ON A CONJECTURE OF ERDŐS AND ČUDAKOV

A.R.D.MATHIAS

Peterhouse, Cambridge

Let f be an arbitrary function from the set of positive integers to the set $\{-1, +1\}$, C a negative integer and D a positive one. We write $f \ll D$ if for all positive m and d, $\sum_{k=1}^{m} f(kd) < D$, and $C \ll f$ if for all positive m and d, $C < \sum_{k=1}^{m} f(kd)$.

In this terminology, a question popularised by Erdős runs:

$$\text{Are there } \quad f, C, D \quad \text{with} \quad C \ll f \ll D\ ?$$

The purpose of this note is to prove that for $D = 2$, no such f and C exist:

Proposition Let f: $\{1, 2, 3, \ldots\} \to \{-1, +1\}$ be such that for all m, d, $\sum_{k=1}^{m} f(kd) \leq 1$. Then for all $C < 0$, there are m, d with

$$\sum_{k=1}^{m} f(kd) \ < \ C.$$

Proof For each $x \geq 1$, $f(x) + f(2x) = -2, 0$ or 2, but the last possibility is excluded by the condition on f, so $f(x) + f(2x) \leq 0$. Thus $\phi(n) = \sum_{x=1}^{n}(f(x) + f(2x))$ is a weakly decreasing function of n. If for some n, $\phi(n) < 2C$, then either $\sum_{x=1}^{n} f(x) < C$ or $\sum_{x=1}^{n} f(2x) < C$. We may therefore suppose that for all n, $\phi(n) \geq 2C$, and hence that $\phi(n)$ is eventually constant. Thus there is a d such that for all $x \geq d$, $f(x) + f(2x) = 0$, and because f takes the value -1 infinitely often, we may, without loss of generality, assume that $f(d) = -1$.We now consider only multiples of d, and since the original condition on f, together with the fact that for all k, $f(2kd) = -f(kd)$, imply that for each $m \geq 1$ and each multiple d' of d, $-1 \leq \sum_{k=1}^{m} f(kd') \leq +1$, we find that the values of $f(kd)$ and $f(2kd)$ for $k = 1, \ldots, 12$ must be as in the following table; but then $\sum_{k=1}^{4} f(3kd) = +2$, a contradiction.

k	1	2	3	4	5	6	7	8	9	10	11	12
$f(kd)$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$f(2kd)$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$

Remark 1 By compactness the problem may be stated in terms of finite sequences, as is done in [2] Problème 49, a negative answer to the above question being equivalent to the assertion that for all $D > 0$ there is an $N > 0$ such that for all $f : \{1, 2, \ldots, N\} \to \{-1, 1\}$ there are $m \geq 1$, $d \geq 1$ with $md \leq N$ and $\left|\sum_{k=1}^{m} f(kd)\right| \geq D$. In [2] reference is made to the paper [1] which studies related questions.

Remark 2 Repeated attempts to improve the proposition to the case $D = 3$ have failed. Indeed the following is still open:

$$\text{Is there an } f \text{ with } \ -3 \ll f \ll +3\ ?$$

REFERENCES

[1] Čudakov, N.G. Theory of the Characters of Number Semigroups, Journal Ind. Math. Soc. 20 (1956) 11-15.

[2] Erdos, P. Quelques Problèmes de la Théorie des Nombres (Monographies de l'Enseignement mathématique, No 6, Génève, 1963).