}}$$

That class is $\dot{\Delta}_1^{\mathsf{KP}}$ in the parameter $V_{\bar{e}}$ since ϱ is a Δ_1^{KP} function and ϑ is $\dot{\Delta}_0$. It contains all non-standard ordinals. By $\dot{\Delta}_1^{\mathsf{KP}}$ Separation in $\mathbf{N}$ (or by $\dot{\Pi}_1$ Foundation there) any non-empty initial segment of it will be a set and it therefore has a minimal element. Therefore that minimal element is a standard ordinal η say. Thus

$$\mathbf{N} \models \bigwedge x\colon \epsilon\, a \bigvee y\colon \epsilon\, V_\eta\ \vartheta[a, b, V_{\bar{e}}].$$

Replacing η if necessary by a larger standard ordinal so that all parameters a, b, of φ lie in V_η, and using the equivalence of $\vartheta[V_\eta]$ and $\vartheta[V_{\bar{e}}]$, the fact that $V_\eta \in \mathbf{M}$ since it is a well-founded member of $\mathbf{N}$, and the fact that Δ_0 formulæ are absolute between $\mathbf{N}$ and $\mathbf{M}$, we have

$$\mathbf{M} \models \bigwedge x\colon \epsilon\, a \bigvee y\colon \epsilon\, V_\eta \vartheta[a, b, V_\eta].$$

But the equivalence between ϑ and φ needs almost no set theory beyond the fact of V_η being a set, and so we have

$$\mathbf{M} \models \bigwedge x\colon \epsilon\, a \bigvee y\colon \epsilon\, V_\eta \varphi[a, b]$$

and we have proved that $\mathbf{M}$ satisfies $\Delta_0^{\mathcal{P}}$ Collection for φ. ⊣ (6·40)

Note that in that second proposition the non-standard model is not required to satisfy Σ_1 Separation.

For $a \subseteq \omega$, we write ω_1^a for the least ordinal not recursive in a. We write $\leqslant_T$ for Turing reducibility and ω_1^{CK}, for the Church–Kleene ordinal, that is, the first non-recursive von Neumann ordinal.

Let $R \subseteq \omega \times \omega$ code an ω-model of a set theory extending $\mathsf{KP} + \omega \in V$; we suppose that $2k$ represents k in the model $\mathbf{M} = (\omega, R)$. Let $\varpi_o(\mathbf{M})$ be the standard ordinal of $\mathbf{M}$.

6·41 LEMMA *If $\eta < \varpi_o(\mathbf{M})$ then $\eta \leqslant_T R$; so $\varpi_o(\mathbf{M}) \leqslant \omega_1^R$.*

(For let η be represented by ℓ. Then η is isomorphic to $(\{m \mid (m, \ell) \in R\}, R)$.)

6·42 LEMMA *If $a \subseteq \omega$ is represented in $\mathbf{M}$, then $\omega_1^a \leqslant \varpi_o(\mathbf{M})$.*

Proof : Let e be an index such that the linear ordering $\{e\}^a$ is actually a well-ordering. That linear ordering will be represented in $\mathbf{M}$, which will attempt to build an isomorphism between it and an ordinal. The complement of the domain of that attempt need not be a set of $\mathbf{M}$, but externally to $\mathbf{M}$ we can, if it is non-empty, find its minimal element, which will be some integer of $\mathbf{M}$, and then can argue, once we have the minimal failure, that the induction could have continued, and therefore the complement is indeed empty. The ordinal of $\mathbf{M}$ isomorphic in $\mathbf{M}$ to that linear ordering is therefore in the standard part of $\mathbf{M}$ since that linear ordering is actually a well-ordering. ⊣ (6·42)

6·43 PROPOSITION $\sup\{\omega_1^a \mid a$ *is represented in* $\mathbf{M}\} \leqslant \varpi_o(\mathbf{M}) \leqslant \omega_1^R$.

6·44 COROLLARY *If $\omega_1^R = \omega_1^{CK}$, then $\varpi_o(\mathbf{M}) = \omega_1^{CK}$.*

We shall use the Gandy basis theorem ([K4], or Corollary III.1.9 of [K5]), for the proof of which MOST is more than enough:

6·45 PROPOSITION (Gandy) *A non-empty $\Sigma_1^1(a)$ set of reals has a member x with $\omega_1^x \leqslant \omega_1^a$.*

We shall later use the Shoenfield absoluteness theorem:

6·46 PROPOSITION (Shoenfield) *A Π_2^1 sentence true in L is true in V.*

6·47 THEOREM $\mathsf{KP}^{\mathcal{P}} + V = L$ *proves the consistency of* $\mathtt{KP}^{\mathcal{P}} +$ `Class Foundation`.

Proof : reason in $\mathsf{KP}^{\mathcal{P}} + V = L$, of which MOST is a subsystem, for by 5·0, 5·1 and 3·18, we know that $\forall\kappa\exists\kappa^+$ and that Σ_1 Separation and Axiom H hold.

From Proposition 6·36 we know that there is an L_ζ which models `KPR + V = L`. Therefore there is by Gandy an ω-model $\mathbf{M} = (\omega, R)$ of `KPR+V = L` with $\omega_1^R = \varpi_o(\mathbf{M}) = \omega_1^{CK}$. Such an $\mathbf{M}$ cannot be well-founded, since $L_{\omega_1^{CK}}$ does not model `KPR`. Hence it is ill-founded; but then its standard part, which Σ_1 Separation suffices to construct, is by Proposition 6·40 a model of $\mathtt{KP}^{\mathcal{P}} +$ `Class Foundation`. ⊣ (6·47)

Hence $\mathsf{KP}^{\mathcal{P}}$ does not prove $(\mathsf{KP}^{\mathcal{P}})^L$, in contrast to KP which proves $(\mathsf{KP})^L$. For completeness, we show now that the yet stronger system $\mathsf{KP}^{\mathcal{P}} + \forall\kappa\exists\kappa^+$ does prove its own truth in L ! First a lemma:

6·48 LEMMA (KLZ_0) *Let ζ be an uncountable limit cardinal. Then $L_\zeta \preceq_{\Delta_0^{\mathcal{P}}} L$.*

Proof : let Φ be $\Delta_0^{\mathcal{P}}$. By Takahashi (6·12), there are a Π_1 formula Ψ_1 and a Σ_1 formula Ψ_2 such that

$$\mathsf{MOST} \vdash \big(\Phi \iff \exists y\ \Psi_1\big)\ \&\ \big(\Phi \iff \forall y\ \Psi_2.\big)$$

Now work in KLZ_0, which equals $\mathsf{MOST} + V = L$, let ζ be an uncountable limit cardinal — we do not, of course, allege that KLZ_0 proves that any such exist — and let Φ be a $\Delta_0^{\mathcal{P}}$ formula with all parameters in L_ζ. We must show that $\Phi \iff (\Phi)^{L_\zeta}$.

L_ζ models MOST, by 5·1. Hence if $(\Phi)^{L_\zeta}$, then for some $y \in L_\zeta$, $(\Psi_1(y))^{L_\zeta}$; since $L_\kappa \preceq_{\Sigma_1} L$ for every uncountable cardinal κ, $\Psi_1(y)$ holds in L. Therefore Φ holds.

In the reverse direction we use Ψ_2. ⊣ (6·48)

6·49 THEOREM $\mathsf{KP}^{\mathcal{P}} + \forall\kappa\exists\kappa^+ \vdash (\mathsf{KP}^{\mathcal{P}} + \forall\kappa\exists\kappa^+)^L$.

Proof : $\mathsf{KP} + \omega \in V + \forall\kappa\exists\kappa^+$ proves its own truth in L. Our first problem, therefore, will be to show that $\Delta_0^{\mathcal{P}}$ Collection holds in L. Let $a \in L$, let Φ be $\Delta_0^{\mathcal{P}}$ and suppose that $\forall x{:}{\in}a\ (\Phi)^L$. We know from 6·33 that there are arbitrarily large limit alephs, which will therefore be limit cardinals in L. Furthermore, for such ζ, L_ζ correctly computes the (constructible) power set of each of its members. So with the device of 6·16 and 6·33 in mind, of quantifying over the subsets of a transitive pairing-closed set containing the relevant parameters, we have

$$\forall x{:}{\in}a\ \exists\zeta\big(\zeta > \omega\ \&\ \underbrace{\forall\alpha{:}{<}\zeta\ \exists\beta{:}{<}\zeta\ \beta = \alpha^+}_{\Sigma_1^{\mathcal{P}},\text{at worst}}\ \&\ \underbrace{L_\zeta \models \dot\Phi}_{\Delta_1^{\mathsf{KP}}}\big).$$

We may therefore apply $\Sigma_1^{\mathcal{P}}$ Collection in V to deduce that

$$\exists\eta\forall x{:}{\in}a\ \exists\zeta{:}{<}\eta\ \big(\zeta > \omega\ \&\ \forall\alpha{:}{<}\zeta\ \exists\beta{:}{<}\zeta\ \beta = \alpha^+\ \&\ L_\zeta \models \dot\Phi\big),$$

which tells us that $\forall x{:}{\in}a\ \exists y{:}{\in}L_\eta\ \Phi$, as required.

Finally we should show that $\Pi_1^{\mathcal{P}}$ Foundation holds in L. Suppose that $\Phi(x, y, a)$ is $\Delta_0^{\mathcal{P}}$, that $a \in L$, and that the class $A = \{x \mid x \in L\ \&\ (\forall y\Phi(x, y, a))^L\}$ is non-empty. We must find an $\in$-minimal element. Let $\bar x \in A$. Pick $\bar\eta$ infinite with $\{a, \bar x\} \subseteq L_\eta$. Consider the class

$$\big\{x \mid x \in L_{\bar\eta}\ \&\ \forall y\forall\zeta\big(\zeta > \bar\eta\ \&\ \zeta \text{ a limit cardinal }\ \&\ y \in L_\zeta \implies L_\zeta \models \dot\Phi[x, y, a]\big)\big\}.$$

That is $\Pi_1^{\mathcal{P}}$, has a member, $\bar x$, and therefore by $\Pi_1^{\mathcal{P}}$ Foundation in V has an $\in$-minimal element, which will be $\in$-minimal for A. ⊣ (6·49)

6·50 COROLLARY *The theory $\mathsf{KP}^{\mathcal{P}} + \forall\kappa\exists\kappa^+$ proves the consistency of* $\mathtt{KP}^{\mathcal{P}} +$ `Class Foundation`*. as do* $\mathsf{KP}^{\mathcal{P}} + \Sigma_1$ Separation *and* $\mathsf{KP}^{\mathcal{P}} + \mathsf{H}$.

Proof : The first statement follows from Theorems 6·47 and 6·49; the other two then follow from Proposition 2·0 and and Remark 3·21. ⊣ (6·50)

6·51 REMARK Hence $\mathsf{KP}^{\mathcal{P}}$, if consistent, cannot prove its own truth in the H-model.

A transitive model of $\mathbf{KP}^{\mathcal{P}}$ + AC in which every ordinal is recursive

To save our energies we work in ZF but that is certainly too strong. The following theorem was first proved by Harvey Friedman in [K2] using the Barwise compactness theorem.

6·52 THEOREM (ZF) *There is a countable transitive model of* $\mathtt{KP}^{\mathcal{P}} + \mathtt{AC} +$ `every von Neumann ordinal is recursive` $+$ `Class Foundation`*. More generally, every real a is a member of a countable transitive model of* $\mathtt{KP}^{\mathcal{P}} + \mathtt{AC} +$ `Class Foundation` *in which every von Neumann ordinal is recursive in a.*

We give first a weaker result, the proof of which may amuse the reader:

6·53 PROPOSITION (ZF) *There is a countable transitive model of* KP + MAC + every von Neumann ordinal is countable + Class Foundation.

Proof: the statement we are to prove is Σ^1_2; by Shoenfield it will suffice to prove it in the theory ZF + "there is a non-constructible real". So fix a non-constructible real, a.

Note that in L, every real is a member of a countable ω-model of KPL + Power Set + Class Foundation. That is a Π^1_2 statement, and so it is true in V. Hence there is a countable ω-model $\mathbf{N}$ of KPL + Power Set + Class Foundation with $a \in \mathbf{N}$. $\mathbf{N}$ will model $\dot{\Sigma}_1$ Foundation by 3·20 and 3·19.

Let $\mathbf{M}$ be the standard part of $\mathbf{N}$. Then a is in $\mathbf{M}$. We shall prove that $\mathbf{M}$ is the desired model of KP + MAC+ every von Neumann ordinal is countable + Class Foundation. In view of Proposition 6·39, it is enough to show that $\mathbf{M} \models$ every von Neumann ordinal is countable. Note that

$$\mathbf{N} \models \bigwedge\xi\big(\neg(a \,\epsilon\, \dot{L}_\xi) \longrightarrow \xi \texttt{ is countable}\big).$$

For we have seen in 5·0 that KLZ_0 proves that ω_1 exists; in that theory we may also carry out Gödel's argument and prove that every real is constructed at a countable stage; the stages are cumulative, hence a stage by which a given real has not been constructed is necessarily countable.

Let $\zeta \in \mathbf{M}$. Then $a \notin L_\zeta$ for ζ is well-founded and a is not constructible. So $\mathbf{N} \models a \notin L_\zeta$. So $\mathbf{N} \models \zeta$ is countable; so $\mathbf{N}$ thinks that there is a subset of $\zeta \times \omega$ which is the graph of a bijection between ω and ζ. But that graph is a well-founded object, being of rank at most $\zeta + 2$, and so is in $\mathbf{M}$. ⊣ (6·53)

Now we shall prove the sharper form:

Proof of Theorem 10: let a be in L. There is certainly a countable well-founded M which models KPR+ V = L with $a \in M$. The class of codes of such ω-models in which a is represented is therefore non-empty and $\Sigma^1_1(a)$, so therefore contains an M with $\omega_1^M \leqslant \omega_1^a$. Hence $\varpi_o(M) = \omega_1^a$.

M cannot be standard, as the only standard model of KPL of height ω_1^a, should it contain a, cannot model KPR. So M is non-standard; its standard part is therefore a model of $\mathsf{KP}^{\mathcal{P}}$ + AC, contains a and is of height ω_1^a.

We have just proved a Π^1_2 sentence assuming $V = L$; it is by Shoenfield therefore true in V. ⊣ (6·50)

⊣ (Theorem 10)

6·54 REMARK Thus there is a power admissible set N of height ω_1^{CK} in which WO is true. Then N models $\mathsf{KP}^{\mathcal{P}}$ + WO+ every ordinal is recursive, and $L^N = L_{\omega_1^{CK}}$, so $\mathsf{KP}^{\mathcal{P}}$ fails to prove that there is a non-recursive ordinal — giving the sharper version of Theorem 9 — and fails to prove $(\mathsf{Power\ Set})^L$. In particular it fails to prove Σ_1 Separation, whereas it proves $\Delta_1^{\mathcal{P}}$ Separation; which shows that there are Σ_1 formulæ which are not contained in $(\Delta_1^{\mathcal{P}})^{\mathrm{MAC}}$, and thus establishes the need for Axiom H in Takahashi's reduction 6·12.

6·55 REMARK Friedman in his paper obtains Theorem 10, and many other results, by using the Barwise compactness theorem.

A transitive model of $\mathbf{KP}^{\mathcal{P}}$ + AC + $\forall\kappa\exists\kappa^+$ with a long well-ordering of the continuum

We shall now use forcing over ill-founded models, to show that in the absence of $V = L$, the system $\mathsf{KP}^{\mathcal{P}} + \mathsf{AC} + \forall\kappa\exists\kappa^+$ does not prove that every well-ordering is isomorphic to an ordinal, and *â fortiori* fails to prove Σ_1 Separation.

We start then from the assumption that there is a countable transitive power-admissible set A in which $V = L$ and, therefore, $\forall\kappa\,\exists\kappa^+$ are true. The reader unused to forcing over weak systems of set theory may like to assume on a first reading that $A = L_\theta$ where θ is a countable ordinal that in L is strongly inaccessible. But a suitable θ satisfying a weaker condition may easily be found by applying Lévy reflection to an appropriate conjunction of finitely many axioms inside L.

By H. Friedman, [K2], Theorem 2.3, there is a non-standard power-admissible set $\mathfrak{N}$ of which A is the standard part: $\mathfrak{N}$ can be chosen to satisfy AC. So all the initial ordinals of A are initial ordinals of $\mathfrak{N}$. Working in $\mathfrak{N}$ we pick a non-standard initial ordinal ℓ, which is regular in the opinion of $\mathfrak{N}$, and add ℓ Cohen reals, so the enlarged model $\mathfrak{N}'$ has the same ordinals as $\mathfrak{N}$, models AC and thinks the continuum is size ℓ;

further its cardinals are the same as $\mathfrak{N}$. The partial ordering, call it $\mathbb{P}$, which does that will be a member of $\mathfrak{N}$.

There are three problems here: one is to show that the generic filter can be built, then that the model can be defined, and finally that truth respects forcing.

We suppose that we are treating forcing in the manner of Shoenfield, in which every element of the ground model is interpreted as a name for a member of the extension.

If one were dealing with a well-founded model N, one would proceed by first choosing a filter G that meets every subclass of $\mathbb{P}$ that is definable over N, so that we may call G $(N, \mathbb{P})$-*generic*, and then making the following recursive definition:

6·56 Putative Definition Define (externally to N) $\phi_G : N \to V$ by

$$\phi_G(b) = \{\phi_G(a) \mid \exists p{:}{\in} G \ \ (a,\, p) \in b\}.$$

Then we would prove the following

6·57 Putative Lemma *For all a and b the following hold:*

$$\phi_G(a) \in \phi_G(b) \iff \exists p{:}{\in} G \ \ p \Vdash \underline{a} \mathrel{\epsilon} \underline{b} \tag{6·58}$$

$$\phi_G(a) \subseteq \phi_G(b) \iff \exists p{:}{\in} G \ \ p \Vdash \underline{a} \mathrel{\dot\subseteq} \underline{b} \tag{6·59}$$

$$\phi_G(a) = \phi_G(b) \iff \exists p{:}{\in} G \ \ p \Vdash \underline{a} = \underline{b} \tag{6·60}$$

In our present context, the model $\mathfrak{N}$ is ill-founded, and so *prima facie* we cannot carry out that recursive definition. However we may choose G as before, meeting every $\mathfrak{N}$-definable subclass of $\mathbb{P}$. Then we treat the above Lemma as a definition:

6·61 Definition Define for all a and b in $\mathfrak{N}$ the following equivalence relation:

$$a \equiv_G b \iff \exists p{:}{\in} G \ \ p \Vdash \underline{a} = \underline{b}$$

Let $\mathfrak{Q} = \mathfrak{Q}_G$ be the set of equivalence classes. Write $[a]_G$ for the $\equiv_G$-equivalence class of $a \in \mathfrak{N}$.

Define a relation $\in_G$ on $\mathfrak{Q}$ by

$$[a]_G \in_G [b]_G \iff \exists p{:}{\in} G \ \ p \Vdash \underline{a} \mathrel{\epsilon} \underline{b}$$

That that relation is independent of the chosen representives a, b, of their equivalence classes follows from general facts about forcing established within $\mathfrak{N}$.

Then $(\mathfrak{Q}, \in_G)$ is a perfectly reasonable countable set with a two-place relation on it, and we can ask which of the sentences of the language of set theory are true in that model when we interpret = by equality and ϵ by $\in_G$.

We establish the familiar principle that what is true in this model is what is forced by some member of G: but the proof of that relies entirely on the fact that G meets all the necessary dense classes, and makes no use of the well-foundedness of the model under consideration.

Once that has been done, we may strengthen the ties between $\mathfrak{Q}$ and $\mathfrak{N}$, by showing that we may treat $\mathfrak{Q}$ as an extension of $\mathfrak{N}$ by considering the map $x \mapsto [\hat{x}]$; we may also show that G is in $\mathfrak{Q}$, being $[\dot{G}]$. Here $\hat{x}$ is the canonical forcing name for the member x of the ground model, defined recursively inside $\mathfrak{N}$, (using which we may define a predicate $\hat{V}$ of the forcing language for membership of the ground model) and $\dot{G}$ is the canonical forcing name for the generic being added.

We show that every name has a unique rank of $\mathfrak{N}$ attached, chosen from all possible by the completeness of G. Those ranks are simply the ordinals of $\mathfrak{N}$.

So loosely we may say that the extension $\mathfrak{Q}$ is no more ill-founded than is the starting model $\mathfrak{N}$. Further, $\mathfrak{Q}$ considers itself to be a generic extension of $\mathfrak{N}$ via $\mathbb{P}$ and G, the corresponding statement about $\hat{\mathbb{P}}$ and $\dot{G}$ being forced. Hence inside $\mathfrak{Q}$ the recursive definition of $\phi_G : \mathfrak{N} \to \mathfrak{Q}$ by

$$\phi_G(b) = \{\phi_G(a) \mid \exists p{:}{\in} G \ \ \langle a,\, p\rangle \in b\}.$$

succeeds, using the predicate $\hat{V}$ identifying the members of $\mathfrak{N}$.

Since the forcing $\mathbb{P}$ is a member of $\mathfrak{N}$, $\mathfrak{Q}$ will be a model of KP + AC; that $\mathfrak{Q} \models$ `ranks are sets` will follow from the observation that $\mathsf{KP}^{\mathcal{P}}$ can support the following transfinite recursion and prove that each K_ν is a set:

$$K_0 = \varnothing$$
$$K_{\nu+1} = \mathcal{P}(K_\nu \times \mathbb{P})$$
$$K_\lambda = \bigcup_{\nu<\lambda} K_\nu \qquad \text{for } 0 < \lambda = \bigcup \lambda$$

Now let $\mathbf{M}$ be the well-founded part of $\mathfrak{Q}$. Then $\mathbf{M}$ is of height θ, since the ordinals of $\mathfrak{Q}$ are the same as those of $\mathfrak{N}$, and since $\mathfrak{Q} \models$ `ranks are sets`, it follows from 6·40 that $\mathbf{M}$ is power admissible.

Now the ordinals of $\mathbf{M}$ are the standard ordinals of $\mathfrak{N}$, in short the ordinals of A. Cofinalities (and therefore cardinalities) in A are preserved in $\mathfrak{N}$, since A is exactly the standard part of $\mathfrak{N}$; the extension from $\mathfrak{N}$ to $\mathfrak{Q}$ preserves cofinalities, being from the point of view of $\mathfrak{N}$ a c.c.c. extension, and are further preserved in the restriction to the standard part $\mathbf{M}$ of $\mathfrak{Q}$. Thus $\mathbf{M}$ is an extension of A in which all cofinalities and cardinalities are preserved; thus in $\mathbf{M}$, the initial ordinals are cofinal in the ordinals. However, every real of $\mathfrak{Q}$ is in $\mathbf{M}$, and every well-ordering in $\mathfrak{Q}$ of the continuum of $\mathfrak{Q}$ will lie in $\mathbf{M}$, but although the continuum of the model $\mathbf{M}$ is well-orderable in the opinion of the model $\mathbf{M}$, none of its well-orderings in $\mathbf{M}$ are isomorphic to any ordinal of $\mathbf{M}$. Those well-orderings are all pseudo-wellorderings, being isomorphic to an ill-founded ordinal of $\mathfrak{Q}$. $\dashv$ (Theorem 11)

6·62 REMARK Another version of Theorem 10 can be proved by the method of forcing over non-standard models. As before let A be a countable power-admissible set, $\mathfrak{N}$ a non-standard power-admissible set with A as its standard part, and ℓ a non-standard ordinal of $\mathfrak{N}$. This time use Lévy forcing to to add a generic subset of ω coding ℓ, and take the standard part of the extension. That standard part will be power admissible, and the real added, g, will have the property that the ordinals in A are precisely the ordinals recursive in g. Thus we have shown that any countable power-admissible set (and therefore by Friedman any countable admissible set) is contained in a power-admissible set in which there is a real in which every ordinal is recursive.

7: Stratifiable formulæ

We continue the study of stratifiable formulæ, a concept that was defined in §0. We follow M. Boffa, [B2].

First, suppose that a and b are disjoint sets, ϕ a bijection between them, and that the class $F : V \longleftrightarrow V$ is defined by

$$F(x) = \begin{cases} \phi(x), & \text{for } x \in a; \\ \phi^{-1}(x) & \text{for } x \in b; \\ x & \text{otherwise.} \end{cases}$$

Define a sequence of classes $F_\mathfrak{i}$ for each concrete natural number $\mathfrak{i}$ thus:

$$F_0 = F; \quad F_{\mathfrak{i}+1}(x) = F_\mathfrak{i}``x.$$

7·0 PROPOSITION SCHEME (KF) $F_0 : V \longleftrightarrow V;$ $\quad F_\mathfrak{i}``x \in V;$ $\quad F_{\mathfrak{i}+1} : V \longleftrightarrow V.$

Proof : The only problem is in showing that each $F_\mathfrak{i}``x \in V$. Note first that the chain

$$\begin{aligned} x_0 \in x_1 \in x_2 \in x_3 \in \ldots &\implies F_0(x_0) \subseteq x_1 \cup a \cup b \subseteq (\bigcup x_2) \cup a \cup b \subseteq (\bigcup\bigcup x_3) \cup a \cup b \ldots \\ &\implies F_1(x_1) \subseteq \mathcal{P}\big((\bigcup x_2) \cup a \cup b\big) \subseteq \mathcal{P}\big((\bigcup\bigcup x_3) \cup a \cup b\big) \ldots \\ &\implies F_2(x_2) \subseteq \mathcal{P}\mathcal{P}\big((\bigcup\bigcup x_3) \cup a \cup b\big) \ldots \end{aligned}$$

will show that for each $\mathfrak{k}$, once $F_\mathfrak{k}$ is known to be a total function,

$$F_\mathfrak{k}``(x_{\mathfrak{k}+1}) \subseteq \mathcal{P}^{\mathfrak{k}+1}\big((\bigcup{}^{\mathfrak{k}+1} x_{\mathfrak{k}+2}) \cup a \cup b\big).$$

We then prove successively that

$$
\begin{array}{rll}
F_0(x_0) \in V & & \text{by examination of cases;} \\
y_0 = F_0(x_0) \iff & \big[\big(x_0 \in a_1 \,\&\, y_0 = \phi_3(x_0)\big) \text{ or } \big(x_0 \in b_1 \,\&\, y_0 = \phi_3^{-1}(x_0)\big) \text{ or } \big(x_0 \notin a_1 \,\&\, x_0 \notin b_1 \,\&\, y_0 = x_0\big)\big]; & \\
y_0 \in F_1(x_1) \iff & \exists z_0 :\in x_1 \; y_0 = F_0(z_0) \;\text{ and }\; F_0\text{“}(x_1) \subseteq \mathcal{P}\big((\bigcup x_2) \cup a \cup b\big); & \text{hence} \\
F_1(x_1) = F_0\text{“}(x_1) \in V & & \text{by stratifiable } \Delta_0 \text{ Separation;} \\
y_1 = F_1(x_1) \iff & [\forall z_0 :\in y_1 \; \exists t_0 :\in x_1 \; z_0 = F_0(t_0) \,\&\, \forall t_0 :\in x_1 \; \exists z_0 :\in y_1 \; z_0 = F_0(t_0)]; & \\
y_1 \in F_2(x_2) \iff & \exists z_1 :\in x_2 \; y_1 = F_1(z_1) \;\text{ and }\; F_1\text{“}(x_2) \subseteq \mathcal{P}\mathcal{P}\big((\bigcup\bigcup x_3) \cup a \cup b\big); & \text{hence} \\
F_2(x_2) = F_1\text{“}(x_2) \in V & & \text{by stratifiable } \Delta_0 \text{ Separation;} \\
\ldots\ldots & & \\
F_{\mathfrak{k}}(x_{\mathfrak{k}}) \in V & & \\
y_{\mathfrak{k}} = F_{\mathfrak{k}}(x_{\mathfrak{k}}) \iff & [\forall z_{\mathfrak{k}-1} :\in y_{\mathfrak{k}} \; \exists t_{\mathfrak{k}-1} :\in x_{\mathfrak{k}} \; z_{\mathfrak{k}-1} = F_{\mathfrak{k}-1}(t_{\mathfrak{k}-1}) \,\&\, \forall t_{\mathfrak{k}-1} :\in x_{\mathfrak{k}} \; \exists z_{\mathfrak{k}-1} :\in y_{\mathfrak{k}} \; z_{\mathfrak{k}-1} = F_{\mathfrak{k}-1}(t_{\mathfrak{k}-1})]; & \\
y_{\mathfrak{k}} \in F_{\mathfrak{k}}\text{“}(x_{\mathfrak{k}+1}) \iff & \exists z_{\mathfrak{k}} :\in x_{\mathfrak{k}+1} \; y_{\mathfrak{k}} = F_{\mathfrak{k}}(z_{\mathfrak{k}}) \;\text{ and }\; F_{\mathfrak{k}}\text{“}(x_{\mathfrak{k}+1}) \subseteq \mathcal{P}^{\mathfrak{k}+1}\big((\bigcup^{\mathfrak{k}+1} x_{\mathfrak{k}+2}) \cup a \cup b\big); & \text{hence} \\
F_{\mathfrak{k}+1}(x_{\mathfrak{k}+1}) = F_{\mathfrak{k}}\text{“}(x_{\mathfrak{k}+1}) \in V & & \text{by stratifiable } \Delta_0 \text{ Separation;} \\
\ldots\ldots & &
\end{array}
$$

⊣ (7·0)

From the bijectivity of $F_{\mathfrak{i}}$ will follow for each $\mathfrak{i}$ that

$$\forall x, y\,((x \in y \iff F_{\mathfrak{i}}(x) \in F_{\mathfrak{i}+1}(y)) \;\&\; (x = y \iff F_{\mathfrak{i}}(x) = F_{\mathfrak{i}}(y)))$$

whence a straightforward induction on the length of formulæ leads to a proof of the

7·1 Proposition Scheme *For each stratifiable formula $\Phi(x_1, \ldots x_{\mathfrak{k}})$ with free variables as shown, and for each stratification of Φ by type assignments $i_1, \ldots i_{\mathfrak{k}}$ to $x_1 \ldots x_{\mathfrak{k}}$:*

$$\vdash_{\mathsf{KF}} \Phi(x_1, \ldots x_{\mathfrak{k}}) \iff \Phi(F_{i_1}(x_1), \ldots, F_{i_{\mathfrak{k}}}(x_{\mathfrak{k}}))$$

An embedding elementary for stratifiable formulæ

We saw in §1 that the function tcl is available in M_1* as a provably total Δ_1 function, and therefore so is the function T defined by

$$T(x) = (x, tcl(\{x\}), \in\restriction tcl(\{x\}));$$

and we recall the translation $(\cdot)^1$ of formulæ introduced in Definition 2·27.

7·2 Proposition Scheme *For any stratifiable formula $\Phi(x_1, \ldots, x_{\mathfrak{k}})$,*

$$\vdash_{\mathsf{M}_1} (\Phi)^1(\overrightarrow{T(x)}) \iff \Phi(\vec{x})$$

Proof by induction on the length of Φ.

(i) $x \in y \iff T(x)\, E^1\, T(y)$

(ii) $x = y \iff T(x) \equiv^1 T(y)$

The above hold since any partial isomorphism between transitive sets is, by an application of Foundation, a restriction of the identity.

(iii) The induction for propositional connectives is trivial.

* but not in ZBQC, therefore not in M_0, still less in KF.

(iv) The hard case: suppose Φ is $\exists y\Psi(y, x_1, \ldots, x_{\mathfrak{k}})$, where in some stratification of $\Psi(y, x_1, \ldots, x_{\mathfrak{k}})$ the variables are of types $\mathsf{j}, i_1, \ldots, i_{\mathfrak{k}}$ respectively. It will be convenient to present the argument model-theoretically, as we are in effect proving that the mapping T embeds the universe as a submodel of W_1 that is elementary with respect to all stratifiable formulæ.

Let us therefore place ourselves in the larger model, W_1, which we will now call H, and denote by R the image of the smaller model, the original universe, under T. By assumption, the axioms of M_1 are true in R, and therefore by our discussion of Theorem 2, are also true in H; so the function *tcl* is available in both R and H; and it is readily checked that R is a transitive submodel of H, and indeed supertransitive in the sense that $a \subseteq b \in R \Longrightarrow a \in R$.

Our problem is this: given $a_1, \ldots, a_{\mathfrak{k}} \in R$ and $y \in H$ such that $\Psi(y, a_1, \ldots, a_{\mathfrak{k}})$ holds in H, to show that there is a z in R with $\Psi(z, a_1, \ldots, a_{\mathfrak{k}})$ holding again in H.

Let $u = tcl(\{a_1, \ldots, a_{\mathfrak{k}}\})$, and $t = tcl(\{y, a_1, \ldots, a_{\mathfrak{k}}\})$. $u \in R$, so $u \subseteq R$, but if $y \notin R$, as is probable, $t \not\subseteq R$. Pick $z \in R$ disjoint from $t \smallsetminus u$ such that there is a bijection $\phi : t \smallsetminus u \longleftrightarrow z$ — here we use the fact that each element of W_1 is coded by an element of V — and write $y = z \cup u$. Extend ϕ to a permutation $F : H \longleftrightarrow H$, and define the sequence F_i as above for $0 \leq i \leq \mathfrak{m} =_{\mathrm{df}} \max\{j, i_1, \ldots, i_{\mathfrak{k}}\}$.

For $x \in t$, $F_0(x) \in y \in R$, so $F_0(x) \in R$. $F_1(x) = F_0``x \subseteq F_0``t = y$, so $F_1(x) \in \mathcal{P}(y) \in R$ so $F_1(x) \in R$. $F_2(x) = F_1``x \subseteq \mathcal{P}(y)$ so $F_2(x) \in \mathcal{P}(\mathcal{P}(y)) \in R$, so $F_2(x) \in R$, and so on till we have shown for $0 \leq i \leq \mathfrak{m}$ that $F_i(x) \in (\mathcal{P})^i(y)$ and so $F_i(x) \in R$.

On the other hand, for each $x \in u$, $F(x) = x$ and so for each i, $F_i(x) = x$: here we use the fact that u is transitive. In particular, $F_{i_\ell}(a_\ell) = a_\ell$ for $1 \leq \ell \leq k$.

We may now apply the preservation of stratified formulæ under the F's to deduce from $\Psi(y, a_1, \ldots, a_{\mathfrak{k}})$ that $\Psi(F_{\mathsf{j}}(y), F_{i_1}(a_1), \ldots, F_{i_{\mathfrak{k}}}(a_{\mathfrak{k}}))$, *i.e.* that $\Psi(F_{\mathsf{j}}(y), a_1, \ldots, a_{\mathfrak{k}})$. Thus $F_{\mathsf{j}}(y)$ is the desired witness in R, and our proof is complete. ⊣ (7·2)

Proof of Theorem 14: let $\mathfrak{A}$ be a stratifiable formula without free variables. If $\mathfrak{A}$ is provable in MOST then $(\mathfrak{A})^1$ is provable in MAC, by Theorem 3 and Theorem 2·28, and so $\mathfrak{A}$ is provable in MAC, by Proposition 7·2. The corresponding results that for stratifiable sentences, M + H is conservative over M, Z + H over Z and ZC + H over ZC follow from 7·2 and 2·45.

⊣ (Theorem 14)

Stratifiable schemata provable in MAC

7·3 PROPOSITION MAC *proves strong stratifiable* Σ_1 *Collection.*

Proof : We shall apply our conservative extension result. We have a stratifiable $\Psi(x, y, b, c)$ in which u and v have no occurrence, though x and y may do.

We know that since Ψ is Σ_1, MAC + H proves the sentence

$$\forall b \forall c \forall u \exists v \forall x{:}{\in} u \left(\exists y \Psi(x, y, b, c) \implies \exists y{:}{\in} v \; \Psi(x, y, b, c)\right).$$

That is still stratifiable: take a stratification of Ψ and extend it by assigning to u type 1 more than that assigned to x in Ψ, and likewise to v type 1 more than that of y. By Theorem 13, the sentence is a theorem of MAC. ⊣ (7·3)

7·4 COROLLARY MAC *proves strong stratifiable* Σ_1 *Replacement and stratifiable* Σ_1 *Separation.*

⊣ (Theorem 15)

7·5 PROPOSITION MAC *proves stratifiable* Π_1 *Foundation and stratifiable* Σ_1 *Foundation.*

Proof : immediate, using stratifiable Σ_1 separation and TCo. ⊣ (7·5)

7·6 REMARK We shall see in §9 that ZC cannot prove schemes of stratifiable Π_1 or $\Delta_0^{\mathcal{P}}$ Collection.

Coret's argument in KF and MAC

Coret showed in [B1] that the scheme of strong stratifiable Replacement is provable in Z. His proof, as presented by Boffa, proceeds in two steps: the first is to show that if $\Phi(x_1, \ldots x_n, y)$ is stratified with types

$i_1, \ldots, i_n, j$ respectively, and for given $x_1, \ldots x_n$ there is exactly one y such that Φ, then that y is a member of $\mathcal{P}^{j+1}(\bigcup^{i_1} x_1 \cup \ldots \cup \bigcup^{i_n} x_n)$ — were it not to be, there would be a permutation T of the universe with each $T_{i_\ell}(x_\ell) = x_\ell$ but $T_j(y) \neq y$; he then applies elementary set theory plus an instance of the full Separation scheme of Z to conclude that the image of a set under a partial function defined by a stratifiable formula is a set.

The first part of Coret's argument may be carried out in KF. For completeness, here are the details, following a manuscript of Boffa.

We first establish two properties of the sequence of functions F_i:

7·7 PROPOSITION SCHEME *(i) If* $\forall x \in u\ F(x) = x$ *then* $F_1(u) = u$;
(ii) If $\forall x \in \bigcup u\ F(x) = x$ *then* $F_2(u) = u$;
(iii) for each j, *if* $\forall x \in \bigcup^j u\ F(x) = x$, *then* $F_{j+1}(u) = u$.

Proof of (ii): Under the given hypothesis,

$$F_2(u) = \{F_1(v) \mid v \in u\} = \{\{F(w) \mid w \in v\} \mid v \in u\} = \{\{w \mid w \in v\} \mid v \in u\} = \{v \mid v \in u\} = u$$

as each w is in $\bigcup u$. ⊣ (7·7)

7·8 PROPOSITION SCHEME *(i)* $\bigcup F_2(y) = F``(\bigcup y)$;
(ii) $\bigcup^2 F_3(y) = F``(\bigcup^2 y)$;
(iii) for each j, $\bigcup^j F_{j+1}(y) = F``(\bigcup^j y)$.

Proof of (ii):

$$\begin{aligned} a \in \bigcup\bigcup F_3(y) &\iff \exists c :\in F_3(y)\ \exists b :\in c\ \ a \in b \\ &\iff \exists x :\in y\ \exists b :\in F_2(x)\ \ a \in b \\ &\iff \exists x :\in y\ \exists w :\in x\ \ a \in F_1(w) \\ &\iff \exists x :\in y\ \exists w :\in x\ \exists v :\in w\ \ a = F(v) \\ &\iff a \in F``\bigcup\bigcup y \end{aligned}$$

⊣ (7·8)

The following is the nub of Coret's argument, though presented here for functions of several variables.

7·9 PROPOSITION *Let* $\Phi(x_1, \ldots x_n, y)$ *be stratified with types* $i_1, \ldots, i_n, j$ *respectively. Then it is provable in* KF *that if for given* $x_1, \ldots x_n$ *there is exactly one* y *such that* Φ, *then that* y *is a member of* $\mathcal{P}^{j+1}(\bigcup^{i_1} x_1 \cup \ldots \cup \bigcup^{i_n} x_n)$

Proof: For the $x_1, \ldots, x_n, y$ under consideration, it suffices to prove that $\bigcup^j y \subseteq \bigcup^{i_1} x_1 \cup \ldots \cup \bigcup^{i_n} x_n$, for then $y \in \mathcal{P}^{j+1} \bigcup^j y \subseteq \mathcal{P}^{j+1}\big(\bigcup^{i_1} x_1 \cup \ldots \cup \bigcup^{i_n} x_n\big)$.

If the inclusion we want is false, let $a \in \bigcup^j y$, $a \notin \bigcup^{i_1} x_1 \cup \ldots \cup \bigcup^{i_n} x_n$, and $b \notin \bigcup^j y \cup \bigcup^{i_1} x_1 \cup \ldots \cup \bigcup^{i_n} x_n$. Let F be the transposition of a and b. Then by the first property above of the sequence F_i, $F_{i_1+1}(x_1) = x_1, \ldots, F_{i_n+1}(x_n) = x_n$; hence $\Phi(x_1, \ldots x_n, F_{j+1}(y))$ and so by the uniqueness of y, $y = F_{j+1}(y)$. From this and the second property of sequences established above, $\bigcup^j y = \bigcup^j F_{j+1}(y) = F``\bigcup^j y \neq \bigcup^j y$, a contradiction. ⊣ (7·9)

Proof of Theorem 12: continuing the notation of Proposition 7·9, if $x_1 \in a_1, \ldots x_n \in a_n$, then the given y is in $\mathcal{P}^{j+1}(\bigcup^{i_1+1} a_1 \cup \ldots \cup \bigcup^{i_n+1} a_n)$, and we take this last set for our v. ⊣ (Theorem 12)

7·10 COROLLARY KF *proves strong stratifiable* $\Delta_0^{\mathcal{P}}$ *Replacement.*

Proof: We have u, and a stratifiable $\Delta_0^{\mathcal{P}}$ function G; working in KF we have found v such that $G``u \subseteq v$. We now apply stratifiable $\Delta_0^{\mathcal{P}}$ separation to conclude that $v \cap \{x \mid \exists y :\in u\ (x, y) \in G\}$ is a set; but that set is exactly the image of u under G. ⊣ (7·10)

7·11 COROLLARY MAC *proves strong stratifiable* Π_1 *Replacement.*

Proof: we suppose that for all $x \in a$ there is at most one y such that $\Psi(x, y, b)$ where Ψ is stratifiable Π_1, say of the form $\forall w \Phi$ where Φ is stratifiable Δ_0. By Coret there is a v such that $\forall x :\in a\ \forall y(\Psi(x, y, a) \implies y \in v)$.

By Proposition 7·3 we may apply strong stratifiable Δ_0 collection, and we then infer that there is a set A such that

$$\forall x :\in a\ \forall y \in v(\exists w \neg\Phi \implies \exists w :\in A\ \neg\Phi).$$

So $v \cap \{y \mid \exists x{:}\in A\ \forall w{:}\in A\ \Phi\}$ is by Δ_0 separation, a set, and is the desired image of a by the partial function defined by Ψ. ⊣ (7·11)

⊣ (Theorem 13)

8: The simple theory of types.

The simple theory of types was developed independently by Chwistek and Ramsey as a simplification of the earlier *ramified* theory of types proposed by Russell and Whitehead. Of the various formulations found in the literature, we shall consider only two: that studied by Kemeny in his Princeton thesis of 1949, and that by Forster and his collaborators in various writings. We call their systems, when taken without an axiom of infinity, respectively TKT and TST, and, when with, TKTI and TSTI. ¶

That M has something of the flavour of type theory is plain from the previous section. Indeed, many writers have remarked the equiconsistency of the simple theory of types including an axiom of infinity with a system of set theory such as Mac Lane set theory. The equiconsistency, though, is asymmetric: in one direction there is an interpretation in Tarski's sense, and the relative consistency is provable in arithmetic. It might be said that one only needs partial information about a model of M to build an initial segment of a model of TSTI. In the other direction, the difficulty is much greater, and the methods of the present paper require the existence of a model of the whole of TSTI at the outset, from which one can, given a sufficiently strong metatheory, build a model of M. We shall find that the relative consistency in this direction and, therefore, the equiconsistency are provable in analysis, that is, in second-order arithmetic.

In this section we sketch a proof of the equiconsistency, for we shall need that fact for our final mathematical section, in which we establish the unprovability of various hypotheses in MAC.

Our method of proof there is somewhat oblique. We exhibit a set-theoretic formula $\Phi(n)$ such that $\forall n{:}\in\omega\ \Phi(n)$ is derivable in M from various assumptions such as Induction(introduced in §9), Π_2 Foundation, or $\Delta_0^{\mathcal{P}}$ Collection, and we show that the theory $\mathsf{M} + \forall n{:}\in\omega\ \Phi(n)$ proves the consistency of TSTI.

Then we shall recover from that consistency statement the consistency of the theory MAC, and invoke the second Incompleteness Theorem of Gödel to deduce the failure of Induction and the other schemata in MAC. Thus the mathematical aims of the present section are dictated by the needs of that argument, which we shall refine to avoid the Axiom of Infinity where possible and to incorporate other theories such as KF into our equiconsistency results. So our present programme is:

1: to introduce the version TST of the simple theory of types;

2: to show in arithmetic that the consistency of KF implies that of TST, and that the consistency of KFI implies that of TSTI ;

3: to derive in analysis the consistency of M_0 from that of TST, and the consistency of $\mathsf{M}_0 + \mathsf{InfWel}$ from that of TSTI;

4: to sketch Kemeny's variant TKTI;

5: to complete the set of equiconsistencies by showing that TST may be interpreted in TKT, TSTI in TKTI, TKT in M_0, TKTI in M, and MAC in $\mathsf{M}_0 + \mathsf{InfWel}$.

The process needed in the third step, of getting from a model of type theory to a model of set theory, has been considerably illuminated by a paper of Forster and Kaye on their weak set theory KF, and we shall adopt their method.

The system TST

We consider a typed language: each variable and each constant has a type, shown as a subscript, which is a natural number. Thus x_k denotes a variable of type k. For each type n we have a 2-place relation $\in_n$. The atomic formula $x_n \in_n y_m$ is well-formed provided $m = n + 1$. We shall have a equality symbol $=_n$ for each type; $x_m =_n y_k$ is well-formed only if $m = n = k$.

¶ For further reading, Gandy's review [G4] and Forster's book [B4] are suggested.

The *simple theory of types*, TST, has as its axioms *Extensionality* for each type m, and at each type, the axiom scheme of *Comprehension*. Thus the axioms are

$$\forall x_m\big((x_m \in_m y_{m+1} \Longleftrightarrow x_m \in_m z_{m+1})\big) \implies y_{m+1} =_{m+1} z_{m+1}$$

and for each formula $\Phi(x_n)$ of the language, possibly with other free variables and with constants,

$$\exists x_{n+1}\forall x_n \, \big(x_n \in_n x_{n+1} \Longleftrightarrow \Phi(x_n)\big)$$

By extensionality, that x_{n+1} will be unique, and we denote it by $\{x_n \mid \Phi(x_n)\}_{n+1}$, the subscript $n+1$ being a reminder that the object is of type $n+1$. For other constructs, too, we use subscripts to remind readers of their type. Thus $\{x_k, y_k\}_{k+1}$ is the term of type $k+1$ which the axioms of TST prove to exist and to have as sole members the objects x_k and y_k of type k.

We introduce at each positive type the subset relation by definition:

$$y_{n+1} \subseteq_{n+1} z_{n+1} \iff_{\mathrm{df}} \forall x_n\big(x_n \in_n y_{n+1} \implies x_n \in_n z_{n+1}\big)$$

The relation $x_0 \subseteq_0 y_0$ at type 0 is not defined.

An *identity model* of TST is a structure $\mathcal{T} = \langle T^0, T^1, \ldots; E^0, E^1, \ldots\rangle$ which is a sequence of sets T^i and binary relations $E^i \subseteq T^i \times T^{i+1}$, such that the axioms of TST become true when variables of type k are interpreted as ranging over the members of T^k, constants of type k, (if any there be) as denoting some member of T^k, $=_k$ is interpreted as identity restricted to T^k, and $\in_k$ is interpreted as E^k. We write $\sqsubseteq^{k+1}$ for the relation on T^{k+1} that is the induced interpretation of $\subseteq_{k+1}$.

When discussing such a model, we may add a superscript $\mathcal{T}$ in naming the interpretation in the model of a term of TST. Thus $\{x_k, y_k\}^{\mathcal{T}}_{k+1}$ would be the object in T^{k+1} which the model $\mathcal{T}$ believes to have as sole members the objects x_k and y_k in T^k.

Note that the sets T^i need not be disjoint. If for example we take T^0 to be ω, and T^1 to be $\mathcal{P}(\omega)$, T^0 is actually a subset of T^1, ω being transitive; but these strange coincidences are simply not expressible in our presentation of TST, and cause no problem: thus the emphasis is different from those studied by Boffa in his paper [G7] on cumulative models of type theory.

We assume that T^0 is not empty.

The interpretation of TST in KF

8·0 We sketch a proof — whose underlying idea is to be found in Kemeny's thesis, pages 7 and 8 — that if Consis(KF) then Consis(TST) by getting an interpretation of TST in KF. Let Ω be any set. For each concrete type formula Φ let $(\Phi)^3$ be the result of restricting the variables of type 0 to Ω, the variables of type 1 to $\mathcal{P}(\Omega)$, and so on. Then each axiom of TST becomes a theorem of KF: for example, consider the following instance of the comprehension scheme:

$$\forall a_2 \forall b_5 \exists c_4 \forall d_3 \, \big[d_3 \in_3 c_4 \Longleftrightarrow \Phi(a_2, b_5, d_3)\big].$$

That translates to

$$\forall a{:}{\in}\mathcal{P}^2(\Omega)\forall b{:}{\in}\mathcal{P}^5(\Omega)\exists c{:}{\in}\mathcal{P}^4(\Omega)\forall d{:}{\in}\mathcal{P}^3(\Omega) \, \big[d \in c \Longleftrightarrow \Phi^*(a, b, d)\big]$$

where Φ^* is a further translation of Φ.

That is provable, because it is an instance of Δ_0 separation, indeed a stratifiable instance since the formula Φ^* originates in a formula of TST.

If we take Ω to be an infinite set as provided by an Axiom of Infinity for KF, we shall get an interpretation of TSTI —the simple theory of types with a corresponding axiom of infinity — in KFI.

A direct limit of types: the method of Forster and Kaye

8·1 Definition The *Forster–Kaye axiom*, SC, is the assertion that *for each set x the restriction of the map $y \mapsto \{y\}$ to x is a set;*

in symbols, $\forall x\ \mathfrak{S} \restriction x \in V$, where we write $\mathfrak{S}$ for the singleton function $y \mapsto \{y\}$ that assigns to each set y its singleton.

In the terminology of NF-istes, the axiom states that every set is strongly cantorian. SC is easily proved in M_0 using the power set axiom and (unstratified !) Δ_0 separation; as the system KF is the system M_0 with the separation scheme confined to formulæ that are both Δ_0 and stratified, KF + SC is a subsystem of M_0, but not a proper subsystem in view of Theorem 8·3.

8·2 HISTORICAL NOTE The central device in the following proof, that of finding names at lower types for a strongly cantorian object, has long been NF folk-lore.

8·3 THEOREM *The scheme of Δ_0 separation is provable in the system* S_0 + strat-Δ_0 Separation + SC.

Proof : Let $\Phi(x, b)$ be a Δ_0 formula with the distinct free variables x, b. Let a be a variable distinct from both. We shall prove that $\forall a \forall b\ a \cap \{x \mid \Phi(x, b)\} \in V$.

Suppose that Φ is in the natural prenex form for a Δ_0 formula, with all its restricted quantifiers outside the quantifier-free matrix, and that there are exactly $\mathfrak{l}$ occurrences of restricted quantifiers in Φ, and that distinct occurrences of quantifiers bind distinct variables: say for $1 \leqslant i \leqslant \mathfrak{l}$, the i^{th} quantifier, $\mathcal{Q}_i$, counting from the left, binds the variable y^i, which is distinct from the variables x, a and b.

We invoke part of our rewriting system of §6: S_0 proves such equivalences as

$$\begin{aligned} x \in a &\implies [\exists y \mathbin{:\in} x\ \mathfrak{A} \iff \exists y \mathbin{:\in} \textstyle\bigcup a\ (y \in x \ \&\ \mathfrak{A})] \\ y \in \textstyle\bigcup d &\implies [\forall z \mathbin{:\in} y\ \mathfrak{A} \iff \forall z \mathbin{:\in} \textstyle\bigcup\bigcup d\ (z \in y \implies \mathfrak{A})] \\ z \in \textstyle\bigcup^2 d &\implies [\exists w \mathbin{:\in} z\ \mathfrak{A} \iff \exists w \mathbin{:\in} \textstyle\bigcup^3 d\ (w \in z \ \&\ \mathfrak{A})] \end{aligned}$$

Using those, and in particular treating the variable x as restricted by a, we write Φ in Second Limited Normal Form, so that every quantifier is restricted by a term t_i of the form $(\bigcup^{\mathfrak{k}+1} a)$ or $(\bigcup^{\mathfrak{k}} b)$, where $\mathfrak{k}$ is some natural number of the meta-language, dependent on i and determined by the length of the relevant chain of nested restrictions. Note that for each i, S_0 proves $t_i \in V$.

We then give ourselves a supply of new variables, $c^0, c^1, \ldots c^{\mathfrak{l}}$. Our rewriting will have led us to a quantifier-free formula $\Theta(x, y^1, \ldots, y^{\mathfrak{l}}, b, c^0, \ldots, c^{\mathfrak{l}})$ such that S_0 proves the following:

$$[c^0 = a \ \&\ c^1 = t_1 \ \&\ \ldots \ \&\ c^{\mathfrak{l}} = t_{\mathfrak{l}} \ \&\ x \in c^0] \implies \big(\Phi(x, b) \iff \mathcal{Q}_1 y^1 \mathbin{:\in} c^1\ \mathcal{Q}_2 y^2 \mathbin{:\in} c^2\ \ldots \mathcal{Q}_{\mathfrak{l}} y^{\mathfrak{l}} \mathbin{:\in} c^{\mathfrak{l}}\ \Theta(x, \vec{y}, b, \vec{c})\big).$$

It will be convenient to call the letters x, $y^1 \ldots, y^{\mathfrak{l}}$ *variables* and the letters a, b, c^0, $c^1 \ldots, c^{\mathfrak{l}}$ *parameters*. With the help of the formula Θ and three as yet unused letters, say f, g, h, we shall create a Δ_0 formula $\Psi(x, b, \vec{c}, f, g, h)$ that will be stratifiable by assigning type 1 to each variable, type 2 to each parameter, and types 4, 5, 4 respectively to f, g, and h, such that under appropriate conditions on $\vec{c}$, f, g and h, $\Phi(x, b)$ will, for $x \in a$, be equivalent to $\Psi(x, b, \vec{c}, f, g, h)$.

Set $V_\Phi =_{\text{df}} a \cup t_1 \cup \ldots \cup t_{\mathfrak{l}}$ and $P_\Phi =_{\text{df}} \{a, t_1, \ldots, t_{\mathfrak{l}}\}$. S_0 proves that both V_Φ and P_Φ are sets. The intended values of the variables lie in V_Φ, and those of the parameters in P_Φ. Our intention is that $f = \mathfrak{S} \restriction V_\Phi$ and that $g = h = \mathfrak{S} \restriction P_\Phi$, both of which restrictions of $\mathfrak{S}$ are, by SC, sets.

The formula Θ is a Boolean combination of atomic formulæ, each of the form $r \in s$ or $r = s$ for r and s variables or parameters. Let us denote arbitrary, possibly identical, variables by w and z, and arbitrary, possibly identical, parameters by d and e.

Let Ψ_0 be the formula that results from Θ by making simultaneously the following replacements, which include the addition of type suffices to variables, parameters and the letters f, g, and h:

replace	$w = z$	by	$w_1 = z_1$	replace	$w \in e$	by	$w_1 \in e_2$
replace	$w \in z$	by	$f_4(w_1) \subseteq z_1$	replace	$e \in w$	by	$h_4\big(\bigcup g_5(e_2)\big) \subseteq w_1$
replace	$w = e$	by	$w_1 = \bigcup g_5(e_2)$	replace	$d = e$	by	$d_2 = e_2$
replace	$e = w$	by	$\bigcup g_5(e_2) = w_1$	replace	$d \in e$	by	$g_5(d_2) \subseteq e_2$

Finally let $\Psi(x, b, \vec{c}, f, g, h)$ be the formula $\mathcal{Q}_1 y_1^1 {:\in}\, c_2^1\ \mathcal{Q}_2 y_1^2 {:\in}\, c_2^2\ \ldots \mathcal{Q}_\mathfrak{l} y_1^\mathfrak{l} {:\in}\, c_2^\mathfrak{l}\ \Psi_0(x_1, \vec{y}_1, b_2, \vec{c}_2, f_4, g_5, h_4)$. Ψ is Δ_0 and stratified.

The intuition behind those replacements is this: in the world of stratified formulæ a function maintains type values, whereas $\bigcup$ lowers them by 1; and with the Wiener–Kuratowski definition of ordered pair, a unary function must be 3 types higher than its argument. Now $\bigcup \mathfrak{S}(p) = p$, so provided p is in the domain of some set that is a restriction of the function $\mathfrak{S}$, $\bigcup \mathfrak{S}(p)$ names the same object as p but at one type lower.

Hence, provably in $\mathsf{S}_0 + \mathsf{SC}$, and omitting type suffices,

$$\big(c^0 = a \,\&\, c^1 = t_1 \,\&\, \ldots \,\&\, c^\mathfrak{l} = t_\mathfrak{l} \,\&\, f = \mathfrak{S}\restriction V_\Phi \,\&\, g = h = \mathfrak{S}\restriction P_\Phi\big) \implies \forall x {:\in}\, a\ \big(\Phi(x, b) \iff \Psi(x, b, \vec{c}, f, g, h)\big)$$

Now it is an instance of stratifiable Δ_0 separation that

$$\vec{\forall}\vec{c}\forall f \forall g \forall h\ a \cap \{x \mid \Psi(x, b, \vec{c}, f, g, h)\} \in V;$$

since our intended values for $\vec{c}$, f, g and h are all, provably in $\mathsf{S}_0 + \mathsf{SC}$, sets, we may conclude that

$$a \cap \{x \mid \Phi(x, b)\} \in V$$

as required.

We have for the sake of clarity only considered Φ with only one free variable other than x, and that distinct from a; adding further free variables distinct from x and a is easy, and to prove that a class of the form $a \cap \{x \mid \Phi(x, b, a)\}$ is a set, take a new variable d, show that

$$\forall d\ a \cap \{x \mid \Phi(x, b, d)\} \in V$$

and then take the case $d = a$. ⊣ (8·3)

We have established the first part of the following; and the second part is straightforward.

8·4 Theorem (Forster, Kaye) *(i) The theory* M_0 *is exactly* $\mathsf{KF} + \mathsf{SC}$*;*

(ii) Quine's system NF *is exactly* $\mathsf{KF} + V \in V$.

8·5 Suppose we have a structure $\mathcal{T} = \langle T^0, T^1, \ldots; E^0, E^1, \ldots\rangle$ which is a set, and which forms an identity model of TST, the simple theory of types. We shall build a model of KF by defining embeddings $\pi_n^\mathcal{T}$ from each T^n to T^{n+1}, and then taking the direct limit of the resulting directed system.

We begin by defining the sequence of type-raising embeddings. We shall treat them ambiguously as functions $\pi_n^\mathcal{T} : T^n \longrightarrow T^{n+1}$; but also as terms π_n of TST. For $x_0 \in T^0$, $\pi_0^\mathcal{T}(x_0)$ will be $\{x_0\}_1^\mathcal{T}$, the object of type 1 which $\mathcal{T}$ thinks is the singleton of x_0. Thereafter, inductively, we put $\pi_{n+1}^\mathcal{T}(x_{n+1}) = \pi_n^\mathcal{T}``x_{n+1}$, which is an object of type $n+2$, namely the interpretation in $\mathcal{T}$ of

$$\{y_{n+1} \mid \exists z_n\ z_n \in_n x_{n+1} \,\&\, y_{n+1} =_{n+1} \pi_n(z_n)\}_{n+2},$$

an expression from which, for each n, the π's may progressively be eliminated.

Successive π's may be composed: for $\ell < m$ we write $\pi_{\ell,m}$ for the composition $\pi_{m-1} \circ \pi_{m-2} \ldots \circ \pi_\ell$.

8·6 Lemma (TST) *(i)* $x_n =_n y_n$ *iff* $\pi_n(x_n) =_n \pi_n(y_n)$*; thus each* π_n *is an injection.*

(ii) $x_n \in_n y_{n+1}$ *iff* $\pi_n(x_n) \in_{n+1} \pi_{n+1}(y_{n+1})$.

(iii) $x_{n+1} \subseteq_{n+1} y_{n+1}$ *iff* $\pi_{n+1}(x_{n+1}) \subseteq_{n+2} \pi_{n+1}(y_{n+1})$.

8·7 Definition We write $\exists^\infty n \ldots$ to mean that there are infinitely many n such that $\ldots$, and accordingly write $\forall^\infty n \ldots$ to mean $\neg\exists^\infty\neg \ldots$; in other words, that for all sufficiently large n, $\ldots$

8·8 Definition Let $\mathcal{M}_0$ be the set of functions $f \in \prod_{n\in\omega} T^n$ with domain ω such that $\forall^\infty n\ \pi_n^\mathcal{T}(f(n)) = f(n+1)$. On that set we define relations:

$$f \sim g \iff_{\mathrm{df}} \forall^\infty n f(n) = g(n)$$
$$f E g \iff_{\mathrm{df}} \forall^\infty n f(n) E^n g(n+1)$$
$$f \sqsubseteq g \iff_{\mathrm{df}} \forall^\infty n f(n) \sqsubseteq^n g(n).$$

We may consider the model $\mathcal{M}(\mathcal{T})$ to have as its underlying set the equivalence classes of $\mathcal{M}_0$ *modulo* the equivalence relation $\sim$, and to have as its membership relation the relation induced between those equivalence classes by the relation E on $\mathcal{M}_0$. The relation $\sqsubseteq$ then corresponds to the subset relation in $\mathcal{M}(\mathcal{T})$.

8·9 Which axioms of set theory are true in $\mathcal{M}(\mathcal{T})$? There is one obvious failure, given that T^0 is non-empty, in the model that we have built, namely the Axiom of Foundation, for let a_0 be an object of type 0, and let $f(0)$ be a_0 and $f(n+1) = \pi_n(f(n))$ for every n. Then fEf; indeed in our model, f is a *Quine atom*, that is, a set which equals its own singleton.

8·10 DEFINITION The *start* of a member x of $\mathcal{M}(\mathcal{T})$, *start*(x), is the least natural number k such that x is represented by a function, f say, which for all $n \geqslant k$ observes *the π-rule* that $\pi_n^{\mathcal{T}}(f(n)) = f(n+1)$.

We may write such x as $[x_k]$ where $k =$ *start*(x).

8·11 LEMMA *(i)* $x_0 E^0 \pi_0(y_0) \iff x_0 = y_0$;

(ii) $x_{n+1} E^{n+1} \pi_{n+1}(y_{n+1}) \iff \exists x_n [x_n E^n y_{n+1} \;\&\; x_{n+1} = \pi_n(x_n)]$

Rephrasing in terms of members of $\mathcal{M}(\mathcal{T})$ gives us:

8·12 LEMMA *(i)* $[x_0]E[y_0] \iff x_0 = y_0$; *(ii) for* $\ell < m$, $[x_\ell]E[y_m] \iff \pi_{\ell,m-1}(x_\ell)E^{m-1}y_m$;

(iii) if xEy *then either start*$(y) = 0 \;\&\; x = y$ *or start*$(y) > 0 \;\&\;$ *start*$(x) <$ *start*(y).

Proof of (ii):

$$\begin{aligned} x_\ell E y_m &\iff \pi_{\ell,m}(x_\ell) E^m \pi_m(y_m) \\ &\iff \exists z_{m-1} : E^{m-1} y_m \; \pi_{\ell,m}(x_\ell) = \pi_{m-1}(z_{m-1}) \\ &\iff \pi_{\ell,m-1}(x_\ell) E^{m-1} y_m \end{aligned}$$

⊣ (8·12)

8·13 PROPOSITION *(i) The only members of* $\mathcal{M}(\mathcal{T})$ *with start 0 are Quine atoms;*

(ii) the empty set has start 1;

(iii) if start$(y) > 0$ *and* xEy *then start*$(x) <$ *start*(y)*;*

(iv) if xEx *then start*$(x) = 0$ *and* x *is a Quine atom of* $\mathcal{M}(\mathcal{T})$*;*

(v) if start$(y) > 0$ *and* $x \sqsubseteq y$ *then start*$(x) \leqslant$ *start*(y).

Proof : immediate. Parts (ii) and (iv) make the hypothesis *start*$(y) > 0$ necessary for Part (v). ⊣ (8·13)

8·14 REMARK It would be ontologically simpler to think of the universe of the model $\mathcal{M}(\mathcal{T})$ as

$$T^0 \cup \bigcup_n (T^{n+1} \setminus \pi_n``T^n),$$

since that set forms an exact transversal of the equivalence classes; with the above lemma in mind, one would then *define* the relation E on that transversal by the equations

$$x_0 E y_0 \iff x_0 = y_0; \qquad \text{for } \ell < m,\; x_\ell E y_m \iff \pi_{\ell,m-1}(x_\ell)E^{m-1}y_m$$

To this author at least, the "direct limit" version is conceptually simpler and we shall therefore continue to have it in mind; but at the end of the section, where we must minimise the strength of the set theory necessary for this construction, this second approach will be helpful.

We pause briefly, now that we have the concept of a Quine atom, to justify Remark 1·26.

8·15 REMARK Consider an infinite set X of Quine atoms, and iterate the power set operation ω times over X. Call the result Q. AxInf, ExInf and InfWel are true in Q, but $(W_0)^Q$ will be precisely the hereditarily finite sets, V_ω, and hence contain no infinite set.

8·16 Our pause over, we proceed to verify that in $\mathcal{M}(\mathcal{T})$, each axiom of KF is true, as also the Forster–Kaye axiom SC; and to establish that various forms of the axiom of infinity will transfer satisfactorily from $\mathcal{T}$ to $\mathcal{M}(\mathcal{T})$. Our verification takes the form of a series of lemmata of type theory, stated formally and schematically with at most the baldest of proofs, accompanied by a commentary, stated less formally, on the significance of those lemmata for our model.

Extensionality: We must show that if $f \sqsubseteq g$ and $g \sqsubseteq f$ then $f = g$; but that follows from the truth of extensionality at each type level in $\mathcal{T}$.

Empty Set: let $\varnothing_k$ be for $k > 0$ the empty set of type k in T^k, and consider the function $(*, \varnothing_1, \varnothing_2, \ldots)$, where * denotes an arbitrary element from the correct type, here 0.

8·17 LEMMA (TST) $\pi_k(\varnothing_k) =_{k+1} \varnothing_{k+1}$.

That Lemma proves that the function in question is indeed in $\mathcal{M}_0$; it represents the empty set of $\mathcal{M}(\mathcal{T})$.

Pair set: given f and g, set $h(0) = *$, and $h(n+1) = \{f(n), g(n)\}_{n+1}$. That h will be in $\mathcal{M}_0$ if both f and g are, and will represent the pair set of the elements represented by f and g, follows from this

8·18 LEMMA (TST) $\pi_{k+1}\big(\{x_k, y_k\}_{k+1}\big) =_{k+2} \{\pi_k(x_k), \pi_k(y_k)\}_{k+2}$.

Union:

8·19 LEMMA *The union* $\bigcup_{n+1}(x_{n+2})$ *of an object of type* $n+2$ *is of type* $n+1$*, namely*

$$\big\{z_n \mid \exists y_{n+1}\ z_n \in_n y_{n+1} \in_{n+1} x_{n+2}\big\}_{n+1}.$$

8·20 LEMMA (TST) $\pi_{n+1}\big(\bigcup_{n+1}(x_{n+2})\big) =_{n+2} \bigcup_{n+2}\big(\pi_{n+2}(x_{n+2})\big)$.

Proof :

$$\begin{aligned}
\textstyle\bigcup_{n+2}\big(\pi_{n+2}(x_{n+2})\big) &=_{n+2} \textstyle\bigcup_{n+2}\big(\pi_{n+1}``x_{n+2}\big)\\
&=_{n+2} \big\{z_{n+1} \mid \exists t_{n+1}\ \big(t_{n+1} \in_{n+1} x_{n+2}\ \&\ z_{n+1} \in_{n+1} \pi_{n+1}(t_{n+1})\big)\big\}_{n+2}\\
&=_{n+2} \big\{z_{n+1} \mid \exists s_n\ \exists t_{n+1}\ \big(s_n \in_n t_{n+1}\ \&\ t_{n+1} \in_{n+1} x_{n+2}\ \&\ z_{n+1} =_{n+1} \pi_n(s_n)\big)\big\}_{n+2}\\
&=_{n+2} \big\{z_{n+1} \mid \exists s_n :\in_n \textstyle\bigcup_{n+1}(x_{n+2})\ z_{n+1} =_{n+1} \pi_n(s_n)\big\}_{n+2}\\
&=_{n+2} \pi_n``\big(\textstyle\bigcup_{n+1}(x_{n+2})\big)\\
&=_{n+2} \pi_{n+1}\big(\textstyle\bigcup_{n+1}(x_{n+2})\big). \qquad \dashv (8{\cdot}20)
\end{aligned}$$

Thus given a set in $\mathcal{M}(\mathcal{T})$ represented by the function $f \in \mathcal{M}_0$, the function g defined by setting $g(0) = *$ and $g(n+1) = \bigcup_{n+1}(f(n+2))$ will be in $\mathcal{M}_0$ and will represent the union in $\mathcal{M}(\mathcal{T})$ of that set.

Difference:

8·21 LEMMA *The difference of two objects of type* n*, at least 1, is an object of type* n*.*

8·22 LEMMA (TST) $\pi_n(x_n \smallsetminus_n y_n) =_{n+1} \pi_n(x_n) \smallsetminus_{n+1} \pi_n(y_n)$.

Thus if A and B in $\mathcal{M}(\mathcal{T})$ are represented by functions $n \mapsto a_n$, $n \mapsto b_n$, the difference $(A \smallsetminus B)^{\mathcal{M}(\mathcal{T})}$ will be represented by the function $n \mapsto a_n \smallsetminus_n b_n$.

Power Set:

8·23 LEMMA *The power set* $\mathcal{P}_{n+2}(x_{n+1})$ *of an object of type* $n+1$ *is an object of type* $n+2$*, namely*

$$\big\{y_{n+1} \mid y_{n+1} \subseteq_{n+1} x_{n+1}\big\}_{n+2}.$$

8·24 LEMMA (TST) $\pi_{n+2}\big(\mathcal{P}_{n+2}(x_{n+1})\big) =_{n+3} \mathcal{P}_{n+3}(\pi_{n+1}(x_{n+1}))$.

Proof:

$$\begin{aligned}
\pi_{n+2}\big(\mathcal{P}_{n+2}(x_{n+1})\big) &=_{n+3} \pi_{n+1}``\big(\mathcal{P}_{n+2}(x_{n+1})\big)\\
&=_{n+3} \{\pi_{n+1}(y_{n+1}) \mid y_{n+1} \subseteq_{n+1} x_{n+1}\}_{n+3};\\
\mathcal{P}_{n+3}(\pi_{n+1}(x_{n+1})) &=_{n+3} \{z_{n+2} \mid z_{n+2} \subseteq_{n+2} \pi_{n+1}(x_{n+1})\}_{n+3};
\end{aligned}$$

By Lemma 8·6, part (iii), $y_{n+1} \subseteq_{n+1} x_{n+1} \iff \pi_{n+1}(y_{n+1}) \subseteq_{n+2} \pi_{n+1}(x_{n+1})$; and each $z_{n+2} \subseteq_{n+2} \pi_{n+1}(x_{n+1})$ is of the form $\pi_{n+1}(y_{n+1})$ for some y_{n+1} with $y_{n+1} \subseteq_{n+1} x_{n+1}$. ⊣ (8·24)

Thus if given $f \in_M \mathsf{M}_0$, we set $g(0) = *$, $g(1) = *$ and $g(n+2) = \mathcal{P}_{n+2}(f(n+1))$, g will be in $\mathcal{M}_0$ and will represent the power set of the set represented by f in $\mathcal{M}(\mathcal{T})$.

The scheme of stratifiable Δ_0 separation:

8·25 LEMMA *Let $\varphi(x_n, a_k)$ be a stratified Δ_0 formula. Then*

$$\vdash_{\mathsf{TST}} \varphi(x_n, a_k) \iff \varphi^+\big(\pi_n(x_n), \pi_k(a_k)\big),$$

where φ^+ is the result of increasing the type of each bound variable in φ by 1.

Proof : by induction on the length of φ, much as for Proposition 7·1. For atomic stratified formulæ apply Lemma 8·6, parts (i) and (ii). Propositional connectives present no problem. For restricted quantifiers, note that for ψ a proper sub-formula of φ, the induction hypothesis gives

$$\psi(x_n, y_\ell, b_{\ell+1} \ldots) \iff \psi^+(\pi_n(x_n), \pi_\ell(y_\ell), \pi_{\ell+1}(b_{\ell+1}) \ldots),$$

whence, and from the definition of $\pi_{\ell+1}$,

$$\exists y_\ell :\in_\ell b_{\ell+1}\ \psi(x_n, y_\ell, b_{\ell+1} \ldots) \iff \exists y_{\ell+1} :\in_{\ell+1} \pi_{\ell+1}(b_{\ell+1})\ \psi^+(\pi_n(x_n), y_{\ell+1}, \pi_{\ell+1}(b_{\ell+1}) \ldots) \qquad \dashv (8{\cdot}25)$$

8·26 REMARK We have not yet defined the intersection $x_n \cap_n y_n$ of two objects of the same type, n, but can easily do so as $x_n \smallsetminus_n (x_n \smallsetminus_n y_n)$: evidently another object of type n.

8·27 PROPOSITION *Let $\varphi(x_n, a_k)$ be a stratified Δ_0 formula. Then*

$$\vdash_{\mathsf{TST}} \pi_{n+1}\big(b_{n+1} \cap_{n+1} \{x_n \mid \varphi(x_n, a_k)\}_{n+1}\big) =_{n+2} \pi_{n+1}(b_{n+1}) \cap_{n+2} \big\{x_{n+1} \bigm| \varphi^+\big(x_{n+1}, \pi_k(a_k)\big)\big\}_{n+2}.$$

Proof : by Lemma 8·25, again with the fact in mind that the members of $\pi_{n+1}(b_{n+1})$ are the objects x_{n+1} of the form $\pi_n(x_n)$ for some $x_n \in_n b_{n+1}$. ⊣ (8·27)

The Proposition shows that if Φ is a stratifiable Δ_0 formula of the language of set theory, and B, A are members of $\mathcal{M}(\mathcal{T})$, represented by the functions $n \mapsto b_n$, $n \mapsto a_n$ in $\mathcal{M}_0$, $B \cap \{x \mid \Phi^{\mathcal{M}(\mathcal{T})}(x, A)\}$ will be a member of $\mathcal{M}(\mathcal{T})$, represented by the function

$$0 \mapsto *$$
$$n+1 \mapsto \begin{cases} b_{n+1} \cap_{n+1} \{x_n \mid \varphi(x_n, a_{\ell+n})\}^{\mathcal{T}}_{n+1}, & \text{if } \ell + n \geqslant 0 \\ * & \text{otherwise} \end{cases}$$

where the lag ℓ, some positive or negative integer, is determined by the chosen stratification φ of Φ.

The Forster–Kaye axiom: let A in the model $\mathcal{M}(\mathcal{T})$ be represented by a function starting with the member a of $\mathcal{T}$ of type $n+1$. For $x \in_n a$, x of type n, we define functions to represent x, $\{x\}$, ... by giving one value of the function and agreeing that later values of the function are governed by the π-rule.

$$\begin{aligned} g_x(n) &= x \\ g_{\{x\}}(n+1) &= \{x\}^{\mathcal{T}}_{n+1} \\ g_{\{\{x\}\}}(n+2) &= \{\{x\}^{\mathcal{T}}_{n+1}\}^{\mathcal{T}}_{n+2} \\ g_{\{\{x\},\,x\}}(n+2) &= \{g_{\{x\}}(n+1),\ g_x(n+1)\}^{\mathcal{T}}_{n+2} \\ g_{\langle\{x\},\,x\rangle}(n+3) &= \{g_{\{\{x\},\,x\}}(n+2),\ g_{\{\{x\}\}}(n+2)\}^{\mathcal{T}}_{n+3} \\ g_{\mathfrak{S}\restriction A}(n+4) &= \{g_{\langle\{x\},\,x\rangle}(n+3) \mid x \in_n a\}^{\mathcal{T}}_{n+4} \end{aligned}$$

That final term may be written in more detail as

$$\big\{y_{n+3} \bigm| \exists x_n \; x_n \in_n a \;\&\; y_{n+3} =_{n+3} g_{\langle\{x_n\},\, x_n\rangle}(n+3)\big\}^{\mathcal{T}}_{n+4}$$

where we rely on the uniformity of our previous definitions in x to eliminate the g's in favour of type-theoretical expressions and thus show that a member of T^{n+4} is indeed being defined. Our previous lemmata will show that the function $g_{\mathfrak{S} \restriction A}$ is in $\mathcal{M}_0$ and represents $\mathfrak{S} \restriction A$ in the model $\mathcal{M}(\mathcal{T})$.

If A can be represented by some element a_0 of type 0, we are in the case of Quine atoms, and in $\mathcal{M}(\mathcal{T})$, $A = \{A\}$. Hence in that model, we may build $\mathfrak{S} \restriction A$ by hand, as it proves to be $\{\langle A,\, A\rangle\}$.

We conclude from Theorem 8·3 that we have built a model, $\mathcal{M}(\mathcal{T})$, of M_0.

The axiom of infinity:

8·28 First let us remark that if $\mathcal{T}$ fails to satisfy an axiom of infinity, so will $\mathcal{M}(\mathcal{T})$: for if T^0 is empty, we may still create a model of M_0 by requiring our functions only to have domain $\omega \smallsetminus \{0\}$: the result will be the familiar hierarchy of hereditarily finite sets; and if T^0 has exactly k elements for some positive natural number k, the model $\mathcal{T}$ will consist solely of finite sets, and in the model $\mathcal{M}(\mathcal{T})$ the axiom of infinity will be false but the statement that there are exactly k Quine atoms will be true.

In all other cases, there will be an infinite well-ordering in the final model, even if T^0 is non-standard finite in some sense. For then T^2 will certainly have a genuinely infinite well-ordering.

8·29 Now we wish, more specifically, to see how the three forms of the axiom of infinity, AxInf, ExInf and InfWel, that we have proposed for use with KF, will survive the journey from a model of type theory to one of set theory. Each has a natural translation into a type-theoretical formula, expressing that there is, respectively, a set of objects of type 0 with an injection that is not a bijection; a set of objects of type 0 that has no double-well-ordering; a non-empty set of objects of type 0 that has a well-ordering with no last element.

AxInf is the easiest of the three statements to transfer, simply because statements about bijections and injections reduce to statements about various ordered pairs.

The other two statements require an examination of the manner in which subsets of elements of $\mathcal{M}(\mathcal{T})$ arise. We shall see that a well-ordering in the sense of $\mathcal{T}$ will transfer to a well-ordering in the sense of $\mathcal{M}(\mathcal{T})$, and, conversely, every well-ordering in $\mathcal{M}(\mathcal{T})$ will arise from a well-ordering in the sense of $\mathcal{T}$. The following lemma, which amplifies part (v) of Proposition 8·13, is what we need.

8·30 Lemma (TST) *(i)* $z_1 \subseteq_1 \pi_0(y_0) \implies z_1 =_1 \varnothing_1$ *or* $z_1 = \pi_0(y_0)$;

(ii) If $z_{n+2} \subseteq_{n+2} \pi_{n+1}(y_{n+1})$, *then* $\exists t_{n+1}(\pi_{n+1}(t_{n+1}) = z_{n+2})$.

Proof : the first part is immediate, since $\pi_0(y_0)$ is the singleton of y_0. For the second part, take t_{n+1} to be $\{w_n \mid \pi_n(w_n) \in z_{n+2}\}_{n+1}$, and use the fact that $\pi_{n+1}(y_{n+1}) =_{n+2} \pi_n “y_{n+1}$. ⊣ (8·30)

So if b is an element of $\mathcal{M}(\mathcal{T})$ that starts at level k, each "subset" a of b in the sense of that model starts at level k or earlier. Hence the concept of well-ordering is preserved between the two models.

Finally, let us see how the infinite von Neumann ordinals fail to emerge in this context. We have seen how the representative of the empty set starts at type 1 with $\varnothing_1$; and we have seen how to form representatives of unordered pairs. To simulate the von Neumann ordinals $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, \{0\}\}$, $3 = \{0, \{0\}, \{0, \{0\}\}\}$, & c, while preserving correct type levels, we must begin thus:

$$\begin{aligned}
0 &= (*, \varnothing_1, \pi_1(\varnothing_1), \ldots)\\
1 &= (*, *, \{\varnothing_1\}_2, \pi_2(\{\varnothing_1\}_2), \ldots)\\
2 &= (*, *, *, \{\varnothing_2, \{\varnothing_1\}_2\}_3, \pi_3(\{\varnothing_2, \{\varnothing_1\}_2\}_3), \ldots)\\
3 &= (*, *, *, *, \{\varnothing_3, \{\varnothing_2\}_3, \{\varnothing_2, \{\varnothing_1\}_2\}_3\}_4, \pi_4(\{\varnothing_3, \{\varnothing_2\}_3, \{\varnothing_2, \{\varnothing_1\}_2\}_3\}_4) \ldots)
\end{aligned}$$

The impending difficulty with finding ω in the model is apparent, and indeed our model $\mathcal{M}(\mathcal{T})$ will not think that $\omega \in V$: where could its representing function start ?

We may now summarise the effect of our construction.

8·31 Theorem *To any identity model* $\mathcal{T}$ *of* TST *we may associate a model* $\mathcal{M}(\mathcal{T})$ *of* M_0 *in which:*

if T^0 is empty, the axiom of foundation will hold;

if T^0 is non-empty, the axiom of foundation will fail and there will be Quine atoms;

if T^0 is finite, AxInf and other axioms of infinity will be false;

if an axiom of infinity is true in $\mathcal{T}$, the corresponding axiom of infinity will be true;

each finite von Neumann ordinal will exist;

in no case will the existence of the von Neumann ordinal ω be true.

8·32 REMARK A model containing Quine atoms cannot be well-founded, but $\mathcal{M}(\mathcal{T})$ will be well-founded in a weak sense: if X is a non-empty subset of $\mathcal{M}(\mathcal{T})$ containing no elements of start 0, then any member y of X with minimal start will have no members in common with X, by Proposition 8·13; in particular, if T^0 is empty, $\mathcal{M}(\mathcal{T})$ will be well-founded.

8·33 We must consider the strength of the metatheory needed to turn the above into a relative consistency proof. Really there are two questions of interest:

(8·33·0) *How strong a set theory is needed to construct $\mathcal{M}(\mathcal{T})$ given $\mathcal{T}$?*

(8·33·1) *How strong a theory is needed to derive Consis(M) from Consis(TSTI) ?*

The chief problem in each case is that of defining the sequence of functions π_k and proving it to be a set. A sufficient set theory for the construction of the model $\mathcal{M}(\mathcal{T})$ from $\mathcal{T}$ is the theory MAC + Induction, but we can do better: write T for $\bigcup_k T^k$; then each π_n is a partial map from T to T and therefore an element of $\mathcal{P}(T \times T)$; hence, using $\mathcal{P}(T \times T)$ as a parameter, we may show that π_n is defined for all n by appeal to set Foundation rather than Π_2 Foundation: so MAC suffices.

For the second question, the best answer this paper can supply — proof theorists can do better — is that second-order arithmetic is sufficient. One is told that TSTI is consistent; one builds a model $\mathcal{T}$ of TSTI following the procedures of Gödel's completeness theorem; the underlying universe of that model is ω. One then sets up the sequence of definitions of the functions $\pi_i^{\mathcal{T}}$, forms the transversal Q given in Remark 8·14, defines the relation E on Q following the equations given in Remark 8·14, verifies that the outcome is a model of M_0 + InfWel, and infers that that latter theory is consistent, whence, essentially by Proposition 2·43, there follows, in arithmetic, the consistency of M.

The derivation of Consis(M_0) from Consis(TST) is similar but easier.

Kemeny's Princeton thesis of 1949

Kemeny in his thesis (Princeton, 1949) [G1] studied versions of the simple theory of types and of Zermelo's set theory. Of his system of type theory, which he calls T, though we shall usually call it TKTI, or, shorn of its axiom of infinity, TKT, he states that *the basic ideas of* T *are taken from a system due to Tarski* [G6]. T *differs from this system in that it contains axioms of infinity and choice, and it has a description operator.* He credits the main ideas of the formalisation of his version, which he calls Z, of Zermelo's system to Skolem: Z too is equipped with a description operator and axioms of infinity and choice. He recommends Quine's paper [G5] for a good discussion of the history of the systems he considers, but stresses that Quine's systems contain no axiom of infinity or of choice. We need not, in our discussion, consider the fine detail of Z and therefore shall state his results as though he had adopted our formalisation Z.

In his thesis, which, regrettably, was never published, though an abstract [G2] exists, Kemeny refuted the belief that Z and TKTI were of more or less the same strength by showing that Z would prove the consistency of TKTI.

He wrote to the author on November 27th 1989 as follows:

Dear Professor Mathias,

My thesis was going to be written up jointly with a PhD student of mine. But his interests shifted, and when he admitted that he would never get to the paper, my interests had shifted too. So the only publication was the abstract.

Princeton used to provide photographic copies, at a fee. I know that several people had gotten copies – but that was years ago. I hope that the statute of limitations has not expired! The title was "Type Theory vs. Set Theory", and the thesis is dated 1949.

Incidentally it not only shows that one can prove the consistency of TT in ST, but the latter is stronger in every sense. One can even give a truth definition (à la Tarski) for TT in ST. And there were similar results for various transfinite extensions of the two theories.

Sorry that I can't be more helpful.

Sincerely yours,
John G. Kemeny
Dartmouth College

Intuitively the system T of Kemeny's thesis concerns objects of type 0, 1, …, and each object of a given type counts as an object of every higher type. To illustrate its style, we quote from pages 1–3 of his thesis, giving Kemeny's text in *slanted type*, with his footnotes incorporated, and our comments in Roman.

T *is a system usually described as a singulary theory of types of type* ω. *It is a simple (not ramified) theory of types having only one-place predicates in it, but of all finite types. This system is as strong as the Russell–Whitehead system.*

Primitive symbols: (,), [,], $\sim$, $\supset$, $\forall$, ι, and for $n = 0, 1, \ldots$, *variables* x_n, y_n … with subscript n. *These are individual and functional variables (i.e. set variables); thus no propositional variables are used.*

Well-formed formulæ *and terms of type* n*: (Definition by recursion.)*

1. If a_n *is a variable with subscript* n*, then* (a_n) *is a term of type* n.

2. If a_n *is a variable with subscript* n *and* $\mathfrak{A}$ *is a well-formed formula, then* $(\iota a_n \mathfrak{A})$ *is a term of type* n.

3. If $\mathfrak{A}$, $\mathfrak{B}$ *are w.f.f. and* a_n *is a variable then* $[\sim \mathfrak{A}]$, $[\mathfrak{A} \supset \mathfrak{B}]$, $[\forall a_n \mathfrak{A}]$ *are w.f.f.*

4. If A_{n_1}, B_{n_2} *are terms of type* n_1, n_2, *and* $n_2 < n_1$, *then* $[A_{n_1} B_{n_2}]$ *is a w.f.f.*

5. The sets of w.f.f. and of terms of type n *are the smallest sets having all four of the above properties.*

[Thus every term is enclosed in round brackets, and every w.f.f. in square ones.]

Convention: a_n, b_n, … *are used to stand for variables with subscript* n*;*

A_n, B_n, … *are used to stand for terms of type* n.

$\mathfrak{A}$, $\mathfrak{B}$, … *are used to stand for w.f.f.*

We introduce all the usual abbreviations. In particular we introduce the abbreviations:

$$a_{n_1} = b_{n_2} \ \ \textit{to stand for} \ \ [c_{n_3} a_{n_1}] \equiv_{c_{n_3}} [c_{n_3} b_{n_2}] \ \textit{where} \ n_3 = \max(n_1, n_2) + 1$$

In this, and similar definitions some convention, only too well known, must be adopted as to which variable c_{n_3} *is.*

[Note that equality is defined as anti-extensionality, since the intended meaning of $[s\ t]$ is $t \in s$. Note also that $\equiv$ is not among the primitive symbols.]

$$\underset{\cdot}{\mathrm{S}} \begin{matrix} (a_{n_1}) & \ldots & (a_{n_k}) \\ A_{m_1} & \ldots & A_{m_k} \end{matrix} \mathfrak{A}|$$

to stand for the result of replacing all free occurrences of the (a_{n_i}) *by* A_{m_i}*, simultaneously for all* i*, in* $\mathfrak{A}$.

[Note that there is no requirement that $n_i = m_i$.]

For each $n > 0$, we introduce 0_n *to stand for* $\iota x_n . \forall y_{n-1} . \sim [x_n y_{n-1}]$

$\{a_n, b_n\}$ *to stand for* $\iota c_{n+1}[c_{n+1} d_n] \equiv_{d_n} .d_n = a_n \vee d_n = b_n$.

$\langle a_n, b_n \rangle$ *to stand for* $\{\{a_n, a_n\}, \{a_n, b_n\}\}$

Axiom schemata:

(1) $\mathfrak{A}$

where $\mathfrak{A}$ *is a substitution instance of a tautology.*

$$\mathfrak{A} \supset_{a_n} \mathfrak{B} \supset .\mathfrak{A} \supset .\forall a_n \mathfrak{B} \tag{2}$$

where a_n *is not free in* $\mathfrak{A}$.

$$[\forall a_n \mathfrak{A}] \supset [\underset{B_{n_2}}{\overset{(a_{n_1})}{\d{S}}} \mathfrak{A} \,|] \tag{3}$$

where $n_2 \leqslant n$, *and no free variable of* B_{n_2} *is bound in* $\mathfrak{A}$.

8·34 REMARK Kemeny's comma, written on a type-writer, and looking like a 1, may have misled him. I suggest that (3) should read

$$[\forall a_{n_1} \mathfrak{A}] \supset [\underset{B_{n_2}}{\overset{(a_{n_1})}{\d{S}}} \mathfrak{A} \,|] \tag{3$'$}$$

where $n_2 \leqslant n_1$ *and no free variable of* B_{n_2} *is bound in* $\mathfrak{A}$.

[The next is the axiom of extensionality:]

$$[(b_{n+1})(a_n) \equiv_{a_n} (c_{n+1})(a_n)] \supset [b_{n+1} = c_{n+1}] \tag{4}$$

[The next three define the workings of the ι symbol: really it would now be called Hilbert's ε-symbol, for uniqueness is not required. In particular this functions as an axiom of choice.]

$$[\exists a_n \mathfrak{A}] \supset \big[\underset{(\iota a_n \mathfrak{A})}{\overset{(a_n)}{\d{S}}} \mathfrak{A}|\big] \tag{5$*$}$$

where *no variable is both free and bound in* $\mathfrak{A}$.

[Footnote: *This axiom, the choice axiom, may be weakened into a description axiom*

$$\big[\exists a_n.\mathfrak{A} \,\&\, .\underset{(b_{n_1})}{\overset{(a_n)}{\d{S}}} \mathfrak{A}| \supset_{b_{n_1}} .b_{n_1} = a_n\big] \supset \big[\underset{(\iota a_n \mathfrak{A})}{\overset{(a_n)}{\d{S}}} \mathfrak{A}|\big] \tag{5}$$

where no variable is both free and bound in $\mathfrak{A}$, $n_1 \leqslant n$, *and* b_{n_1} *does not occur in* $\mathfrak{A}$.]

$$[\forall a_n \sim \mathfrak{A}] \supset [(\iota a_n \mathfrak{A}) = 0_n] \qquad n > 0 \tag{6}$$

$$[\mathfrak{A} \equiv_{a_n} \mathfrak{B}] \supset [(\iota a_n \mathfrak{A}) = (\iota a_n \mathfrak{B})] \tag{7}$$

[The next is the scheme of Comprehension:]

$$\exists b_{n+1} \forall a_n.[(b_{n+1})(a_n) \equiv \mathfrak{A}] \tag{8}$$

where b_{n+1} is *not free in* $\mathfrak{A}$.

[Finally Kemeny formulates an axiom of infinity:]

$$\begin{aligned} &\exists a_3.\exists a_0.\forall b_0\big[\sim [(a_3)(\langle b_0,\, a_0\rangle)]\big] \,\& \\ &\quad \& \,\forall a_1\Big[\exists c_0\big[(a_1)(d_0) \supset_{d_0} .(a_3)(\langle e_0, d_0\rangle) \supset_{e_0} (a_3)(\langle e_0,\, c_0\rangle)\big] \supset \\ &\qquad \supset \exists f_0[(a_3)(\langle g_0,\, f_0\rangle) \equiv_{g_0} (a_1)(g_0)]\Big] \end{aligned} \tag{9}$$

[If we write $X_a = \{b \mid b \text{ is of type } 0 \text{ and } \langle b,\, a\rangle \in a_3\}$, we may paraphrase the axiom thus:

$$\exists a_3 \exists a_0 \Big(X_{a_0} \text{ is empty } \ \&\ \forall a_1 \Big[\ \textbf{if}\ \ \exists c_0 \forall d_0 {:}{\in} a_1\ X_{d_0} \subseteq X_{c_0}\ \ \textbf{then}\ \ \exists f_0 X_{f_0} = a_1 \Big]\Big)$$

Thus the axiom generates objects f_0^0, f_0^1, f_0^2, ..., of type 0, with $X_{f_0^0} = \varnothing$, $X_{f_0^1} = \{f_0^0\}$, $X_{f_0^2} = \{f_0^0,\, f_0^1\}$, and so on, and we are therefore guaranteed an infinity of objects of type 0, and thence of every larger type.]

Rules of inference:

[I] From $\mathfrak{A}$ *and* $[\mathfrak{A} \supset \mathfrak{B}]$ *infer* $\mathfrak{B}$.

[II] From $\mathfrak{A}$ *infer* $[\forall a_n \mathfrak{A}]$.

[Note that Axiom 3 gives copies of terms at higher types: let $\mathfrak{A}$ be $\exists b_n a_n = b_n$. Let $m < n$, and let a_m be a variable with subscript m. Then Axiom 3 yields, since $\forall a_n \mathfrak{A}$, by substitution, $\exists b_n (a_m) = b_n$. By considering a more complicated formula involving ordered pairs of objects, we can ensure that if two terms are "equal" their copies are "equal".]

Completing the circles

We see that Kemeny's system TKTI differs from TSTI by permitting the membership relation to hold between a term and another term of any higher type, rather than just the next one; by defining an equality relation which may hold between any two terms, even of different types; by having a different formulation of the axiom of infinity; and by having ι- or rather ε-operators.

Nevertheless it may be verified that, provably in analysis, his system is equiconsistent with M and TSTI, and his system without infinity is equiconsistent with M_0, KF and TST. We outline the necessary steps.

First, we should show that each instance of our axioms of equality corresponds to a theorem of Kemeny's system. Then we should show that our chosen axiom of infinity in TSTI follows from that in TKTI .

That done, we may say that TST is a subsystem of TKT and TSTI of TKTI .

Then we may verify that TKTI is interpretable in M + KPL, for then we may simply interpret an iota term as giving the first constructible example of whatever the formula might say. Here the fact that ω is transitive is useful, for it enables us to accommodate the cumulative character of Kemeny's types without difficulty, since $\omega \subseteq \mathcal{P}(\omega) \subseteq \mathcal{PP}(\omega)\ldots$

For the case without the axiom of infinity, we must invoke the second epsilon theorem of Hilbert and Bernays to know that the iota terms have not increased the strength of Kemeny's system. For details, see Chapter III of the treatise of Leisenring [J1].

We complete one circle of equiconsistency by applying the result of Theorem 8·31, that a model of M_0 is recoverable from a model of TST. For models of the axiom of infinity, there is one minor point that perhaps should be repeated.

We have seen that armed with $\mathcal{T}$, a model of TSTI, we may build a model of M_0 plus an axiom of infinity which we may take in the form InfWel, which model will however contain Quine atoms and therefore fail to model Foundation, and moreover will contain no infinite von Neumann ordinal.

If we move to the union of all transitive sets, we retain the Quine atoms, since each Quine atom is a member of a transitive set, namely itself. So the Axiom of Foundation will remain false.

We might be tempted to move from $\mathcal{M}(\mathcal{T})$ to the union of all its well-founded transitive sets, which will give us foundation and exclude the Quine atoms, but, as we saw in Remark 8·15, we might thereby have lost the axiom of infinity altogether. Therefore to pick up ω it is necessary to go directly from $\mathcal{M}(\mathcal{T})$ to our model built from well-founded extensional relations. Thus we use our Axiom H spectacles to go from a model of M_0 to one of $\mathsf{M}_1 + \mathsf{H}$, and the infinite well-ordering will form a well-founded extensional relation, and by Proposition 2·43 the vague form InfWel of the axiom of infinity will have become true in the precise form $\omega \in V$. Hence the importance, in §2, of working in M_0 rather than M_1.

8·35 Remark A curiosity of our discussion is that to derive the consistency of TKTI from that of TSTI, we have first to establish the consistency of the set-theoretic system M, and thus our derivation is in analysis, though we are assured that the machinery of proof theory can supply a derivation in primitive recursive arithmetic. One natural idea does not work: the system TSTI is designed to relate well to stratified formulæ, but if one liberalises the notion of stratification to accept all formulæ which admit a type assignment such that in an atomic formula $x_i \in y_j$ the type j is any integer strictly greater than the type i, NF would become inconsistent, since, as noted by Kirmayer, the Russell class would become a set, being then definable as

$$\{x_3 \mid \exists y_2[\neg(y_2 \in x_3) \ \& \ \forall z_1(z_1 \in y_2 \iff z_1 \in x_3)]\}.$$

8·36 HISTORICAL NOTE The detailed arguments of this section have been developed by the author starting from the outlines given in unpublished work of Forster and Kaye. Flattered though the author of the present paper is by the kind comment of the referee that it is the first to give a complete proof of the equiconsistency of TSTI and MAC, he must allow that the detail is not yet complete, for he is unable, with his techniques and their reliance on the axiom of infinity for the Gödelisation of formal languages and appeal to model-theoretic notions, to reduce the requisite metatheory to primitive recursive arithmetic, though proof theorists assure him that that can be done.

Jensen in [G8] mentions that the said equiconsistency is part of the folk-lore of the subject; he gives the beginning of a sketch of a proof, but as correctly remarked by Lake [G9], his definitions and his arguments are incomplete.

Lake in his very compressed paper [G9] shows how to define a model of M_0 starting from a model of TST; it is possible to see in his paper the precursors of the ideas underlying the proof sketched by Forster and Kaye.

9: Limitations of MAC and Z

Failure of stratifiable Π_1 Collection in Z

We begin with the following example of the scheme of Collection. Consider the formula

$$\mathfrak{C}_\ell(a, f) \iff_{\mathrm{df}} \mathrm{Fn}(f) \ \& \ \mathrm{Dom}\,(f) = a \ \& \ \forall x :\in a \ \overline{\overline{f(x)}} \not< \aleph_0 \ \& \ \forall x, y :\in a \ \overline{\overline{f(x)}} \neq \overline{\overline{f(y)}},$$

which says that f associates a set to each member of a, the sets associated being none finite and no two of the same cardinal.

That formula is stratifiable and Π_1. In Z one may prove that $\forall a :\in \omega \ \exists f \mathfrak{C}_\ell(a, f)$, as the set of failures may be formed and the least member taken. That formula is the hypothesis of an instance of the scheme of stratifiable Π_1 Collection; but the conclusion, $\exists c \forall a :\in \omega \ \exists f :\in c \ \mathfrak{C}_\ell(a, f)$, is refutable in KLMZ, for the existence of such a c entails in $\mathsf{ZC} + \mathsf{H}$ the existence of $\aleph_\omega$, which would contradict the minimality axiom of KLMZ.

We have seen that if Consis(Z) then Consis(KLZ); whence the consistency of the theory KLMZ, for if $\aleph_\omega$ exists in a model of KLZ, then in that model we may form $L_{\aleph_\omega}$ which is itself a model of KLMZ. Hence

9·0 METATHEOREM $\mathsf{ZC} + \mathsf{H}$, *if consistent, does not prove the scheme of stratifiable* Π_1 *Collection.*

9·1 REMARK As a curiosity, the conclusion, that there is an infinite set of infinite cardinals, is provable in Z plus the existence of a Dedekind-finite set that is not finite: for if $\mathfrak{p} \neq \mathfrak{p} + 1 \not\leq \aleph_0$, and we pick P disjoint from ω with $\overline{\overline{P}} = \mathfrak{p}$, we may, even in Z, form the family $\{P \cup n \mid n \in \omega\}$. Hence our use of the Axiom of Choice in the above discussion, which use of the following variant, again stratifiable Π_1, would obviate.

$$\mathfrak{C}_m(a, f) \iff_{\mathrm{df}} \mathrm{Fn}(f) \ \& \ \mathrm{Dom}\,(f) = a \ \& \ \forall x :\in a \ f(x) \text{ is a non-empty well-ordering with no maximal element} \ \& \ \forall x,y :\in a \ \overline{\overline{\mathrm{Field}(f(x))}} \neq \overline{\overline{\mathrm{Field}(f(y))}}.$$

9·2 REMARK NF-ists and type-theorists may object to ω; our counterexample may be purified by proving first that for all finite a there is an x with $\mathfrak{C}_m(a, x)$ and then considering a set containing finite sets of all sizes.

We saw in §6 that over KF, stratifiable Π_1 Collection proves stratifiable $\Delta_0^{\mathcal{P}}$ Collection, but even over MOST the converse is not clear. Hence a slightly different approach will be needed to obtain a failure of stratifiable $\Delta_0^{\mathcal{P}}$ Collection in $\mathsf{ZC} + \mathsf{H}$: here is one.

9·3 EXAMPLE Define the relation $\ll$ between sets by

$$A_2 \ll B_2 \iff_{\mathrm{df}} \exists X_1 \exists Y_2 (Y_2 = \mathcal{P}(X_1) \ \&\ \exists f_4 \exists g_4 (f_4 : A_2 \xrightarrow{1-1} \{\{a_0\} \mid a_0 \in X_1\} \ \&\ g_4 : Y_2 \xrightarrow{1-1} B_2).$$

Thus $A \ll B$ expresses the assertion that $2^{\overline{\overline{A}}} \leqslant \overline{\overline{B}}$, and, as the type indices show, does so in a stratifiable way, since in the above context $\{\{a_0\} \mid a_0 \in X_1\} = Y_2 \cap \{y_1 \mid \exists a_0 :\in X_1\ y_1 = \{a_0\}\}$. The relation $\ll$ is $\Sigma_1^{\mathcal{P}}$.

9·4 PROPOSITION (ZF) *If $\zeta > 0$ and $V_\zeta \models$ `stratifiable` $\dot{\Delta}_0^{\mathcal{P}}$ `Collection`, then $\zeta = \beth_\zeta$.*

Proof : ζ cannot be a successor: let $c = V_\eta \in V_{\eta+1}$; then in $V_{\eta+1}$ anything which is a member of anything is a member of c; hence both $\forall z \exists x (x \notin c)$ and $\neg \forall a \exists b \forall z :\in a\ \exists x :\in b\ (x \notin c)$ are true in $V_{\eta+1}$, violating stratifiable $\dot{\Delta}_0$ collection.

Suppose then that ζ is a limit ordinal with $\zeta < \beth_\zeta$. There is an $(I, <_I)$ in V_ζ which is a well-ordering of order type ζ. Let $J = \{X \subseteq I \mid X$ is an initial segment of I under $<_I\}$.

Denote by $\mathfrak{C}_g(X, I, f)$ the assertion that f is a 1-1 function with domain X such that $\forall \xi :\in X\ (f(\xi)$ is infinite) and $\forall \xi, \eta :\in X (\xi <_I \eta \implies f(\xi) \ll f(\eta))$. $\mathfrak{C}_g$ is stratifiable and $\Sigma_1^{\mathcal{P}}$.

Then

$$V_\zeta \models \bigwedge X :\epsilon J \bigvee f\ \dot{\mathfrak{C}}_g(X, f)[I] \ \wedge\ \neg \bigvee K \bigwedge X :\epsilon J \bigvee f :\epsilon K \dot{\mathfrak{C}}_g(X, f)[I]$$

since any such K would have to have cardinality at least $\beth_\zeta$. So we have a failure of $\dot{\Sigma}_1^{\mathcal{P}}$ Collection, which by Proposition 6·14 will yield a failure of $\dot{\Delta}_0^{\mathcal{P}}$ Collection. Contradiction ! ⊣ (9·4)

9·5 COROLLARY *If $V = L$ and $\aleph_\omega < ON$, both stratifiable $\dot{\Delta}_0^{\mathcal{P}}$ Collection and stratifiable $\dot{\Pi}_1$ Collection fail in L_{ω_ω}.*

Proof : by Proposition 9·4 and Corollary 6·15.

9·6 METATHEOREM ZC + H, *if consistent, does not prove the scheme of stratifiable $\Delta_0^{\mathcal{P}}$ Collection.*

Failure of $\Delta_0^{\mathcal{P}}$ and Π_1 Replacement in Z

We have seen three stratified instances of a scheme of Collection where Z can prove the hypothesis, but the conclusion can be refuted in KLMZ, and hence the scheme is unprovable in Z, always assuming that system is consistent.

Knowing from Coret that stratifiable Replacement is provable in Z, we consider an unstratified variant of the formula $\mathfrak{C}_\ell$.

9·7 DEFINITION By a *type sequence* we mean a sequence s such that

$$\mathrm{Dom}\,(s) \in \omega \,\&\, s(0) = \omega \,\&\, \forall k :\in \omega \big(k + 1 \in \mathrm{Dom}\,(s) \implies s(k+1) = \mathcal{P}(s(k))\big).$$

We write $TS(n, s)$ for the formula asserting that s is a type sequence of length $n + 1$, and $TS(n)$ for the formula $\exists s\ TS(n, s)$.

9·8 REMARK The formula $TS(n, s)$ is Π_1 and $\Delta_0^{\mathcal{P}}$; the formula $TS(n)$ is Σ_2 and $\Sigma_1^{\mathcal{P}}$. We shall see that these formulæ are irredeemably unstratifiable.

9·9 METATHEOREM *It is provable in* Z *that $\forall n :\in \omega\ \exists! s\ TS(n, s)$; that $\exists c \forall n :\in \omega\ \exists s :\in c\ TS(n, s)$ is refutable in* KLMZ*; hence the schemes of Π_1 and $\Delta_0^{\mathcal{P}}$ Replacement are refutable in* KLMZ *and in particular are provable neither in* Z *nor in* MOST.

In §7 we saw that MAC proves strong stratifiable Π_1 Replacement and KF proves strong stratifiable $\Delta_0^{\mathcal{P}}$ Replacement.

The following remark owes much to a discussion with Thomas Forster:

9·10 REMARK The two differences between $\mathfrak{C}_\ell(n, f)$ and $TS(n, s)$ is that the former is stratifiable but the latter not, whereas the s is unique given n whereas the f is not. Both of them assert the existence of increasing sequences of infinite cardinals, one vaguely and the other giving a unique construction. Now by Coret, stratifiable Replacement is provable in Z; but Z cannot build an infinite set of alephs. There therefore cannot be a stratifiable version of $TS(n, s)$: thus there is no homogeneous function which, provably in Z, raises cardinals, where we call an operation F *homogeneous* if there is a formula $G(x, y)$ which has a stratification assigning equal types to the variables x and y, such that $G(x, y)$ holds if and only if $x = F(y)$.

That tells us, for example, that there is no stratifiable way to develop the theory of L and its global well-ordering $<_L$. For suppose we have a stratifiable formula that, if $V = L$, means x is the $<_L$-first thing after y to fulfil some stratifiable condition such as being of larger cardinal. Run that formula in KLMZ, and use it to build a stratifiable version of $TS(n)$: for each (stratifiable integer) m the object at $m + 1$ is the next one after the object at m. But then, using stratifiable Replacement, we could build an infinite set of distinct infinite well-ordered cardinals, which we know to be impossible.

9·11 REMARK Compare the above remark with the observation of Forster that Boffa's permutation argument, presented in §7, shows that there is no stratifiable total ordering of *HF*, the set of hereditarily finite sets.

Failure of Induction and other schemata in MAC

Now we look at those instances from the perspective of MAC. First, a remark due essentially to Gödel.

9·12 METATHEOREM *Any arithmetical statement provable in* KLZ_0 *is provable in* Z_0.

Proof : in the process of shrinking to a model of Z_0^+, extending to a model of Axiom H and shrinking again to L, the integers are not affected. ⊣ (9·12)

$\mathsf{Z}_0 + \mathsf{AC}$ is the same system as ZBQC.

9·13 THEOREM *Each of the formulæ* $\exists c \forall n :\in \omega\ \exists f :\in c\ \mathfrak{C}_\ell(n, f)$, $\exists c \forall n :\in \omega\ \exists f :\in c\ \mathfrak{C}_m(n, f)$, $\exists c \forall n :\in \omega\ \exists f :\in c\ \mathfrak{C}_g(n, f)$ *and* $\exists c \forall n :\in \omega\ \exists f :\in c\ TS(n, f)$, *if added to* MAC, *proves the consistency of* Z.

Proof : In the theory $\mathsf{MAC} + \exists c \forall n :\in \omega\ \exists f :\in c\ \mathfrak{C}_\ell(n, f)$, move to the H-model, (treating ω as a parameter about which a stratified statement is being made); we may well-order such a c; the order-type is at least ω_ω, which therefore exists; but now L_{ω_ω} is a model of ZC + H; hence we have proved Consis(Z), an arithmetical fact that we can pull back to the ground model.

The next two cases are handled similarly. In the last case, if $\exists c \forall n :\in \omega\ \exists f :\in c\ TS(n, f)$, argue directly that c will enjoy the same property in the H-model, as follows: let c be a member of the transitive set u, and consider $\mathcal{P}(u)$. Then if $TS(n, f)$ with $n \in \omega$ and $f \in c$, the property $TS(n, f)$ is equivalent to

$$\mathrm{Dom}\,(f) \in \omega \,\&\, f(0) = \omega \,\&\, \forall k :\in \omega \left(k + 1 \in \mathrm{Dom}\,(f) \Longrightarrow \forall v :\in u\ (u \in f(k+1) \Longleftrightarrow u \subseteq f(k))\right),$$

a Δ_0 formula, and hence absolute. Complete the argument as before by well-ordering c in the H-model. ⊣ (9·13)

We essentially saw in §8 the following:

9·14 THEOREM *The theory* $\mathsf{M} + \forall n :\in \omega\ TS(n)$ *proves the consistency of* TSTI *and therefore that of* MAC.

Proof : a contradiction in TSTI will only involve variables of type less than or equal to n, for some n: but that fragment of type theory is interpretable in a type sequence of length $n + 1$. Hence the consistency of TSTI, which, as we have seen, implies, in second-order arithmetic and therefore in M, the consistency of MAC. ⊣ (9·14)

Note that that proof would not work in the theory M + all statements $TS(\mathfrak{n})$ for each concrete $\mathfrak{n}$, since our quantifier over all types (in the metatheory of TSTI) could not be constrained to be standard. That is fortunate, as each statement $TS(\mathfrak{n})$ *is* provable in M.

Indeed we may summarise our remarks in the following pleasing form:

9·15 THEOREM *There are instances of stratifiable* Π_1 *collection and of stratifiable* $\Delta_0^{\mathcal{P}}$ *collection of which*

(i) MAC *plus the hypothesis proves the consistency of* MAC;

(ii) Z *proves the hypothesis;*

(iii) MAC *plus the conclusion proves the consistency of* Z.

Proof : If we are in the theory MAC $+\forall n :\in \omega\ \exists f \mathfrak{C}_\ell(n,f)$, again we may treat ω as a parameter about which a stratified statement is being made, and move to the H-model in which MOST $+ \forall n :\in \omega\ \exists f \mathfrak{C}_\ell(n,f)$ hold. In that theory we may easily prove that each ω_n exists; then we may move to L, where it will still be true that $\forall n \exists \omega_n$; but then we may prove that $\forall n\, TS(n)$ holds in L; hence Consis(MAC) is true there, therefore in our MOST model therefore in the original MAC model, proving Part (i). Parts (ii) and (iii) have already been treated.

⊣ (9·15)

⊣ (Theorem 16)

We know from Theorem 9·14 and the Second Incompleteness Theorem that if MAC is consistent it cannot prove $\forall n :\in \omega\ TS(n)$.

However, let us note various systems which do prove $\forall n :\in \omega\ TS(n)$ or $\forall n :\in \omega\ \exists f \mathfrak{C}_\ell(nf)$.

9·16 Definition By *Induction* we mean the following scheme:

$$\Big(\Phi(0)\,\&\,\forall n :\in \omega\ \big(\Phi(n) \Longrightarrow \Phi(n+1)\big)\Big) \Longrightarrow \forall n :\in \omega\ \Phi(n) \qquad \text{for every wff } \Phi$$

9·17 Proposition *It is provable in* M + Induction *that* $\forall n :\in \omega\ TS(n)$.

Proof : There is a type sequence of length 1, namely $\{\langle \omega, 0\rangle\}$, so $TS(0)$ holds; if s is a type sequence of length $n+1$, $s \cup \{\langle \mathcal{P}(s(n)), n+1\rangle\}$ is a type sequence of length $n+2$, so $\forall n :\in \omega\ \big(TS(n) \implies TS(n+1)\big)$; hence by Induction, $\forall n :\in \omega\ TS(n)$. ⊣ (9·17)

9·18 Metacorollary (Lake) *An instance of Induction is not provable in* MAC.

9·19 Historical Note Metacorollary 9·18 is essentially Corollary 4 of Lake's 1975 paper [G9], which also contains the observations underlying Theorem 9·14 and Proposition 9·17.

Gandy in his 1973 obituary [G3] of Russell as a mathematician, remarks that a paper of Myhill then in preparation would prove that there would be a failure of induction in any implementation of the theory of types within set theory. I have been unable to trace any such paper in print, but perhaps the above discussion is what Myhill had in mind.

9·20 Proposition M *plus stratifiable* Π_2 *Foundation proves the consistency of* MAC.

Proof : in that theory we may prove $\forall n :\in \omega\ \exists f \mathfrak{C}_m(n,f)$, since $\{n \in \omega \mid \neg \exists f \mathfrak{C}_m(n,f)\}$ is a Π_2 class, and therefore if non-empty has a least element, which is not 0 as M proves $\exists f \mathfrak{C}_m(0,f)$, and is therefore of the form $k+1$. So for that k, $\exists f \mathfrak{C}_m(k,f)$ holds, but M proves that $\exists f \mathfrak{C}_m(k,f) \implies \exists f \mathfrak{C}_m(k+1,f)$.

The consistency of TSTI follows, and therefore also the consistency of M and of MAC. ⊣ (9·20)

9·21 Metacorollary *An instance of stratifiable* Π_2 *Foundation is not provable in* MAC.

By working with $\mathfrak{C}_g$ instead, we find

9·22 Proposition M *plus stratifiable* $\Pi_1^{\mathcal{P}}$ *Foundation proves the consistency of* MAC.

9·23 Metacorollary *An instance of stratifiable* $\Pi_1^{\mathcal{P}}$ *Foundation is not provable in* MAC.

9·24 Another natural but more extravagant way to try to prove $\forall n :\in \omega\ \exists f\ \Phi(n,f)$ by seeking the least n for which there is no such f is to use a scheme of Separation, saying that the class of those n in ω that fail is a set, and therefore if non-empty has a least element. Hence we get

9·25 Proposition M *plus either stratifiable* Σ_2 *Separation or stratifiable* $\Sigma_1^{\mathcal{P}}$ *Separation proves the consistency of* MAC.

9·26 Metacorollary *Neither stratifiable* Σ_2 *Separation nor stratifiable* $\Sigma_1^{\mathcal{P}}$ *Separation is provable in* MAC.

9·27 Forster and Kaye in unpublished work have proved a bounding lemma that in effect says that if the hypothesis of an instance of $\Delta_0^{\mathcal{P}}$ collection, or indeed of $\Sigma_1^{\mathcal{P}}$ collection is provable in a system such as KF or MAC or any extension of them obtained by adding $\Pi_1^{\mathcal{P}}$ formulæ as axioms, then the conclusion is provable in the same system: a result that forms a fine contrast to Metatheorem 9·6.

Inspired by their proof, we give here a weak variant of their result:

9·28 METATHEOREM *Let $\Phi(n, w, y)$ be a $\Delta_0^{\mathcal{P}}$ formula. Suppose that*

$$\vdash_{\mathsf{MAC}} \forall n{:}{\in}\omega\ \exists y\, \Phi(n, \omega, y).$$

Then there is a concrete natural number $\mathfrak{k}$ such that

$$\vdash_{\mathsf{MAC}} \forall n{:}{\in}\omega\ \exists y{:}{\in}\mathcal{P}^{\mathfrak{k}}(\omega)\ \ \Phi(n, \omega, y).$$

Proof: essentially a compactness argument, which we present model-theoretically, though with the assistance of techniques from proof theory we could avoid the use of the axiom of infinity in the meta-theory. If the conclusion is false for a given Φ, then for each k there is a possibly ill-founded model $\mathfrak{M}_k$ of $\mathsf{MAC} + \forall y{:}{\in} \mathcal{P}^k[w_k]\ \neg\Phi(y)[n_k, w_k]$, where w_k is the first limit ordinal of the model M_k and n_k is some member of w_k in that model. Let $\mathfrak{M}$ be an ultrapower of all those models by a free ultrafilter on ω. In $\mathfrak{M}$, let $\mathfrak{b}$ be represented by the function $k \mapsto \mathcal{P}^k(w_k)$, $\mathfrak{n}$ by $k \mapsto n_k$ and $\mathfrak{w}$ by $k \mapsto w_k$. Then

$$\mathfrak{M} \models \mathsf{MAC} + \forall y{:}{\in}\mathfrak{b}\ \neg\Phi(y)[\mathfrak{n}, \mathfrak{w}].$$

Note that for each concrete natural number $\mathfrak{l}$, $\mathfrak{M} \models \mathcal{P}^{\mathfrak{l}}(\mathfrak{w}) \subseteq \mathfrak{b}$, since for $\mathfrak{l} \leqslant \mathfrak{k}$, $\vdash_{\mathsf{MAC}} \mathcal{P}^{\mathfrak{l}}(\omega) \subseteq \mathcal{P}^{\mathfrak{k}}(\omega)$.

Now, externally to the model $\mathfrak{M}$, form $\mathfrak{N}$, the union of all the $\mathcal{P}^{\mathfrak{k}}(\mathfrak{w})$'s. By a fundamental result of Forster and Kaye, that set together with the membership relation of $\mathfrak{M}$ restricted to $\mathfrak{N}$ is a $\Delta_0^{\mathcal{P}}$ elementary submodel of $\mathfrak{M}$; hence $\mathfrak{w}$ is also the ω of $\mathfrak{N}$, which is a model of MAC of which $\mathfrak{M}$ is an end extension preserving $\mathcal{P}$. Since each member of $\mathfrak{N}$ is a member of $\mathfrak{b}$, $\mathfrak{N} \models \mathsf{MAC} + \mathfrak{n} \in \omega + \forall y \neg\Phi(\mathfrak{n}, \omega, y)$, contradicting the hypothesis that $\vdash_{\mathsf{MAC}} \forall n{:}{\in}\omega\ \exists y \Phi(n, \omega, y)$. ⊣ (9·28)

Consider now the formula

$$OS(n, \omega, y) \iff_{\mathrm{df}} \mathrm{Fn}(y)\ \&\ \mathrm{Dom}\,(y) = n + 1\ \&\ y(0) = \omega\ \&\ \forall m{:}{\in}\mathrm{Dom}\,(y) - 1\ [y(m+1) = y(m) + 1].$$

That is a Δ_0^{M} formula; plainly there is no $\mathfrak{k}$ for which MAC can prove $\forall n{:}{\in}\omega\ \exists y{:}{\in}\mathcal{P}^{\mathfrak{k}}(\omega)\ OS(n, \omega, y)$, since the rank of $\mathcal{P}^{\mathfrak{k}}(\omega)$ is $\omega + \mathfrak{k}$ and the rank of $\omega + n$ is $\omega + n$. Hence

9·29 METATHEOREM *The formula $\forall n{:}{\in}\omega\ \exists y\ OS(n, \omega, y)$ is unprovable in* MAC.

Since that formula is provable in $\mathsf{MAC} + \Pi_1\mathsf{Foundation}$ or in $\mathsf{MAC} + \Sigma_1\mathsf{Separation}$, we have immediately

9·30 METACOROLLARY *Instances of the schemes of Π_1 Foundation and of Σ_1 Separation are unprovable in* MAC.

We saw, of course, in §6 that Σ_1 Separation is unprovable in $\mathsf{MAC} + \mathsf{KP}$, though by a more complicated argument; Π_1 Foundation is one of the axiom schemata of KP.

9·31 REMARK The formula $OS(n, \omega, y)$ would also furnish examples of failure of Δ_0 Collection and Δ_0 Separation in $\mathsf{ZC} + \mathsf{TCo}$.

An algebraic illustration

9·32 EXAMPLE Let P_0 be the real vector space $\mathbb{R}[t]$ of all real polynomials. Let P_{n+1} be the dual of P_n. Then, using AC and setting β_n to be the size of a basis of P_n, one may show that $\beta_0 = \aleph_0$ and $\beta_{n+1} = 2^{\beta_n}$ for every n. Thus the operation taking each space to its dual, or taking each space to its bidual, necessarily raises the cardinality of the space at each step after the first. We may conclude that the operation of taking the dual of a space is not homogeneous.

It follows that if we write $DS(n)$ to mean that the sequence of spaces $P_0, \ldots P_{n-1}$ exists — in other words a sequence of real vector spaces of length n starting from $\mathbb{R}[t]$ and taking the dual at each step, then MAC cannot prove that $\forall n DS(n)$; Z can prove that but cannot prove the existence of the infinite sequence $\langle P_n \mid n \in \omega\rangle$, nor the existence of the direct limit of the P_{2n}'s under the natural embedding of a space in its bidual, nor of the dual of that space, the projective or inverse limit of the P_{2n+1}'s, any of which existential statements imply in MAC the consistency of Z.

Peroration

We review our discussion of §§6, 7, 8 and 9.

9·33 METATHEOREM MAC *proves the scheme of stratifiable* Σ_1 *Separation, but none of these three strengthenings of that scheme:*

Σ_1 Separation;

stratifiable Σ_2 Separation;

stratifiable $\Sigma_1^{\mathcal{P}}$ Separation.

All those schemes are provable in Z*, which proves the full separation scheme;* KF *proves stratifiable* $\Delta_0^{\mathcal{P}}$ *Separation;* M_0 *proves* $\Delta_0^{\mathcal{P}}$ *Separation;* MOST *proves* Σ_1 *Separation, but* MAC + KP *does not; both stratifiable* Σ_2 *Separation and stratifiable* $\Sigma_1^{\mathcal{P}}$ *Separation, when added to* MAC*, prove the consistency of* MAC*.*

9·34 METATHEOREM MAC *proves the schemes of stratifiable* Π_1 *Foundation and stratifiable* Σ_1 *Foundation, but none of these three strengthenings:*

Π_1 Foundation;

stratifiable Π_2 Foundation;

stratifiable $\Pi_1^{\mathcal{P}}$ Foundation.

All those schemes are provable in Z + TCo*, which proves the full Class Foundation scheme, though by Boffa* [C3] Z *alone cannot prove* Π_1 *Foundation;* M_1 *proves* $\Delta_0^{\mathcal{P}}$ *Foundation;* MOST *proves* Σ_1 *Foundation; and* Π_1 *Foundation; stratifiable* Π_2 *Foundation and stratifiable* $\Pi_1^{\mathcal{P}}$ *Foundation, when added to* MAC*, prove the consistency of* MAC*.*

9·35 PROBLEM What does it take to prove Σ_1 Foundation ?

9·36 METATHEOREM MAC *proves the schemes of strong stratifiable* Σ_1 *Replacement and strong stratifiable* Π_1 *Replacement, but neither of these stronger schemes:*

strong stratifiable Σ_2 Replacement

strong stratifiable $\Pi_1^{\mathcal{P}}$ Replacement

Both those are provable in Z*, which proves the full scheme of strong stratifiable Replacement;* KF *proves the scheme of strong stratifiable* $\Delta_0^{\mathcal{P}}$ *Replacement; over* MAC*, each of the schemes strong stratifiable* Σ_2 *Replacement and strong stratifiable* $\Pi_1^{\mathcal{P}}$ *Replacement proves the consistency of* MAC*.*

9·37 PROBLEM Does MAC prove stratifiable Σ_2 Replacement ?

9·38 PROBLEM Does MAC prove stratifiable $\Pi_1^{\mathcal{P}}$ Replacement ?

9·39 METATHEOREM *Not even* ZC+TCo *can prove* Δ_0 *Replacement, although* M+H *proves* Σ_1 *Replacement; over* MAC*, each of the schemes* Π_1 *Replacement and* $\Delta_0^{\mathcal{P}}$ *Replacement proves the consistency of* Z*.*

9·40 REMARK Although MOST can prove Σ_1 Replacement, MAC + KP cannot; so the problem is not one of consistency strength but of what we might call ordinal strength. $\Delta_0^{\mathcal{P}}$ Replacement, refutable in KLMZ, of which MOST is a subsystem, on the other hand, is a problem of consistency strength.

9·41 METATHEOREM MAC *proves the scheme of strong stratifiable* Σ_1 *Collection, but* ZC + TCo *can prove neither the scheme of stratifiable* Π_1 *Collection nor that of stratifiable* $\Delta_0^{\mathcal{P}}$ *Collection, nor, of course,* Δ_0 *Collection, though* M+H *proves the last named. Over* MAC*, each of stratifiable* Π_1 *Collection and stratifiable* $\Delta_0^{\mathcal{P}}$ *Collection proves the consistency of* Z*.*

9·42 REMARK Again Δ_0 collection poses a problem of ordinal strength, and both stratifiable Π_1 Collection and stratifiable $\Delta_0^{\mathcal{P}}$ Collection problems of consistency strength.

9·43 REMARK From the above results, it follows that MAC proves "strong stratifiable KP", in that it proves, besides Δ_0 Separation, stratifiable Π_1 Foundation and strong stratifiable Δ_0 Collection, though of course the proofs have made used of TCo.

9·44 PROBLEM Are those schemata provable say in $\mathsf{KF} + \mathsf{TCo} + \mathsf{AC} + \omega \in V$, or in even less ?

9·45 PROBLEM Is every stratifiable theorem of KP a theorem of stratifiable KP ?

9·46 PROBLEM There remains the problem of determining the status of $\Sigma_{\mathfrak{k}}$ versions of all the above schemata relative to systems such as $\mathsf{Z}_{\mathfrak{l}}$ and $\mathsf{KZ}_{\mathfrak{l}}$, which this author is content to leave to future generations.

10: Envoi

The author regards this paper as a semi-survey, in that though parts of the paper are original, much in it is not. He would be grateful for information from his readers that would enable him to improve the accuracy of his attributions.

This paper has concentrated on purely mathematical and logical aspects of Mac Lane set theory. It is hoped, though, that it will shed some light on the relationship between, and the several merits of, set-theoretic and category-theoretic foundations of mathematics. At the prompting of the referee, we add some remarks concerning the philosophical motivation of the paper.

10·0 For some years there has been rivalry, not always friendly, between two camps, whom I shall call CAT and SET, both of which are claimed, at least by the extreme members of each faction, to have the one true view of pure mathematics.

This rivalry descends from the two reactions to the foundational crisis of the beginning of the twentieth century: the type-theoretical approach — beginning with Russell, Whitehead, Ramsey, Chwistek, and Quine — and the set-theoretical one, starting with Zermelo, Fraenkel, and Skolem (who despite his important contributions to its development would not, however, have regarded set theory as a remedy) and many since.

10·1 I have written in [L5] about the foundational shortcomings of Bourbaki. A much deeper study is that of Leo Corry [L4], [L7], from whose writings I have formed the view that Mac Lane and his school have done successfully what the Bourbachistes were at times trying to do, namely to give a convenient organisation to the type-theoretic side of mathematics. What I do not believe is that Mac Lane has found an organisation for the whole of mathematics.

Indeed I would view with suspicion any claim by anybody to have an account of the whole of mathematics, because I believe that mathematics flows from at least two distinct intuitions, and that the balance between these intuitions will be different in different mathematicians. I desire the unity of mathematics—meaning the communicability of mathematics—whilst believing that pressure from particular groups who seek to enforce an unhealthy uniformity by stifling alternative approaches should be resisted.

10·2 The CAT camp may with justice claim that category theory brings out subtleties in geometry to which set theory is blind. But that is a far cry from saying such things as “Mathematics has no need of set theory”, or “set theory has been left behind by the tide of history”.

It is entirely reasonable for people to state that they find their kind of mathematics is best served by adopting the category-theoretic style. But those who make that statement must allow an equal liberty to those who find the category-theoretic style alien.

10·3 The SET camp may with equal justice claim that set-theoretic analysis brings out subtleties to which the CAT camp is blind: for examples, see my expository paper “Strong statements of analysis” [F4], where it is argued that the large cardinal assumptions studied by set theorists are inextricably involved in certain concepts, problems, and theorems of analysis. That paper evolved from conversations with mathematicians of various hues who had made rash pronouncements concerning an alleged irrelevance of logic to their work. Its purpose was not to say that people should not do topos theory but to show that much insight would be lost by a refusal to contemplate set theory.

Against the opinion of many, I hold set theory to be the study of well-foundedness and thus to be concerned with recursive constructions in the widest abstract sense. The language is rich and therefore admits translations into it of huge amounts of mathematics; though often by a formal interpretation which fails to translate the underlying intuition. Set theory is at its best when discussing problems which involve (overtly or covertly) well-founded relations of high rank; and the results of Harvey Friedman and his collaborators on the necessary use of abstract set theory show how set-theoretic questions concerning large notions of infinity may arise even in apparently innocent mathematical statements concerning only finite structures.

10·4 It is no part of my purpose to criticise Mac Lane on the grounds that his system is weak. There is nothing wrong with studying weak systems; there is plenty of hard mathematics going on in weak systems; much information may be obtained when something has a proof in a weak system; but sometimes a system cannot handle a problem and one wants to know when that is happening. Friedman's proposal to calibrate theorems through his programme of reverse mathematics promises a systematic critical apparatus: see Simpson's new book [F2] on that subject.

10·5 There are portions of mathematical reasoning for which the language of categories gives a very smooth presentation (Example: treatment of the algebraic structures arising in geometry, which are essentially confined to ω types over the space under consideration); there are portions for which it is clumsy (Example: recursive constructions); there are portions which it cannot handle at all (Example: determinacy).

Hence I believe that mathematics would be enriched if members of the different foundational camps learned something of each other's language and fundamental perceptions.

10·6 A mathematician, or perhaps a group of mathematicians, wishes to work in a certain area of mathematics; he certainly wishes to deepen knowledge of his chosen area; he may even wish to change the way people think about that area. At a given moment, he will be wishing to attack certain problems with certain tools. We may ask

" *What are the ideas he wishes to employ ?* "

" *What is their strength ?* " and

" *What is his chosen style for their presentation ?* "

I see those as different and almost independent questions. I regard the answer to the first as being roughly the specification of a system of thought. If one had a core dump of the mathematician's mind, one could see what are the ideas playing an important rôle in his handling of his problem. Of course, I understand that in research, things are in a state of flux; one does not know what one is doing; one is responding to intuitions and trying to articulate them. The giving of a definition is an attempt to articulate them. It may be the right definition; later work may turn things round and find a better one.

The second question, " *What is their strength ?* " I regard as a question about the comparative strengths of various systems. For a given theorem in a given system, one can ask: did the proof use the whole strength of the system ? could it be proved in a weaker system ? Perhaps it could but with a much longer proof. But then let us ask, is there a stronger system which would furnish a shorter proof ?

The idea of one system being stronger than another has evolved among logicians during the twentieth century. Broadly it is considered that if one system can proved the consistency of another, then the first is stronger. Two systems are regarded as being of the same strength if there is a proof in some third system, weaker than either, of their equiconsistency.

From the history of mathematics we know how hard it is to introduce new ideas; the square root of 2, the square root of -1, the concept of an infinite set, non-Euclidean geometry, Hilbert's "theological", that is, non-constructive inductions; all met with considerable resistance upon their introduction. So there is a psychological if not professional cost to introducing stronger systems. People are suspicious of new-fangled ideas. The purpose of logical investigations is to test the soundness of and necessity for new proposals.

The meaning of the third question, about style, is " *Which, among several equiconsistent, systems has he chosen ?* " For example TST and M_0 are equiconsistent systems, but there is a different feel to them. They are equally strong, but some will feel more at home with one and some with the other. M and $\mathsf{M}+\mathsf{H}$ are equiconsistent, but the one is closer to type theory, the other to set theory. The equiconsistency of two systems is no guarantee that they will be found psychologically to be equally satisfactory.

10·7 The purpose of my paper therefore is to study the relationship of Mac Lane's system, which encapsulates in set-theoretic terms his mathematical world, to the Kripke-Platek system that gives a standard formalisation of a certain kind of abstract recursion. I suggest that they capture two distinct modes of thought, and somewhere between ZBQC and the system I have called MOST is the watershed between the two. Finding a system that accommodates both without violating either is a task perhaps similar to resolving the Continuum Hypothesis: there the continuum, a concept from geometry, is being matched against a concept from transfinite arithmetic, that of the first uncountable ordinal.

MOST might have been called $\mathsf{KZ_1C}$. It is ZBQC with the addition of Δ_0 Collection, Σ_1 Separation, and TCo. It might also be presented as the system MAC $+$ KP$+$ *Mostowski's principle* that "*every well-founded extensional relation is isomorphic to a transitive set*", whence its name. In it, one can prove Axiom H, and one can develop the concept of constructibility.

10·8 ZBQC and MOST are equiconsistent, by the results of sections 2 and 3 of this paper; it is a natural equiconsistency, for there is a natural interpretation of the apparently stronger system MOST in its subsystem ZBQC using what I have called Axiom H spectacles.

In view of the many alternative presentations of MOST given in section 3, it must qualify as a natural system, and it is much more intelligible to the set-theoretical mind than is MAC alone.

Thus my suggestion is that the transition from ZBQC to MOST is the natural bridge between CAT and SET.

10·9 Before discussing who stands at each end of this bridge, let me pause to answer a query of the referee, who asks why I have placed so much emphasis in the paper not on Mostowski's principle mentioned above but on the less intuitive Axiom H.

The answer is that when I began writing the paper, Σ_1 Separation seemed to be the chief object of attention. But then it became clear, given that $\mathsf{M_0}$ proves the existence of universal extensional well-founded relations, that the natural construction in the context of $\mathsf{M_0}$ was to build a model for Axiom H, from which one would derive, first (by the arguments of 3·0) Mostowski's principle, and then, with the help of the axioms of Choice, Infinity and Foundation, Σ_1 Separation. I only noticed Proposition 3·2 when the paper was well advanced, and I am reluctant to run the risk of error by recasting the paper round it, but I entirely agree that Mostowski's principle is the most natural of the many versions which in our context are, by Theorem 3·18, equivalent.

10·10 To return to my metaphor of the bridge between SET and CAT: it would indeed seem that Mostowski's principle is the point of divergence between those with leanings towards SET and those with leanings towards CAT. The CAT camp is structuralist — witness Mac Lane's remark in [L3] that every mathematical notion is protean, and Bell's illuminating essay that forms the Epilogue of his book [H5] — whereas the SET camp is absolutist in something like the sense suggested by the last sentence of [A1], which reads "Hence the consistency of $[V = L]$ seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase." The difference between the two camps is perhaps summarised by their attitudes to a particular case of Mostowski's principle, namely the precept, to set theorists so natural and to category theorists so suspect, that every well-ordering is isomorphic to precisely one von Neumann ordinal, a concept that Mac Lane dismisses as a gimmick. If one wants to do geometry, why bother with von Neumann ordinals ? What do they do that is not achieved by arbitrary well-orderings ? Perhaps nothing; if all geometry is contained in stratifiable mathematics, then certainly nothing, in view of our result that every stratifiable theorem of MOST is provable in MAC.

But if, sated with geometry, one wants to do transfinite recursion theory, why make life hard by avoiding von Neumann ordinals ? Compare the mammoth struggle in section 4 to present the concept of constructibility without using Mostowski's principle with the easy ride you get if you adopt it. And the two systems are equiconsistent, so one is not demanding a stronger system; one is merely presenting Mac Lane's sytem in a style that is more efficient for transfinite recursion theory.

It may well be that the criterion of stratifiability does indeed mark the frontier separating Mac Lane's world from mine. We may point to one highly important topic in set theory from which those who would confine themselves to stratifiable mathematics are excluded, namely the theory of constructibility, which was developed by Gödel to prove the relative consistency of the Axiom of Choice. Rather technical, one might think; but its study led to the discovery by Jensen of the set-theoretic principle $\diamondsuit$ which was applied by Shelah to prove that if $V = L$ all Abelian groups G with $\mathrm{Ext}(G, \mathbb{Z}) = 0$ are free Abelian: for details, and for later developments in this line, see the the book [F5] of Eklof and Mekler.

Further reading

For contrasting views of the foundations of mathematics, the reader may like to read [L2] and [L3]. On page 378 of [L1] Mac Lane makes the excellent remark that proofs are not only a means to certainty but also a means to understanding; but on page 395 he gives the impression that he thinks the phenomenon of incompleteness will go away if mathematics were to shift from set-theoretic to category-theoretic foundations, a belief as unrealistic as that the law of gravity will cease to operate if one travels to the Antipodes. Against his suggestion that ZBQC is "appropriate for most Mathematics," — if he would change "most" to "much", I would agree — I would mention the closing chapter of the book of Adamek and Rosicky [L6] which illustrates the way in which large cardinal questions have infiltrated even category-theoretic concerns, and for more recent evidence of that phenomenon, the papers [F1] and [F3]. The recent book of Corry [L7] treats the history and philosophy of twentieth-century algebra. The paper of Marshall and Chuaqui [G10] gives a characterization of the truths captured by type theory that sheds further light on the rôle of Mac Lane set theory.

The author hopes to treat the philosophical, psychological and sociological aspects of the differences between SET and CAT in greater detail in a forthcoming essay [L8] tentatively entitled *Danish Lectures on Bourbaki, Mac Lane and the Foundations of Mathematics.*

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The penultimate draft of the paper was completed whilst the author held the Sanford Visiting Professorship of Mathematical Logic at the Universidad de los Andes at Santa Fé de Bogotá in Colombia, and the author expresses his gratitude to Leonardo Venegas, Sergio Agarve and most of all Carlos Montenegro for their encouragement.

In this final revision, made at the Université de la Réunion, the later sections, concerning stratifiable formulæ and the theory of types, have been rewritten in greater detail, and several independence results refined or added. The kindness of the logicians at the Albert-Ludwigs-Universität at Freiburg im Breisgau in according the author full access to their superb collection has given him an invaluable opportunity of improving the historical accuracy of this work.

In conclusion the author thanks the unknown referee for the very great trouble he took with this lengthy paper and for his numerous valuable suggestions.

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‖ Significant typographical errors on page 327 are corrected in the footnote on page 139 of [B4].

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