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ON THE EXISTENCE OF LARGE p-IDEALS

WINFRIED JUST, A. R. D. MATHIAS, KAREL PRIKRY AND PETR SIMON

Abstract. We prove the existence of p-ideals that are nonmeagre subsets of $\mathscr{P}(\omega)$ under various set-theoretic assumptions.

§0. Introduction. Throughout this paper by "ideal" we mean a proper ideal on ω that contains Fin—the ideal of finite sets. An ideal I is called a *p-ideal*, if for every countable subfamily $\{A_j: j \in \omega\} \subset I$ there exists an $A \in I$ such that $A_j \setminus A$ is finite for every j. An ideal I is *meagre*, if it is a subset of $\mathscr{P}(\omega)$ of first Baire category, where $\mathscr{P}(\omega)$ is considered to be endowed with the topology of the Cantor set. It is easy to find examples of meagre p-ideals (e.g. Fin is one), but the following remains open.

0.1. *Question.* Does there exist a nonmeagre p-ideal?

The main purpose of this paper is to compile known partial answers. It is easy to observe that maximal p-ideals (also called *p-points* in $\beta\omega\setminus\omega$) are examples of nonmeagre p-ideals (see Corollaries 1.6 and 1.7 of this paper). The existence of p-points is both relatively consistent with and independent of the axioms of ZFC. The former was shown by W. Rudin in [Ru], and the latter by S. Shelah (for a proof see [W] or [Sh]). Motivated by the, then still open, question of whether ZFC proves the existence of p-points, A. R. D. Mathias showed in [M1] that if $0^\#$ does not exist, or if $2^{\aleph_0} \leq \aleph_{w+1}$, then there are nonmeagre p-ideals. Shortly afterwards, K. Prikry extracted a combinatorial principle from Mathias' proof that allows one to weaken the assumption "$0^\#$ does not exist" considerably (see [P] and §4 of this paper). This result was never published and appears here for the first time in print. In 1985, J. Burzyk found a striking application of nonmeagre p-ideals to the theory of Banach spaces (see [B], and §2 of this paper). His result led to renewed interest in Question 0.1. R. Frankiewicz and P. Zbierski proved that if CH holds, then there exists a nonmeagre p-ideal that is not contained in any maximal p-ideal (see [FZ]). P. Simon proved that if $\mathbf{t} = \mathbf{b}$, or $\mathbf{b} < \mathbf{d}$, then there exists a nonmeagre p-ideal ([Sm] and §3 of this paper), and W. Just rediscovered the result of Prikry and also

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proved that $\mathbf{t} = \mathbf{b}$ implies the existence of nonmeagre p-ideals. Also, W. Just observed that all those constructions yield ideals of character $\leq \mathbf{d}$ and showed that it is consistent relative to the existence of some large cardinals that there are no nonmeagre p-ideals of character $\leq \mathbf{d}$ ([J] and §5 of this paper). The compilation of all these results was done by W. Just, and he is the one to be blamed for all flaws of this paper.

§1. Notation and basic facts. By $\exists n^\infty$ and $\forall n^\infty$ we abbreviate "there exist infinitely many n" and "for all but finitely many n", respectively.

By ω^ω we denote the set of all strictly increasing functions from ω into ω. For $f, g \in \omega^\omega$ we write $f \leq^* g$, iff $\forall n^\infty\ f(n) \leq g(n)$. A family $\mathscr{F} \subseteq \omega^\omega$ is called *dominating*, if $\forall f \in \omega^\omega\ \exists g \in \mathscr{F}\ f \leq^* g$. If $\mathscr{F} \subseteq \omega^\omega$, and $g \in \omega^\omega$ is such that $f \leq^* g$ for all $f \in \mathscr{F}$, then we say that g *dominates* $\mathscr{F}$ and write $\mathscr{F} \leq^* g$. A family $\mathscr{F} \subseteq \omega^\omega$ which is not dominated by any $g \in \omega^\omega$ is called an *unbounded* family.

By $\mathbf{b}$ we denote the minimum cardinality of an unbounded family, by $\mathbf{d}$ the minimum cardinality of a dominating family. For $A, B \subseteq \omega$ we write $A \subseteq^* B$ iff $A \backslash B \in \mathrm{Fin}$. A sequence $\{A_\xi: \xi < \kappa\} \subseteq \mathscr{P}(\omega)$ is called a *κ-chain*, if it consists of infinite sets and $A_\eta \subseteq^* A_\xi$ for all $\xi < \eta < \kappa$. It is called a *κ-tower*, if moreover there is no infinite set A such that $A \subseteq^* A_\xi$ for all $\xi < \kappa$. By $\mathbf{t}$ we denote the minimal cardinal κ such that there exists a κ-tower.

1.1. PROPOSITION. $\aleph_0 < \mathrm{cf}(\mathbf{t}) = \mathbf{t} \leq \mathrm{cf}(\mathbf{b}) = \mathbf{b} \leq \mathrm{cf}(\mathbf{d}) \leq \mathbf{d} \leq 2^{\aleph_0}$.

PROOF. Easy. See also [vD]. □

Adopting a terminology introduced in [M2], we call an ideal I *feeble*, iff there exists a sequence $(a_n)_{n \in \omega}$ of nonempty, pairwise disjoint finite subsets of ω such that for all $B \in I$ the set $\{n: a_n \subset B\}$ is finite. The following result of S. A. Jalali-Naini and M. Talagrand is crucial for most of our results.

1.2. LEMMA. *An ideal I is meagre iff it is feeble.*

PROOF. See [T]. □

Let I be an ideal. A set $\mathscr{B} \subset I$ will be called a *base of I*, if $\forall A \in I\ \exists B \in \mathscr{B}\ A \subseteq^* B$. By $\chi(I)$ we denote the *character* of I, i.e. the minimum cardinality of a base of I.

For $\mathscr{A} \subset \mathscr{P}(\omega)$ we denote $I(\mathscr{A}) = \{B \subset \omega: \exists A \in \mathscr{A}\ B \subseteq^* A\}$. Let $\mathscr{A} \subseteq \mathscr{P}(\omega)$. We say that $B \subseteq \omega$ is a *cover* of $\mathscr{A}$, if $A \subseteq^* B$ for every $A \in \mathscr{A}$. The family $\mathscr{A}$ is called *p-closed*, if for every countable subfamily $\mathscr{A}_1 \subseteq \mathscr{A}$ there is a cover B of $\mathscr{A}_1$ in $\mathscr{A}$.

1.3. PROPOSITION. *Let I be an ideal. Then I is a p-ideal iff I is p-closed iff every base of I is p-closed iff there exists a p-closed family $\mathscr{B}$ such that $I = I(\mathscr{B})$.* □

For infinite $A \subset \omega$ we denote by f_A the function enumerating A in increasing order.

1.4. LEMMA. *An ideal I is nonmeagre iff for every base $\mathscr{B}$ of I the family $\mathscr{F}_\mathscr{B} = \{f_A: \omega\backslash A \in \mathscr{B}\}$ is unbounded in ω^ω.*

PROOF. Assume $\mathscr{F}_\mathscr{B}$ is unbounded and let $(a_n)_{n \in \omega}$ be a sequence of pairwise disjoint finite sets. Define $g \in \omega^\omega$ by $g(m) = \max(\bigcup_{n=0}^{2m} a_n)$. Clearly, if $f_A(m) > g(m)$, then $|\{n: a_n \subset g(m)\backslash A\}| \geq m$, hence if $f_A \not\leq^* g$, then $\omega\backslash A$ contains infinitely many a_n's.

On the other hand, let $g \in \omega^\omega$, and define $h \in \omega^\omega$ by $h(0) = 1$ and $h(n+1) = g(h(n) + 2)$. Let $a_n = [h(n), h(n+1))$. Clearly, if $a_n \subset \omega\backslash A$, then $f_A(h(n) + 1) \geq h(n+1) > g(h(n) + 1)$. □

1.5. COROLLARY. *If I is a nonmeagre ideal, then $\chi(I) \geq \mathbf{b}$.* □

The following proposition is due to Sierpiński (see [S]).

1.6. PROPOSITION. *If I is a maximal ideal, then I is nonmeagre.* $\square$

1.7. COROLLARY. *If there exists a p-point in $\beta\omega\setminus\omega$, then there exists a nonmeagre p-ideal.* $\square$

1.8. COROLLARY. *If $\mathbf{d} = 2^{\aleph_0}$, then there exists a nonmeagre p-ideal.*

PROOF. J. Ketonen has shown in [K] that the equality $\mathbf{d} = 2^{\aleph_0}$ implies the existence of p-points. $\square$

§2. An application to the theory of Banach spaces.

2.1. DEFINITION. Let X be a normed linear space over $\mathbf{R}$ ($\mathbf{C}$). We say that X has *property* (N'), if for every sequence $(x_n)_{n\in\omega}$ of elements of X, converging to zero, there exists a subsequence $(x_{n_k})_{k\in\omega}$ such that for every bounded sequence of reals (complex numbers) $(c_k)_{k\in\omega}$ the series $\sum_{k=0}^{\infty} c_k x_{n_k}$ converges to an element of X.

Normed linear spaces with property (N') inherit many properties of Banach spaces; and as far as we know it remains an open problem to construct a linear metric space over $\mathbf{R}$ ($\mathbf{C}$) which is not complete, but has property (N'). J. Burzyk obtained the following consistency result (see [B]).

2.2. THEOREM. *If there exists a nonmeagre p-ideal, then every infinite-dimensional, separable, complete normed linear space over $\mathbf{R}$ ($\mathbf{C}$) has a noncomplete dense subspace that has property (N').*

In his proof, Burzyk uses the existence of an ideal I that has the following property:

(B) If $(A_n)_{n\in\omega}$ is a sequence of elements of I such that $\limsup_{n\to\infty} \min A_n = \infty$, then there exists a subsequence $(A_{n_k})_{k\in\omega}$ such that $\bigcup_{k\in\omega} A_{n_k} \in I$.

2.3. PROPOSITION. *An ideal I has property (B) iff it is a nonmeagre p-ideal.*

PROOF. This is an easy consequence of Lemma 1.2. See also [J]. $\square$

We do not know the answer to the following.

2.4. *Question.* Does the existence of a noncomplete space with property (N') imply the existence of a nonmeagre p-ideal?

2.5. DEFINITION. Let X be a normed linear space over $\mathbf{R}$ ($\mathbf{C}$). We say that X has *property* (K), if for every sequence $(x_n)_{n\in\omega}$ of elements of X, converging to zero, there exists a subsequence $(x_{n_k})_{k\in\omega}$ such that $\sum_{k=0}^{\infty} x_{n_k} \in X$.

Evidently, property (N') implies property (K), but the two properties are not equivalent (see [K1]).

A subfamily $\mathscr{A} \subset [\omega]^{\aleph_0}$ (also called "a set of reals") is said to be *Ramsey*, if there is some $A \in [\omega]^{\aleph_0}$ such that either $[A]^{\aleph_0} \subseteq \mathscr{A}$ or $[A]^{\aleph_0} \cap \mathscr{A} = \varnothing$. The axiom of choice implies the existence of a set of reals which is not Ramsey. It was shown in [M3] and published in [M4] that if the existence of an inaccessible cardinal is consistent with ZFC, then so is the theory ZF + DC + "Every set of reals is Ramsey" (there, the statement "Every set of reals is Ramsey" is abbreviated by $\omega \to (\omega)^{\omega}$). Therefore, the following theorem of Mathias shows that at least a strong form of the axiom of choice is required in all constructions of noncomplete (K)- and (N')-spaces.

2.6. THEOREM (ZF + "Every set of reals is Ramsey"). *Every space with property (K) is complete.*

PROOF. Suppose X is a noncomplete space with property (K). Let $(x_n)_{n\in\omega}$ be a Cauchy sequence that does not converge in X. Define inductively a sequence $(n_k)_{k\in\omega}$: $n_0 = 0$, and given n_k let $n_{k+1} = \min\{n > n_k : \forall m > n\ \|x_m - x_n\| \leq 2^{-k-2}\}$.

Note that the axiom of choice is not used in defining the sequence $(n_k)_{k\in\omega}$. Now denote $y_0 = x_0$ and $y_{k+1} = x_{n_{k+1}} - x_{n_k}$. Then $\|y_k\| \le 2^{-k}$ for almost all k, and $\sum_{k=0}^{\infty} y_k$ does not exist. For a set $A \in [\omega]^\omega$, denote by $A(0), A(1), \ldots$ its monotonic enumeration. Define $E(A) = \bigcup_{n\in\omega}[A(2n), A(2n+1))$, and $O(A) = \omega \setminus E(A)$. Note that if $n \in A$, then $E(A\setminus\{n\})$ differs from $O(A)$ by a finite set. Define

$$\chi(A) = \begin{cases} 0 & \text{if } \sum_{k\in E(A)} y_k \in X, \\ 1 & \text{otherwise.} \end{cases}$$

For $n \in A$, at least one of $\chi(A)$ and $\chi(A\setminus\{n\})$ is equal to 1, as $\sum_{k\in\omega} y_k \notin X$.

Since $\chi^{-1}\{1\}$ is Ramsey, there is $B \in [\omega]^\omega$ such that χ is constant on $[B]^\omega$. By the previous paragraph, $\forall A \in [B]^\omega\ \chi(A) = 1$. Now put

$$Z_i = \sum_{k=B(2i)}^{B(2i+1)-1} y_k.$$

Clearly, $Z_i \to 0$ (as $\|y_k\| < 2^{-k}$ almost always), so by property (K) there is a subsequence $(z_{i_j})_{j\in\omega}$ with $\sum_{j\in\omega} z_{i_j} \in X$. Put $A = \bigcup_{j\in\omega}\{B(2i_j),\ B(2i_j+1)\}$. Since $A \in [B]^\omega$, we have $\chi(A) = 1$, so $\sum_{k\in E(A)} y_k \notin X$, but $\sum_{k\in E(A)} y_k = \sum_{j\in\omega} z_{i_j}$, a contradiction. □

As shown in [M2], if every set of reals is Ramsey, then every ideal is feeble, and therefore, by Lemma 1.2, meagre.

2.7. *Question.* Can one replace in Theorem 2.6 the assumption "every set of reals is Ramsey" by "every ideal is meagre"?

§3. Nonmeagre p-ideals of character $\mathbf{b}$.

3.1. PROPOSITION. *Suppose $\mathscr{A} = \{A_\xi : \xi < \kappa\}$ is a κ-chain, and let $\mathscr{B} = \{\omega\setminus A_\xi : \xi < \kappa\}$. Then $I(\mathscr{B})$ is an ideal. Moreover, if $\operatorname{cf}(\kappa) > \aleph_0$, then $I(\mathscr{B})$ is a p-ideal.* □

The following is due to Rothberger (see [R]).

3.2. PROPOSITION. (a) *If $A \subseteq^* B$, then $f_B \le^* 2\cdot f_A$.*

(b) *If $\mathscr{A} = \{A_\xi : \xi < \kappa\}$ is a κ-chain such that the family $\{f_{A_\xi} : \xi < \kappa\}$ is unbounded in ω^ω, then $\mathscr{A}$ is a κ-tower.*

PROOF. Easy. □

3.3. THEOREM. *If $\mathbf{t} = \mathbf{b}$, then there exists a nonmeagre p-ideal.*

PROOF. Let $\mathscr{F} = \{f_\xi : \xi < \mathbf{b}\}$ be an unbounded subfamily of ω^ω. By induction, construct a $\mathbf{t}$-chain $\mathscr{A} = \{A_\xi : \xi < \mathbf{t}\}$ such that $f_{A_\xi} {}^*\!\ge f_\xi$ for all $\xi < \mathbf{b}$. Let $\mathscr{B} = \{\omega\setminus A_\xi : \xi < \mathbf{t}\}$. By Proposition 3.1, $I(\mathscr{B})$ is a p-ideal, and it follows from Lemma 1.4 that $I(\mathscr{B})$ is nonmeagre. □

3.4. THEOREM. *Assume there exists an unbounded family $H = \{h_\xi : \xi < \kappa\} \subset \omega^\omega$ which is well-ordered by the relation $\le^*$ (i.e. $h_\xi \le^* h_\eta$ for $\xi < \eta < \kappa$) and not dominating. Then there exists a nonmeagre p-ideal of character κ.*

PROOF. Let H be as in the assumption, and let $g \in \omega^\omega$ be such that $\forall\xi < \kappa\ \exists n^\infty\ g(n) > h_\xi(n)$. We put $A_\xi = \{n : h_\xi(n) < g(n)\}$. Clearly, $\mathscr{A} = \{A_\xi : \xi < \kappa\}$ is a κ-chain, and $\operatorname{cf}(\kappa) > \aleph_0$.

3.5. *Claim. $F = \{f_{A_\xi} : \xi < \kappa\}$ is unbounded.*

Theorem 3.4 is an immediate consequence of the above claim, Proposition 3.1 and Lemma 1.4.

Proof of the Claim. Suppose towards a contradiction that $f^* > \mathscr{F}$, and define $g_1 \in \omega^\omega$ by $g_1(n) = g(f(n+2))$. Let $h_\xi \in H$. If $f_{A_\xi}(n+2) \leq f(n+2)$, then by definition, there are at least $n+2$ numbers $i \leq f(n+2)$ such that $g(i) > h_\xi(i)$. Hence we may pick $i_0 > n$ such that $g_1(n) = g(f(n+2)) \geq g(i_0) > h_\xi(i_0) > h_\xi(n)$, so g_1 eventually dominates every function of H, contradicting our assumption that H was unbounded. □□

3.6. COROLLARY. *If there are no nonmeagre p-ideals of character* $\mathbf{b}$, *then* $\mathbf{t} < \mathbf{b} = \mathbf{d} < 2^{\aleph_0}$; *and every* $<^*$*-increasing sequence of elements of* ω^ω *which is unbounded in* ω^ω *is also dominating.* □

§4. Nonmeagre p-ideals of character $\leq \mathbf{d}$. Let κ, λ, ν be cardinals. By $[\kappa]^\nu$ we denote the family of subsets of κ of size ν. Let $\mathrm{COV}_\nu(\kappa, \lambda)$ be the following statement:

"There is a family $T \subseteq [\kappa]^\nu$ such that $|T| \leq \lambda$ and for each $X \in [\kappa]^\nu$ there is a $Y \in T$ such that $X \subseteq Y$."

4.1. THEOREM. *Suppose* $\mathrm{COV}_{\aleph_0}(\mathbf{d}, \mathbf{d})$ *holds.*

(a) *There exists a nonmeagre p-ideal of character* $\leq \mathbf{d}$.

(b) *Every ideal of character* $< \mathbf{d}$ *can be extended to a nonmeagre p-ideal of character* $\leq \mathbf{d}$.

(c) *Every nonmeagre p-ideal* I_1 *contains a nonmeagre p-ideal I of character* $\leq \mathbf{d}$.

PROOF. The proof rests on two lemmas.

4.2. LEMMA. *Let I be an ideal of character less than* $\mathbf{d}$, *and let* $\mathscr{A}$ *be a countable subset of I. Then there exists a* $B \subset \omega$ *such that* $A \subseteq^* B$ *for all* $A \in \mathscr{A}$, *and* $I \cup \{B\}$ *generates an ideal.*

PROOF. See [K, the proof of Proposition 1.3] or [Je, p. 258, Lemma 24.11]. □

4.3. LEMMA. *Suppose* $\bar{a} = (a_n)_{n\in\omega}$ *is a sequence of pairwise disjoint, nonempty, finite subsets of* ω. *Then there exists a function* $g \in \omega^\omega$ *such that whenever* $f \geq^* g$, *and* $h \in \omega^\omega$ *is defined by* $h(0) = f(0)$ *and* $h(k+1) = f(h(k)+1)$, *then* $\forall k^\infty\ \exists n\ a_n \subset [h(k), h(k+1))$.

PROOF. Let g be such that $g(k) \geq k$, and $\forall k\ \exists n\ a_n \subset [k, g(k))$. If f and h are as in the assumptions, $f(h(k)) \geq g(h(k))$ for almost all k, and if $a_n \subset [h(k), g(h(k)))$, then clearly $a_n \subset [h(k), h(k+1))$. □

Now we are ready to prove Theorem 4.1. We pick a dominating family $\{f_\xi: \xi < \mathbf{d}\}$, and define h_ξ as in Lemma 4.3: $h_\xi(0) = f_\xi(0)$ and $h_\xi(k+1) = f(h_\xi(k)+1)$. Also, we fix a family $T \subset [\mathbf{d}]^{\aleph_0}$ such that for all $X \in [\mathbf{d}]^{\aleph_0}$ there exists $Y \in T$ such that $X \subseteq Y$.

Now we construct inductively a base $\{A_\xi: \xi < \mathbf{d}\}$ for an ideal I. At stage ξ of the construction we take alternatively care that:

(i) a generator of the form $\bigcup_{k\in B}[h_\xi(k), h_\xi(k+1))$ is added, or

(ii) if Y_ξ is the ξth element of T, and the A_η for $\eta \in Y_\xi$ have already been constructed, then A_ξ is a cover of the family $\{A_\eta: \eta \in Y_\xi\}$.

Lemma 4.2 tells us that we can apply procedure (ii) at any stage of the construction. It is clear that we can always apply procedure (i), since either $\bigcup_{k\in\omega}[h_\xi(2k), h_\xi(2k+1))$ or $\bigcup_{k\in\omega}[h_\xi(2k+1), h_\xi(2k+2))$ can be added as a new generator without causing improperness. By careful bookkeeping we ensure that the resulting family is a p-closed base of a nonmeagre ideal. This proves (a) and (b). For the proof of (c), we choose all the generators from I_1. Since I_1 was a nonmeagre p-ideal, there will always be a suitable candidate around. □

Theorem 4.1(c) shows that if $\mathrm{COV}_{\aleph_0}(\mathbf{d},\mathbf{d})$ holds, then all nonmeagre p-ideals are "essentially" of character $\leq \mathbf{d}$. We shall see in the next section that this cannot be proved in ZFC alone.

We conclude this section with an investigation of the consistency strength of $\mathrm{COV}_{\aleph_0}(\mathbf{d},\mathbf{d})$.

4.4. Proposition. $\mathrm{COV}_{\aleph_0}(2^{\aleph_0}, 2^{\aleph_0})$. $\square$

As a corollary, we get the result of Ketonen mentioned already in §1.

4.5. Corollary. *If $\mathbf{d} = 2^{\aleph_0}$, then there is a nonmeagre p-ideal, and in fact a maximal p-ideal.* $\square$

4.6. Lemma. *If $\kappa = \bigcup\{\lambda_\xi \colon \xi < \mathrm{cf}(\kappa)\}$, where $\lambda_\xi < \kappa$ are ordinals, $\mathrm{cf}(\kappa) > \aleph_0$, and $\mathrm{COV}_{\aleph_0}(|\lambda_\xi|, \kappa)$ holds for every $\xi < \mathrm{cf}(\kappa)$, then $\mathrm{COV}_{\aleph_0}(\kappa,\kappa)$ holds.*

Proof. Notice that $[\kappa]^{\aleph_0} = \bigcup\{[\lambda_\xi]^{\aleph_0} \colon \xi < \mathrm{cf}(\kappa)\}$. $\square$

4.7. Corollary. $\mathrm{COV}_{\aleph_0}(\kappa, \kappa^+) \to \mathrm{COV}_{\aleph_0}(\kappa^+, \kappa^+)$. $\square$

4.8. Corollary. $\mathrm{COV}_{\aleph_0}(\aleph_n, \aleph_n)$ *holds for every $n < \omega$.* $\square$

4.9. Proposition. *If $\kappa, \lambda \geq \aleph_1$, then $\mathrm{COV}_{\aleph_1}(\kappa,\lambda) \to \mathrm{COV}_{\aleph_0}(\kappa,\lambda)$.*

Proof. Suppose $T \subset [\kappa]^{\aleph_1}$ witnesses $\mathrm{COV}_{\aleph_1}(\kappa,\lambda)$. For every $X \in T$ choose a family $T_X \subset [X]^{\aleph_0}$ such that $\forall Y \in [X]^{\aleph_0}\, \exists Z \in T_X\; Y \subseteq Z$. The family $\bigcup\{T_X \colon X \in T\}$ witnesses $\mathrm{COV}_{\aleph_0}(\kappa,\lambda)$. $\square$

4.10. Lemma. *If the covering lemma holds with respect to an inner model for GCH, then $\mathrm{COV}_{\aleph_0}(\kappa,\kappa)$ holds for all κ of cofinality $> \aleph_0$. In particular, the covering lemma implies $\mathrm{COV}_{\aleph_0}(\mathbf{d},\mathbf{d})$.*

Proof. Let M be the inner model with respect to which the covering lemma holds. Fix a cardinal κ. Since we know already that $\mathrm{COV}_{\aleph_0}(\aleph_1,\aleph_1)$ holds in ZFC, we may without loss of generality assume that $\kappa > \aleph_1$. The covering lemma asserts that $M \cap [\kappa]^{\aleph_1}$ witnesses $\mathrm{COV}_{\aleph_1}(\kappa, |M \cap [\kappa]^{\aleph_1}|)$. Notice that since $M \models \mathrm{GCH}$, the cardinality $|M \cap [\kappa]^{\aleph_1}| \leq \kappa^+$. We infer from Proposition 4.9 that $\mathrm{COV}_{\aleph_0}(\kappa,\kappa^+)$ holds. This is true for all κ, and now it follows from Lemma 4.6 that $\mathrm{COV}_{\aleph_0}(\kappa,\kappa)$ holds, provided $\mathrm{cf}(\kappa) > \aleph_0$. $\square$

4.11. Corollary. *If every p-ideal is meagre, then there exists a measurable cardinal in an inner model.*

Proof. See [DJ]. $\square$

4.12. Remark. Corollary 4.11 is far from being the strongest possible statement that can be made along these lines. Since the question of the precise consistency strength of the negation of the covering lemma is still not ultimately settled, the formulation of Lemma 4.10 seems the most "durable" way to state our result.

§5. No nonmeagre p-ideals of character $\leq \mathbf{d}$. In this section we prove the following.

5.1. Theorem. *If it is consistent that the singular cardinal hypothesis fails, then it is consistent with ZFC that every p-ideal of character $\leq \mathbf{d}$ is meagre.*

5.2. Lemma. *Suppose $\lambda > \aleph_0$ is a strong limit cardinal of countable cofinality such that $2^\lambda > \lambda^+$. Then $\mathrm{COV}_{\aleph_0}(\lambda^+, \lambda^+)$ does not hold.*

Proof. If λ is as above, then $\lambda^{\aleph_0} > \lambda^+$. On the other hand, if $T \subseteq [\lambda]^{\aleph_0}$, then $|\{X \colon \exists Y \in T\; X \subseteq Y\}| = |T| \cdot 2^{\aleph_0}$. $\square$

5.3. Lemma. *Suppose $V \models \neg\mathrm{COV}_{\aleph_0}(\kappa,\lambda)$, and that $\mathbf{P}$ is a c.c.c. forcing notion. Then $V^{\mathbb{P}} \models \neg\mathrm{COV}_{\aleph_0}(\kappa,\lambda)$.*

PROOF. Suppose $V^{\mathbf{P}} \models \mathrm{COV}_{\aleph_0}(\kappa, \lambda)$, and let $\dot{T}$ be a $\mathbf{P}$-name for a witness. In particular, we have

$$\Vdash_{\mathbf{P}} \text{“}\dot{T} \subseteq [\kappa]^{\aleph_0} \ \&\ |\dot{T}| = \lambda \ \&\ \forall X \in ([\kappa]^{\aleph_0})^V \ \exists Y \in \dot{T}\ X \subseteq Y\text{”}.$$

Let $\dot{F}$ be a $\mathbf{P}$-name for an enumeration of $\dot{T}$, i.e. $\Vdash_{\mathbf{P}}$“$\dot{F}\colon \lambda \xrightarrow[\text{onto}]{1:1} \dot{T}$”. For every $\xi < \lambda$ we find a $\mathbf{P}$-name $\dot{Y}_\xi$ such that $\Vdash_{\mathbf{P}}$“$\dot{F}(\xi) = \dot{Y}_\xi$”. Since $\mathbf{P}$ satisfies the c.c.c., for every $\xi < \lambda$ we find in V a set $Y_\xi^* \in [\kappa]^{\aleph_0}$ such that $\Vdash_{\mathbf{P}}$“$\dot{Y}_\xi \subseteq Y_\xi^*$”. Now $\{Y_\xi^*\colon \xi < \lambda\}$ witnesses that $V \models \mathrm{COV}_{\aleph_0}(\kappa, \lambda)$, contradicting our assumption. □

We denote $\mathbf{Q} = \{(s, f)\colon s \in \omega^{<\omega}, f \in \omega^\omega, s \text{ is increasing}\}$, partially ordered by the relation: $(s,f) \le (t,g)$ iff $s \supseteq t \ \&\ \forall n\ f(n) \ge g(n) \ \&\ (n \ge \mathrm{lh}(t) \to s(n) > g(n))$. By $\mathbf{Q}_\alpha$ we denote the finite support iteration of $\mathbf{Q}$ of length α.

The following is well known.

5.4. *Claim.* $\mathbf{Q}_\alpha$ *satisfies the c.c.c. for every* α. *Moreover, if* $\mathrm{cf}(\alpha) > \aleph_0$, *then forcing with* $\mathbf{Q}_\alpha$ *adds an* α-***scale***, *i.e. a dominating subfamily of* ω^ω *ordered by* $<^*$ *in order type* α. □

Now Theorem 5.1 is an immediate corollary of the following.

5.5. THEOREM. *Suppose that* λ *is a strong limit cardinal of cofinality* $\aleph_0$ *such that* $2^\lambda > \lambda^+$. *Let* $\mathbf{P} = \mathbf{Q}_{\lambda^+}$. *Then* $V^{\mathbf{P}} \models$ “$\mathbf{d} = \lambda^+$ *& every nonmeagre p-ideal is of character* $> \mathbf{d}$”.

PROOF. We call a sequence $(A_\xi)_{\xi<\kappa}$ of subsets of ω *eventually interfering*, iff $\forall A \in [\omega]^{\aleph_0}\ \exists \xi < \kappa\ \forall \eta > \xi\ |A \cap A_\eta| = \aleph_0$.

5.6. LEMMA. $V^{\mathbf{P}} \models$ “*For every nonmeagre p-ideal I there exists a sequence* $(A_\alpha)_{\alpha<\lambda^+}$ *of elements of I such that every subsequence of length* ω_1 *is eventually interfering.*”

Before we prove Lemma 5.6, we show how it implies Theorem 5.5. Assume I is a nonmeagre p-ideal in $V^{\mathbf{P}}$. Let $\mathscr{B}$ be a base of I. By Corollary 1.5, $|\mathscr{B}| \ge \mathbf{b} = \mathbf{d}$. Let $\bar{A} = (A_\xi)_{\xi<\lambda^+}$ be the sequence that exists by Lemma 5.6. For $B \in \mathscr{B}$ define $Y_B = \{\xi < \lambda^+\colon A_\xi \subseteq^* B\}$. Notice that Y_B is countable for all B, as otherwise the sequence $(A_\xi)_{\xi \in Y_B}$ would contain a subsequence of order type ω_1 that does not interfere with $\omega \backslash B$. Since I is a p-ideal, the family $T = \{Y_B\colon B \in \mathscr{B}\}$ witnesses $\mathrm{COV}_{\aleph_0}(\lambda^+, |B|)$. On the other hand, it follows from 5.2 and 5.3 that $\mathrm{COV}_{\aleph_0}(\lambda^+, \lambda^+)$ does not hold in $V^{\mathbf{P}}$, and it follows that $|\mathscr{B}| > \lambda^+ = \mathbf{d}$. So our task reduces to the

PROOF OF LEMMA 5.6. In $V^{\mathbf{P}}$ there exists a sequence $(h_\alpha)_{\alpha<\lambda^+}$ of functions from ω^ω such that for every $\beta \le \lambda^+$ of uncountable cofinality, the sequence $(h_\alpha)_{\alpha<\beta}$ is a scale in $V^{\mathbf{Q}_\beta}$. Fix such a sequence, and fix a sequence $(A_\alpha)_{\alpha<\lambda^+}$ of elements of I such that $f_{\omega\backslash A_\alpha} \not\le^* h_\alpha$ for every α. This is possible by Lemma 1.4. By passing to a closed unbounded subset of λ^+ if necessary, we may without loss of generality assume that $A_\alpha \in V^{\mathbf{Q}_{\alpha+1}}$ and $(A_\xi)_{\xi<\alpha} \in V^{\mathbf{Q}_\alpha}$ for every $\alpha < \lambda^+$.

5.7. *Claim. If* $\delta \le \lambda^+$ *is a limit ordinal of uncountable cofinality, then* $V^{\mathbf{Q}_\delta} \models$ “$(A_\alpha)_{\alpha<\delta}$ *is eventually interfering*”.

Proof. Recall that $(h_\alpha)_{\alpha<\delta}$ is a scale in $V^{\mathbf{Q}_\delta}$; hence for every infinite $A \in P(\omega) \cap V^{\mathbf{Q}_\delta}$, there is $\eta < \delta$ such that if $\alpha > \eta$, then $\exists k^\infty\ f_{\omega\backslash A_\alpha}(k) > f_A(2k)$. Notice that if $n = f_{\omega\backslash A_\alpha}(k) > f_\alpha(2k)$, then $|(n+1)\backslash A_\alpha| = k$, and $|(n+1) \cap A| \ge 2k$, hence $|(n+1) \cap A \cap A_\alpha| \ge k$. This proves the claim. □

5.8. SUBLEMMA. *If* $(A_\alpha)_{\alpha<\gamma} \in V^{\mathbf{Q}_\delta}$ *for some* $\delta < \lambda^+$, *and* $V^{\mathbf{Q}_\delta} \models$ “$(A_\alpha)_{\alpha<\gamma}$ *is an eventually interfering sequence*”, *then* $V^{\mathbf{P}} \models$ “$(A_\alpha)_{\alpha<\gamma}$ *is an eventually interfering sequence*”.

PROOF. Baumgartner and Dordal call a sequence $(A_\xi)_{\xi<\eta}$ *eventually narrow*, iff $\forall A \in [\omega]^{\aleph_0}\ \exists \xi < \eta\ \forall \eta > \xi\ |A \setminus A_\eta| = \aleph_0$. Clearly, the sequence $(A_\xi)_{\xi<\eta}$ is eventually narrow iff the sequence $(\omega \setminus A_\xi)_{\xi<\eta}$ is eventually interfering. Now the sublemma follows immediately from Theorem 3.3 of [BD]. □□

In the remainder of this paper we show that nevertheless, in the model $V^{\mathbb{P}}$ constructed above, there exist nonmeagre p-ideals.

5.9. DEFINITION. By SP ("scale property") we denote the following statement: "There exists a sequence of models $(M_\xi)_{\xi<\mathbf{d}}$ and a sequence $(f_\xi)_{\xi<\mathbf{d}}$ of elements of ω^ω such that for all $\xi < \eta < \mathbf{d}$:

(i) $M_\xi \models$ a sufficiently large fragment of ZFC,

(ii) $M_\xi \subseteq M_\eta$ & $f_\xi \in M_\xi$,

(iii) $g <^* f_\eta$ for all $g \in M_\eta \cap \omega^\omega$,

(iv) $\bigcup_{\xi<\mathbf{d}} M_\xi \cap H(\aleph_1) = H(\aleph_1)$."

As usual, $H(\aleph_1)$ is the family of all hereditarily countable sets.

5.10. *Claim.* $V^{\mathbb{P}} \models SP$. □

5.11. THEOREM. *If SP holds, then there exists a maximal p-ideal.*

PROOF. Suppose $(M_\xi)_{\xi<\mathbf{d}}$ and $(f_\xi)_{\xi<\mathbf{d}}$ witness (i)–(iv). Inductively, we construct an increasing sequence of ideals $(I_\xi)_{\xi\leq\mathbf{d}}$ such that I_ξ has a base contained in $M_\xi \cap \mathscr{P}(\omega)$. If ξ is a limit ordinal, then let $I_\xi = \bigcup_{\eta<\xi} I_\eta$. If $\xi = \eta + 1$, choose $I_\xi^- \geq I_\eta$ generated by elements of $M_\eta \cap \mathscr{P}(\omega)$ such that for every $X \in \mathscr{P}(\omega) \cap M_\eta$, either X or $\omega \setminus X$ is in I_ξ^-. Now let $\mathscr{C}_\eta$ be the family of all sequences $\bar{C} = (C_n)_{n\in\omega}$ such that $C_n \subset C_{n+1} \in I_\xi^-$ for all n, and $\bar{C} \in M_\eta$. For $\bar{C} \in \mathscr{C}_\eta$ we denote $A_{\bar{C}} = \bigcup\{C_n \setminus f_\xi(n) : n \in \omega\}$, and let I_ξ be the ideal generated by $I_\xi^- \cup \{A_{\bar{C}} : \bar{C} \in \mathscr{C}_\eta\}$.

It follows from (iv) that if $I_{\bar{d}}$ constructed as above is proper, then it is a maximal p-ideal. We show that we do not cause improperness at any stage of the construction. Suppose $I_{\bar{d}}$ were improper and let $\xi = \eta + 1$ be the least ξ such that $\omega \in I_\xi$. Then there exist sequences $(C_n^1)_{n\in\omega}, (C_n^k)_{n\in\omega} \in \mathscr{C}_\eta$ and a $C \in I_\xi^-$ such that $C \cup \bigcup\{(C_n^1 \cup C_n^2 \cup \cdots \cup C_n^k) \setminus f_\xi(n) : n \in \omega\} = \omega$. Since I_ξ^- has a base contained in M_η, we may assume $C \in M_\eta$. Moreover, since I_η was assumed to be proper, I_ξ^- can still be chosen to be proper, and we may define $g(n) = \min\{i : i \notin C \cup C_n^1 \cup \cdots \cup C_n^k\}$. Then $g \in M_\eta \cap \omega^\omega$, but it is not hard to see that $g \not\leq^* f_\xi$, contradicting (iii). □

5.12. REMARK. Both Theorem 5.11 and its proof generalize Ketonen's result.

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UNIVERSITY OF WARSAW
WARSAW, POLAND

ERINDALE COLLEGE
UNIVERSITY OF TORONTO
TORONTO, CANADA

PETERHOUSE
CAMBRIDGE CB2 1RD, ENGLAND

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455

DEPARTMENT OF MATHEMATICS
CHARLES UNIVERSITY
186 00 PRAGUE 8, CZECHOSLOVAKIA