Delays, Recurrence and Ordinals

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Abstract We apply set-theoretical ideas to an iteration problem of dynamical systems. Among other results, we prove that these iterations never stabilise later than the first uncountable ordinal; for every countable ordinal we give examples in Baire space and in Cantor space of an iteration that stabilises exactly at that ordinal; we give an example of an iteration with recursive data which stabilises exactly at the first non-recursive ordinal; and we find new examples of complete analytic sets simply definable from concepts of recurrence.

0: Introduction.

Let $\mathcal{X}$ be a Polish space — that is, a complete, separable metric space — and let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous function. We write ω for the set of natural numbers $\{0, 1, 2 \ldots\}$, and for $k \in \omega$, f^k for the k^{th} iterate of f, so that for each $x \in \mathcal{X}$, $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. For x and y in $\mathcal{X}$ we define

$$x \curvearrowright_f y \iff_{\text{df}} \exists \text{ strictly increasing } \alpha : \omega \to \omega \text{ with } \lim_{n\to\infty} f^{\alpha(n)}(x) = y.$$

We write $\omega_f(x)$ for the set $\{y \mid x \curvearrowright_f y\}$: so the members of $\omega_f(x)$ are the accumulation points of the forward orbit of x under f, including the periodic points. When f is fixed in a discussion, we write $x \curvearrowright y$ for $x \curvearrowright_f y$, and we sometimes write $y \curvearrowleft x$ for $x \curvearrowright y$. We read $x \curvearrowright y$ as "*x attacks y*". Officially the distance between two points x and y of $\mathcal{X}$ is written $d(x, y)$, but we are prone to denote it by $|x - y|$, to bring concepts of undergraduate analysis to mind. If $A \subseteq \mathcal{X}$ is such that $(x \in A \,\&\, x \curvearrowright y) \implies y \in A$, we call A *$\curvearrowright$-closed.*

0·0 PROPOSITION *(i) if $x \curvearrowright y$ and $y \curvearrowright z$ then $x \curvearrowright z$.*
(ii) $\omega_f(x)$ is a closed and $\curvearrowright_f$-closed subset of $\mathcal{X}$.

Proof : by elementary analysis. ⊣ (0·0)

We define an operator Γ_f on subsets of $\mathcal{X}$ by

$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\}.$$

Using this operator and starting from a given point $a \in \mathcal{X}$, we define a transfinite sequence of sets:

$$\begin{aligned} A^0(a, f) &= \omega_f(a) \\ A^{\beta+1}(a, f) &= \Gamma_f(A^\beta(a, f)) \\ A^\lambda(a, f) &= \bigcap_{\nu<\lambda} A^\nu(a, f) \qquad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

$\Gamma(X)$ is always $\curvearrowright_f$-closed, and if A is $\curvearrowright_f$-closed, then $\Gamma(A) \subseteq A$. Hence $A^0(a, f) \supseteq A^1(a, f)$, and so by repeated application of the trivial principle that $\mathcal{X} \supseteq B \supseteq C \implies \Gamma_f(B) \supseteq \Gamma_f(C)$, we have

$$A^0(a, f) \supseteq A^1(a, f) \supseteq A^2(a, f) \ldots \ ;$$

as we take intersections at limit ordinals we shall have that for all ordinals α, β,

$$\alpha < \beta \implies A^\alpha(a, f) \supseteq A^\beta(a, f).$$

0·1 LEMMA *$x \in A^\nu(a,f) \implies f(x) \in A^\nu(a,f)$; $A^\nu(a,f)$ is $\curvearrowright_f$-closed.*

Proof : by induction on ν using the continuity of f and properties of Γ. ⊣ (0·1)

0·2 LEMMA *For each ordinal μ, $A^\mu(a,f) = \omega_f(a) \cap \bigcap_{\nu<\mu} A^{\nu+1}(a,f)$.*

Proof : by induction on μ. ⊣ (0·2)

0·3 DEFINITION The *escape set* is the union over all ordinals β of the set of those points in $\omega_f(a)$ eliminated at stage β of the iteration:

$$E(a,f) =_{\rm df} \bigcup_\beta \bigl(A^\beta(a,f) \setminus A^{\beta+1}(a,f)\bigr).$$

Here $X \setminus Y$ is the set-theoretic difference $\{x \mid x \in X \;\&\; x \notin Y\}$.

0·4 DEFINITION For $x \in E(a,f)$, we write $\beta(x,a,f)$ for the unique β with $x \in A^\beta(a,f) \setminus A^{\beta+1}(a,f)$.

$E(a,f)$ is a set, being a subset of $\mathcal{X}$, so the axiom of replacement will tell us that the image of the **set** $E(a,f)$ under the map $x \mapsto \beta(x,a,f)$ cannot be the whole of the **proper class** of all ordinals: so there will exist an ordinal θ, which using the axiom of choice may be easily seen to be less than $(2^{\aleph_0})^+$, such that $A^\theta(a,f) = A^{\theta+1}(a,f)$, and accordingly we may define

0·5 DEFINITION $\theta(a,f) =_{\rm df}$ the least ordinal θ with $A^\theta(a,f) = A^{\theta+1}(a,f)$.

Then for all $\delta \geqslant \theta$, $A^\delta(a,f) = A^\theta(a,f)$.

0·6 DEFINITION We write $A(a,f)$ for this final set $A^{\theta(a,f)}(a,f)$. We call $A(a,f)$ the *abode* and the ordinal $\theta(a,f)$ the *score* of the point a under f.

Thus $E(a,f) = \omega_f(a,f) \setminus A(a,f)$. We say that points in $A(a,f)$ *abide*, and points in $E(a,f)$ *escape*.

This paper studies the closure ordinal $\theta(a,f)$ with the help of ideas from descriptive set theory.

Statement of the results

In §1 we review some set-theoretical notions, chiefly the concept of the well-foundedness of a tree of finite sequences.

In §2, we show how, given $\mathcal{X}$, f and a, to associate to each point x in $\omega_f(a)$ a tree of finite sequences; we prove using DC, the Axiom of Dependent Choice, which is explained in §1, that $x \in E(a,f)$ iff the tree associated to x is well-founded, and we use that characterisation to obtain our first theorem:

0·7 THEOREM (DC) *For every $\mathcal{X}$, f and a, $\theta(a,f)$ is at most the first uncountable ordinal, ω_1; if $A(a,f)$ is empty or, more generally, is a Borel set, then $\theta(a,f) < \omega_1$.*

Our proof proceeds by showing that for each point x in $E(a,f)$, $\beta(x,a,f) < \omega_1$. Set-theorists will recognise that we are here applying the classical boundedness theorem of descriptive set theory; we shall prove that theorem in the special case that we use. In fact this section of the paper would easily generalise: the ordinal $\theta(a,R)$ may be defined for any transitive relation R in place of the relation $\curvearrowright_f$, and when R is in addition Borel or indeed analytic, $\theta(a,R)$ will again be at most ω_1.

In §§4 and 5 we shall give examples of $\mathcal{X}$ and f where $\omega_1 = \sup_{a\in\mathcal{X}} \theta(a,f)$: the author conjectures that that bound can never be attained.

In §3, we characterise the abode $A(a,f)$ in terms of *recurrent points*, which are points z such that $z \curvearrowright_f z$, by proving for an arbitrary Polish $\mathcal{X}$, continuous f and point a, the following

PROPOSITION $\quad y \in A(a,f) \iff \exists z\,[a \curvearrowright z \curvearrowright z \curvearrowright y]$

so in particular recurrent points exist in $\omega_f(a)$ if and only if $A(a,f)$ is non-empty. Implicit in our proof is the idea of searching for points at the end of infinite paths through ill-founded trees. The argument here relies on particular properties of the relation $\curvearrowright_f$ and seems not generalise to an arbitrary transitive relation R. We also establish in that section that whenever recurrent points exist, there also exist *maximal* recurrent points in the sense of the following, which again holds for arbitrary $\mathcal{X}$, f, and a:

PROPOSITION *If $a \curvearrowright y \curvearrowright y$, then there is a z such that $a \curvearrowright z \curvearrowright z \curvearrowright y$ and whenever $a \curvearrowright w \curvearrowright w \curvearrowright z$, then $z \curvearrowright w$.*

In words, z is recurrent and there is no recurrent w in $\omega_f(a)$ strictly above z in the relation $\curvearrowright$.

In §4, we consider *Baire space,* , which is called variously $\mathcal{N}$, ω^ω or ${}^\omega\omega$ in different mathematical cultures, and is the set of all functions $\alpha : \omega \to \omega$. $\mathcal{N}$ is a Polish space. On that space we define the shift function $\mathfrak{s}$ by the formula $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$. We show how to associate to each countable well-founded tree T a point x_T in Baire space with $\theta(x_T, \mathfrak{s})$ exceeding the rank of the tree, and we explain how to modify that construction to find a point $a \in \mathcal{N}$ with $\theta(a, \mathfrak{s})$ exactly a given countable ordinal. We end the section with an example showing that whilst the abode is always contained in the set of *non-wandering points* in the sense of Birkhoff [3], it need not coïncide with it.

In §5, we adapt our construction to a compact subspace of $\mathcal{N}$, closed under $\mathfrak{s}$. A difficulty is caused by the fact that in a compact space the abode cannot be empty. Further obstacles emerge when one attempts to construct points which score a given countable limit ordinal, with the result that though such points exist in the compact space in question, they all must attack infinitely many recurrent points.

The observations in this paper leave open the question whether there exist $\mathcal{X}$, f, and a with $\theta(a, f) = \omega_1$. In §6, we give this answer to an effective version of that question:

THEOREM *Let $\mathcal{N}$ be Baire space and $f : \mathcal{N} \to \mathcal{N}$ the shift function. There is a recursive point $a \in \mathcal{N}$ such that $\theta(a, f) = \omega_1^{CK}$, the first non-recursive ordinal.*

That result is inspired by the theorem of Kreisel (see [8] and, for historical detail, [16]) that there is a recursively coded closed set whose Cantor–Bendixson sequence of derivatives runs through all recursive ordinals. The essence of the proof is an adaptation of our constructions on well-founded trees to the case of a recursive ill-founded tree. We shall also see that when $A(a, f)$ is empty, $\theta(a, f)$ will be strictly less than the first ordinal not recursive in the pair (a, f).

In §7, we exploit the continuity of our association, implicit in earlier sections, of points in Baire space to countable linear orderings to obtain a new complete Σ^1_1 set. Specifically, the set

$$P = \{\alpha \in \mathcal{N} \mid \exists \rho{:}\in\mathcal{N}\ \alpha \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho\}$$

is complete in the sense that to any $\mathbf{\Sigma^1_1}$ set $Q \subseteq \mathcal{N}$ there is a continuous function $h : \mathcal{N} \longrightarrow \mathcal{N}$ with $x \in Q \iff h(x) \in P$; known results will then extend that to supplying for any $\mathbf{\Sigma^1_1}$ subset of an arbitrary Polish space $\mathcal{X}$ a Borel function $h : \mathcal{X} \longrightarrow \mathcal{N}$ reducing it to P.

To find such a set in a compact space, the above definition, which is of the set of points with non-empty abode, must be modified, since in a compact space every point attacks at least one recurrent point. Various modifications prove to succeed in suitable spaces, for example the set of points with uncountable abode, the set of points with infinite abode, the set of points which attack at least one non-periodic recurrent point, the set of points with abode of size at least three, or even, in ${}^\omega 4$, the set of points which attack at least one recurrent point distinct from 0^∞ and 2^∞; the complement of the second of those, namely the set of points in ${}^\omega 7$ that attack only finitely many recurrent points, forms a space, closed under shift and attack but not a Polish space, in which our iteration question is intelligible but the only ordinals scored are 0 and the countable successor ordinals.

Finally in §8 we list some open problems.

Alternative terminology

In a recent preprint, *On the structure of the ω-limit sets for the continuous maps of the interval*, Lluis Alsedà, Moira Chas and Jaroslav Smítal use the term *centre* for what I have called the *abode* of a point, and *depth* for what I have called its *score*.

In discussion of the results of the present paper at the Bonn logic seminar the terms *kernel* and *boundary* were suggested as alternatives for my *abode* and *escape set*.

Attractive as these alternatives are, I have thought it prudent not to attempt to change the terminology of my paper at this late stage.

Acknowledgments

A lecture around 1977 on strange attractors by Sir Peter Swinnerton-Dyer, Bar[t], in the short-lived Colloquium of the Pure Mathematics Department at Cambridge, alerted the author to the possibility of set-theoretical ideas being applicable in this area.

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1: Set theoretic preliminaries

We write $\emptyset$ for the empty sequence: technically of course it is the same as the empty set, which we write as $\varnothing$; and also the same as the number zero, which we write as 0, since set-theorists customarily identify each natural number n with the set $\{0, 1, \dots n-1\}$.

We denote by ${}^{<\omega}X$ the set of finite sequences of points in the set X, including the empty sequence.

When s is a finite sequence, we write $\ell h(s)$ for its length, so that $s = \langle s(0), s(1), \dots s(\ell h(s)-1)\rangle$. We also write $\ell(s)$ for its last element, $s(\ell h(s)-1)$. Concatenation is denoted by $^\frown$, so $\ell h(s^\frown\langle p\rangle) = \ell h(s)+1$.

A two-place relation R on a set X is called *well-founded* if every non-empty subset Y of X has a minimal element under R:

$$\forall Y \subseteq X \big(Y \neq \varnothing \implies \exists y{:\in} Y\ \forall z{:\in} Y\ \neg(z\, R\, y)\big).$$

For example, a well-ordering is a well-founded linear ordering. An important simplification in the definition of well-foundedness is afforded by the *Axiom of Dependent Choice*, *DC*, which is the following assertion:

given a relation S on a non-empty set X such that

$$\forall x{:\in} X\ \exists y{:\in} X\ x\, S\, y,$$

there is a function $h : \omega \longrightarrow X$ such that

$$\forall n{:\in}\omega\ h(n)\, S\, h(n+1).$$

DC is strictly weaker than the full Axiom of Choice but nevertheless unprovable from the other axioms of set theory.

1·0 PROPOSITION *Assuming DC, a relation R is well-founded iff there exist no infinite descending sequences*

$$\ldots R\, y_3\, R\, y_2\, R\, y_1\, R\, y_0.$$

Proof : Such a sequence would yield (even without DC) a non-empty subset $\{y_i \mid y \in \omega\}$ with no minimal element. Conversely, suppose Y to be a non-empty subset of X with no minimal element: then (writing S for R^{-1}), we have $\forall x :\in Y\ \exists y :\in Y\ x\, S\, y$ and so by DC there will be a sequence $\langle y_i \mid i \in \omega\rangle$ with $\forall i\ y_i\, S\, y_{i+1}$, yielding a descending sequence in R, as desired. ⊣ (1·0)

Our constructions will be based on well-founded relations of a particular kind, *well-founded trees of finite sequences.*

For a non-empty set X — for example, let $X = \omega$ — we define a relation on the set ${}^{<\omega}X$.

1·1 DEFINITION $t \preccurlyeq s \iff_{\mathrm{df}} t$ is an extension of s; $t \prec s \iff_{\mathrm{df}} t$ is an proper extension of s; $s \succcurlyeq t \iff_{\mathrm{df}} s$ is an initial segment of t; $s \succ t \iff_{\mathrm{df}} s$ is a proper initial segment of t.

1·2 REMARK Thus $s \succcurlyeq t \iff t \preccurlyeq s$, and so on. $\emptyset$ has no proper initial segments, but is itself a proper initial segment of every finite sequence of positive length. Note that longer sequences are lower in this ordering.

1·3 DEFINITION A *tree* in this paper will mean a subset of ${}^{<\omega}X$ which is *closed under shortening* in the sense that $s \succcurlyeq t \in T \implies s \in T$. Thus if T is non-empty it will contain the empty sequence $\emptyset$. We shall refer to the members of T as its *nodes.*

1·4 DEFINITION A tree T is *well-founded* if whenever C is a non-empty subset of T there is an $s \in C$ such that no $t \prec s$ is in C. Such an s is termed a T-*minimal* element of C.

1·5 REMARK If X has a well-ordering, as is the case with the two main examples, or if we assume DC, then saying that T is well-founded is equivalent to the requirement that there be no infinite path through T: that is, that there is no function $f : \omega \to X$ such that for each n, the finite sequence $f \restriction n =_{\mathrm{df}} \langle f(0), f(1), \ldots, f(n-1)\rangle$ is in T.

Given a well-founded tree T that is closed under shortening we may define a *rank function* ϱ_T on it by recursion:

$$\varrho_T(s) = \sup\{\varrho_T(t) + 1 \mid t \in T \;\&\; t \prec s\}$$

Some comments on this definition: if T consists solely of the empty sequence, $\varrho_T(\emptyset) = 0$. For any non-empty well-founded T there will be by definition of well-foundedness nodes of T with no proper extension in T; such nodes, which we term *bottom nodes* of the tree, will have rank 0. Should ϱ_T not be defined for all nodes of the tree, we may by well-foundedness find a node s such that ϱ_T is not defined at s but is defined for each proper extension of s. But then the recipe tells us how to proceed to define ϱ_T at s.

The above illustrates the process of *definition by induction* on a well-founded tree. There is also available a method of *proof by induction* on a well-founded tree:

1·6 PROPOSITION *Let T be a well-founded tree, and $\Phi(s)$ some property. If* $\forall s :\in T \left[(\forall t \prec s\ \Phi(t)) \Longrightarrow \Phi(s)\right]$ *then* $\forall s :\in T\ \Phi(s)$.

That may be proved by supposing $\{s \mid \neg\Phi(s)\}$ to be non-empty, considering a T-minimal element thereof, and reaching a contradiction. It may also be proved by using the rank function ϱ_T and considering a counterexample s with $\varrho_T(s)$ minimal. Just such an argument proves the following

1·7 PROPOSITION *Let T be a well-founded tree and $s \in T$. For each $\nu < \varrho_T(s)$ there is a $t \prec s$ with* $\varrho_T(t) = \nu$.

2: Linking escape to well-foundedness

Characterising the escape set by well-foundedness of certain trees

We introduce the trees we shall use to calculate $\beta(b)$ for $b \in E(a,f)$. We shall define for our fixed a and for each $b \in \mathcal{X}$ a tree $T^a_b = T^a_b(f)$ of finite sequences and show using DC that $b \in A(a,f) \iff T^a_b$ is ill-founded.

2·0 DEFINITION For $b \in \mathcal{X}$, set

$$T^a_b(f) =_{\rm df} \big\{s \in {}^{<\omega}\mathcal{X} \mid \ell h(s) > 0 \Longrightarrow \big(s(0) = b \,\&\, \forall i{:}{<}\ell h(s)\ (a \curvearrowright_f s(i)) \,\&\, \forall i{:}{<}\ell h(s){-}1\ (s(i+1) \curvearrowright_f s(i))\big)\big\}.$$

Note that if $t \succ s \in T^a_b$, then $t \in T^a_b$, so that T^a_b is closed under shortening. Our definition is of most interest when $b \in \omega_f(a)$, since $b \notin \omega_f(a) \iff T^a_b = \{\emptyset\}$.

2·1 LEMMA (DC) $b \in A(a,f) \iff \exists$ *an infinite sequence* $\langle x_i | i < \omega\rangle$ *such that* $\forall i{:}{\in}\omega\ a \curvearrowright x_i$ *and*

$$b = x_0 \curvearrowleft x_1 \curvearrowleft x_2 \curvearrowleft \ldots \quad .$$

Proof : given such a sequence, one checks easily by induction on ξ that each of its members is in $A^\xi(a,f)$, hence is in $A(a,f)$; in particular $b = x_0$ is in $A(a,f)$. If no such sequence exists for a given b, then by DC the tree T^a_b will be well-founded under $\prec$, and hence we may define a rank function $\varrho = \varrho^a_b$ mapping T^a_b to the ordinals by

$$\varrho^a_b(s) = \sup\{\varrho^a_b(s{}^\frown\langle r\rangle) + 1 \mid r \in \mathcal{X} \;\&\; s{}^\frown\langle r\rangle \in T^a_b\}.$$

and show by induction on ξ that $\varrho^a_b(s) = \xi \implies \ell(s) \notin A^{\xi+1}(a,f)$: hence $b \notin A^{\varrho^a_b(\langle b\rangle)+1}(a,f)$.

Here are the inductive arguments: if we are given $x_0 \curvearrowleft x_1 \curvearrowleft \ldots$, with $\forall i\ a \curvearrowright x_i$, what is the least ordinal ζ such that not all x_i are in $A^\zeta(a,f)$? It cannot be 0 since $\forall i\ a \curvearrowright x_i$; it cannot be a successor ordinal $\xi+1$, for $x_{j+1} \in A^\xi(a,f) \implies x_j \in A^{\xi+1}(a,f)$ and so if each $x_i \in A^\xi(a,f)$ then each $x_i \in A^{\xi+1}(a,f)$, and it cannot be a limit ordinal since at limits we take intersections. Hence ζ does not exist, and so each $x_i \in A(a,f)$.

For the second argument, we know the tree is well-founded, so that we can define the rank function ϱ, and we want to show that for each s, $\ell(s) \notin A^{\varrho(s)+1}(a,f)$. If $\rho(s) = 0$, s cannot be extended within the tree, hence there is no $y \in A^0(a,f)$ with $y \curvearrowright \ell(s)$, and so $\ell(s)$ is not in $A^1(a,f)$. If $\rho(s) = \xi > 0$, then every extension $s{}^\frown\langle y\rangle$ in T has rank less than ξ, and so by the induction hypothesis no such y is in $A^\xi(a,f)$, and therefore $\ell(s) \notin A^{\xi+1}(a,f)$. $\dashv$ (2·1)

2·2 COROLLARY (DC) *For* $b \in \omega_f(a)$, $b \in E(a,f) \iff T^a_b$ *is well-founded.*

2·3 REMARK The structure of our argument is this: without DC we prove that if there is an infinite $\curvearrowleft$ sequence starting from b, as in 2·1, then $b \in A(a,f)$; if $T^a_b = \{\emptyset\}$ then $b \notin \omega_f(a)$; if $\{\langle b\rangle\} \in T^a_b$ and T^a_b is well-founded then $b \in E(a,f)$; and then DC tells us that these possibilities are exhaustive.

Bounding the rank of well-founded trees

We turn to the proof of the following

2·4 PROPOSITION *For each* $b \in E(a,f)$, $\varrho^a_b(\langle b\rangle) < \omega_1$.

from which the first part of Theorem 0·7 follows immediately:

2·5 COROLLARY $\theta \leqslant \omega_1$

Proof : by 2·4, each b in $E(a,f)$ leaves the A-sequence at the countable stage $\varrho^a_b(\langle b\rangle)+1$. Hence by stage ω_1 all those points that are to escape have already done so. $\dashv$ (2·5)

We defer the proof of the second part of Theorem 0·7, that if $A(a,f)$ is a Borel set, $\theta<\omega_1$.

2·6 DEFINITION Let $R_f=\{(x,\,y)\mid y\curvearrowright x\}$.

The relation R_f is G_δ, since it equals $\bigcap_n\bigcap_q\bigcup_m X_{n,q,m}$ where for natural numbers n, q and m, $X_{n,q,m}\ =_{\rm df}\ \{(x,y)\in\mathcal{X}\times\mathcal{X}\mid m>n\ \&\ |f^m(y)-x|<\frac{1}{q+1}\}$, but will not in general be closed (nor, as we shall see later, even F_σ):

2·7 EXAMPLE Take $\mathcal{X}$ to be the closed unit interval $[0,1]$, and $f(x)\equiv x^2$. Then

$$R_f=\{(1,\,1)\}\cup\{(0,y)\mid 0\leqslant y<1\}.$$

Our aim, nevertheless, is to show that many sequences of points in R_f are convergent to a point in R_f. To that end we define a sequence of maps ϕ_n, each mapping R_f to the countable set ${}^3\omega$ of all triples of natural numbers.

Recall that we are working in a Polish space $\mathcal{X}$: there is therefore a countable basis of open neighbourhoods, enumerated as $\langle N_p\mid p\in\omega\rangle$, such that for each $\varepsilon>0$ and each $x\in\mathcal{X}$, there is a p with $x\in N_p$ and $\mathrm{diam}(N_p)<\varepsilon$.

Given $(x,\,y)\in R_f$ and $n\in\omega$, let $\phi_n(x,y)$ be the first triple (k,p,q) in some natural well-ordering of ${}^3\omega$ such that

$$k>n\ \&\ |f^k(y)-x|<\frac{1}{n+1},\quad \mathrm{diam}(N_p)<\frac{1}{n+1}\ \&\ x\in N_p,\quad\text{and}\quad \mathrm{diam}(N_q)<\frac{1}{n+1}\ \&\ y\in N_q.$$

2·8 PROPOSITION *Let $(x_i,\,y_i)$ be a sequence of points in R_f such that for each n the sequence $\phi_n(x_i,y_i)$ is eventually constant. Then there are points x and y such that*

$$\lim_i x_i=x\ \&\ \lim_i y_i=y\ \&\ y\curvearrowright x.$$

Proof : let (k_n,p_n,q_n) be the eventual value of $\phi_n(x_i,y_i)$. For each n, the sequence (x_i) lies eventually in N_{p_n} and the diameter of that neighbourhood is less than $(n+1)^{-1}$; so the sequence (x_i) is Cauchy; since we are in a complete metric space, the (unique) limit point x will exist. Similarly $\lim_i y_i$ will exist, call it y.

For each n and all sufficiently large i,

$$|f^{k_n}(y_i)-x_i|<\frac{1}{n+1}$$

so by the continuity of f^{k_n},

$$|f^{k_n}(y)-x|\leq\frac{1}{n+1}$$

As k_n tends to infinity with n, $y\curvearrowright x$ as required. $\dashv$ (2·8)

2·9 REMARK A special case of the above is when each $x_i=b$: then our conclusion is that $y\curvearrowright b$. Note also that if $a\curvearrowright y_i$ for each i, then since all y_i lie in the closed set $\omega_f(a)$, so does y, and thus $a\curvearrowright y$.

Now we define a map $\sigma=\sigma^a_b$ from the tree T^a_b to a tree S^a_b of finite sequences of elements of ${}^3\omega$. We define σ progressively:

$$\begin{aligned}
\sigma(\langle\,\rangle)&=\langle\,\rangle\\
\sigma(x_0)&=\langle(0,0,0)\rangle\\
\sigma(x_0,x_1)&=\langle(0,0,0),\phi_0(x_0,x_1)\rangle\\
\sigma(x_0,x_1,x_2)&=\langle(0,0,0),\phi_0(x_0,x_1),\phi_0(x_1,x_2),\phi_1(x_1,x_2),\phi_1(x_0,x_1)\rangle\\
\sigma(x_0,x_1,x_2,x_3)&=\sigma(x_0,x_1,x_2)^\frown\langle\phi_0(x_2,x_3),\phi_1(x_2,x_3),\phi_2(x_2,x_3),\phi_2(x_1,x_2),\phi_2(x_0,x_1)\rangle
\end{aligned}$$

and so on. The order of listing is according to this diagram:

$$
\begin{array}{c|c|c|c|c}
 & \phi_0 & \phi_1 & \phi_2 & \\
x_0, x_1 & 1 & 4 & 9 & \dots \\
x_1, x_2 & 2 & 3 & 8 & \dots \\
x_2, x_3 & 5 & 6 & 7 & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Note that

$$u \succ v \implies \sigma(u) \succ \sigma(v) \tag{$*$}$$

The members of S^a_b are by definition the initial segments of the image of σ^a_b:

$$S^a_b \ =_{\mathrm{df}} \ \{t \mid \exists u {:}{\in} T^a_b \ t \succcurlyeq \sigma^a_b(u)\}$$

2·10 Proposition *If T^a_b is well-founded then so is S^a_b.*

Proof : suppose S^a_b ill-founded, with a descending sequence $t_0 \succ t_1 \succ t_2 \succ \ldots$, where by interpolating terms if necessary we may assume that $\ell h(t_i) = i$. So $t_0 = \langle\,\rangle$, and each t_i for $i \geqslant 1$ is an initial segment of a sequence of the form $\sigma(x^i_0, x^i_1, \ldots x^i_{j_i})$, where $j_i \geqslant i-1$, for each i, $x^i_0 = b$ and for each $j < j_i - 1$ $x^i_j \curvearrowleft x^i_{j+1}$.

Then for each j and n, the sequence $\phi_n(x^i_j, x^i_{j+1})$ is for sufficiently large i defined and constant: *e.g.* the first term of t_i for $i \geqslant 1$ is $(0,0,0)$, the third term of t_i for $i \geqslant 3$ will be $\phi_0(x^3_1, x^3_2)$ and the seventh term of t_i for $i \geqslant 7$ will be $\phi_1(x^7_2, x^7_3)$. By Proposition 2·8 and Remark 2·9, there are points x_j for $j \in \omega$ such that $\lim_i x^i_j = x_j$, $a \curvearrowright x_j$ and

$$b = x_0 \curvearrowleft x_1 \curvearrowleft x_2 \curvearrowleft \ldots$$

contradicting the well-foundedness of T^a_b. ⊣ (2·10)

Thus for $b \in E(a, f)$, the image of σ generates a well-founded tree, S^a_b: but as ${}^3\omega$ is countable, so is S^a_b, and hence its rank is less than ω_1. We shall show that the height of T^a_b is at most that of S^a_b by proving by induction on the well-founded tree T^a_b the following

2·11 Proposition *For each $b \in E(a, f)$ and $u \in T^a_b$,*

$$\rho_T(u) \leq \rho_S(\sigma(u)).$$

Proof:

$$
\begin{aligned}
\rho_T(u) &= \sup\{\rho_T(v) + 1 \mid u \succ v\} && \\
&\leqslant \sup\{\rho_S(\sigma(v)) + 1 \mid u \succ v\} && \text{by the induction hypothesis} \\
&\leqslant \sup\{\rho_S(t) + 1 \mid \sigma(u) \succ t\} && \text{since by } (*)\ \{\sigma(v) | u \succ v\} \subseteq \{t | \sigma(u) \succ t\} \\
&= \rho_S(\sigma(u))
\end{aligned}
$$

⊣ (2·11)

Thus $\varrho^a_b(\langle b \rangle) \leqslant \rho_{S^a_b}(\langle (0,0,0) \rangle) < \omega_1$. ⊣ (2·4)

A confession

Logicians will have noticed that this section of the paper reduces the original dynamical question to a result of classical descriptive set theory, and that the argument just given is, apart from the simplifications that the present special case permits, simply a modern proof of a modern generalisation of that classical result; namely Kunen's proof of the Kunen–Martin theorem, as presented in Moschovakis' treatise [14], page 101, Theorem 2G.2.

That classical result may be formulated thus:

> if R is an analytic relation on a Polish space, and X is an analytic subset of that space, and the restriction $R \cap (X \times X)$ of R to X is well-founded then that restriction is of countable height.

We refer the reader to Moschovakis' notes on pages 113–5 for the history of that result, and to his Chapter I for a definition of the notion of an *analytic* subset of a Polish space; a *co-analytic* set is the complement of an analytic set. Every Borel set is analytic; an analytic set is Borel if and only if it is also co-analytic.

These concepts arose in the development of descriptive set theory in the first quarter of this century; lately this subject has as described by Moschovakis, been re-worked as an effective theory by the adoption of concepts from recursion theory. In terms of the notation of the effective theory for definable subsets of Polish spaces we have, from the definition of $\curvearrowright$ and using 2·1 and 2·2:

2·12 PROPOSITION *(i) The relation $y \curvearrowright z$ is a Π^0_2 property of y, z,and f.*
(ii) $A(a,f)$ is a $\Sigma^1_1(a,f)$ set; $E(a,f)$ is a $\Pi^1_1(a,f)$ set.

So in the older terminology, $A(a,f)$ is an analytic set and $E(a,f)$ is co-analytic.

2·13 REMARK We have seen that $\omega_f(y)$ is closed, and thus for each y the property $y \curvearrowright z$ is $\Pi^0_1(y,z,f,\alpha)$ for some real α **dependent on** y. Example 2·7 shows that there can be no uniform Π^0_1 formula $\Phi(y,z,f)$ such that for an arbitrary continuous f and arbitrary points y and z, $y \curvearrowright z \iff \Phi(y,z,f)$, nor indeed a Σ^0_2 one, as shown by the following example where the relation $\curvearrowright$ fails to be F_σ, for supplying which the author thanks a colleague of the referee.

> Consider the space $X = \mathbf{T}^2$, where $\mathbf{T}$ is the one-dimensional unit circle $\{e^{2\pi ix} \mid x \in [0,1]\}$. Define $f : X \to X$ by $f(\xi,\eta) = (\xi, \xi\eta)$. Let $y = e^{2\pi ix}$. The set $\{z \mid y \curvearrowright z\}$ is finite if x is rational, and is $\mathbf{T}$ otherwise. Hence the set $\{(y,z) \in X \mid y \curvearrowright z\}$ is a dense G_δ in X and therefore, being co-meagre, cannot be F_σ.

The author's original example will be mentioned in 3·5.

Turning now to the proof of the last part of Theorem 0·7: the relation R_f is Borel, being indeed G_δ; on $A(a,f)$ it is ill-founded, but on $E(a,f)$ it is well-founded. The obstacle to proving that $\theta < \omega_1$ is therefore that we do not know *prima facie* that the set $E(a,f)$ is Borel, only that it is co-analytic; but if it is Borel, (as it will be if $A(a,f)$ is Borel, since $\omega_f(a)$ is closed and $E(a,f) = \omega_f(a) \setminus A(a,f)$) then the classical result applies and we may indeed conclude that $\theta < \omega_1$. $\dashv$ (0·7)

2·14 REMARK A converse result holds: if $\theta < \omega_1$ in a particular case, then $E(a,f)$ will be Borel.

We shall not in this paper prove the classical result, though we have in the proof of 2·4 given the main ideas of the proof of the Kunen–Martin theorem from which it follows. It is proved as Theorem 31.1 on page 239 of Kechris' text [7], which is an excellent work despite its concealing from its readers on page 132 the dangerous knowledge that Mathias forcing — many variants of which are discussed in *Happy Families*, in the *Annals of Mathematical Logic*, **12** (1977), pp 59–111, reviewed as *Mathematical Reviews* **58** # 10462 — was developed for the express purpose of proving the consistency with *ZF* + *DC* of the statement that all sets of "reals" are completely Ramsey.

The Kunen–Martin theorem may also be derived from Theorem 1.6 of the paper [4] by Cenzer and Mauldin. These last authors are concerned with sets defined as the closure points of iterated operators on Polish spaces, such as our Γ; they have general results for those operators that they call Borel, analytic or co-analytic; and they establish the countability of an ordinal corresponding to our $\beta(x)$ by a stage comparison theorem, again using ideas of Kunen. The "working mathematician" may find the style of much of their paper more approachable than more "logical" treatments.

3: Recurrent points, minimal sets, and maximal recurrent points

Recurrent points

3·0 DEFINITION A *recurrent* point is a b such that $b \curvearrowright b$.

It has long been known that the existence of recurrent points is neither certain nor impossible:

3·1 EXAMPLE Let $\mathcal{X} = \mathbb{R}$, and $f(x) \equiv x + 1$. Then f has no recurrent points.

3·2 THEOREM (AC) *Let $\mathcal{X}$ be a compact Polish space and $f : \mathcal{X} \longrightarrow \mathcal{X}$ continuous. Then recurrent points exist: indeed each $x \in \mathcal{X}$ attacks at least one recurrent point.*

3·3 REMARK The above use of AC could be reduced to an application of DC by working in $L[a, f]$ and appealing to Shoenfield's absoluteness theorem, which appears as Theorem 8F.10 on page 526 of [14].

We may use the following lemma since in a metric space second countability and separability are equivalent conditions.

3·4 LEMMA (AC) *In a second countable space $\mathcal{X}$ there can exist neither a strictly descending sequence $\langle C_\nu \mid \nu < \omega_1 \rangle$ nor a strictly ascending sequence $\langle D_\nu \mid \nu < \omega_1 \rangle$ of non-empty closed subsets of $\mathcal{X}$.*

Proof: given a descending counter-example in a space with countable basis $\{N_s \mid s \in \omega\}$, pick $p_\nu \in C_\nu \setminus C_{\nu+1}$, and $s_\nu \in \omega$ with $p_\nu \in N_{s_\nu}$ and $N_{s_\nu} \cap C_{\nu+1}$ empty. There will be $\nu < \delta < \omega_1$ with $s_\nu = s_\delta$. But then $p_\delta \in C_\delta \cap N_{s_\delta} \subseteq C_{\nu+1} \cap N_{s_\nu} = \varnothing$, a contradiction. In the ascending case, pick $p_\nu \in D_{\nu+1} \setminus D_\nu$, and $s_\nu \in \omega$ with $p_\nu \in N_{s_\nu}$ and $N_{s_\nu} \cap D_\nu$ empty. Again there will be $\nu < \delta < \omega_1$ with $s_\nu = s_\delta$. But then $p_\nu \in D_{\nu+1} \cap N_{s_\nu} \subseteq D_\delta \cap N_{s_\delta} = \varnothing$, another contradiction. ⊣ (3·4)

3·5 REMARK Hausdorff in §27 of his book *Mengenlehre* [6] proves with a beautiful argument that, more generally, there cannot be an uncountable sequence, whether strictly increasing or strictly decreasing, of sets that are simultaneously F_σ and G_δ. That may be used to prove that in Baire space, neither the set $\{\beta \mid \beta \curvearrowright_{\mathfrak{s}} \beta\}$ nor for any ε the set $\{\beta \mid \beta \curvearrowright_{\mathfrak{s}} \varepsilon\}$ is $\mathbf{\Sigma^0_2}$, and therefore the relation $R_{\mathfrak{s}}$ cannot be, either; see Proposition 7·6 of [13] for the details, but note that in two places in the proof, β_n is printed instead of $\beta \restriction n$.

Proof of 3·2: We know that each $\omega_f(x)$ is a closed set, which, by sequential compactness is non-empty, and that if $y \in \omega_f(x)$ then $\omega_f(y) \subseteq \omega_f(x)$. Start from x, and set $C_0 = \omega_f(x)$. We shall define a shrinking sequence of closed sets all of the form $\omega_f(z)$.

If $C_\xi = \omega_f(x_\xi)$ ask if there is a $y \in C_\xi$ such that $\omega_f(y)$ is a proper subset of C_ξ: if not, then every element of $\omega_f(x_\xi)$ is recurrent (in a strong sense, indeed). If there is, pick some such and call it $x_{\xi+1}$, and take $C_{\xi+1} = \omega_f(x_{\xi+1})$.

At limit stages, take the intersection, call it C'_λ: by compactness it will be non-empty. Pick x_λ in it. Then for each $\nu < \lambda$ $x_\nu \curvearrowright x_\lambda$; so $\omega_f(x_\lambda) \subseteq C'_\lambda$. Set $C_\lambda = \omega_f(x_\lambda)$ and continue.

By the Lemma this process breaks down before stage ω_1: when it does, we have reached a z such that $\forall v, w :\in \omega_f(z)\ v \curvearrowright w$: in particular, each member of the non-empty set $\omega_f(z)$ is recurrent, and is attacked by our original x. ⊣ (3·2)

I am grateful to Marianne Morillon for correcting a slip in my original presentation of the above proof, stemming from my forgetting that at each stage the chosen point y might not be a member of $\omega_f(y)$; Christian Delhommé suggests that it might therefore be better to present the proof as proceeding by choice of sequences of closed sets, rather than of points.

3·6 DEFINITION The set $\omega_f(z)$ reached at the final step, or any point w with $w \in \omega_f(w) = \omega_f(z)$, will be termed *minimal*.

The minimal sets are pairwise disjoint closed sets, which might, but need not, partition the set of recurrent points. Here are some examples.

3·7 EXAMPLE Let $\mathcal{X} = [0, 1]$ and $f(x) \equiv x^2$: then $A^\infty = \{0, 1\}$, although f is onto and 1-1 and $\mathcal{X}$ is compact. The minimal sets in this case are $\{0\}$ and $\{1\}$.

3·8 EXAMPLE Let $\mathcal{X} = \{x \in \mathbb{R}^2 \mid d(x, 0) \leqslant 1\}$. Let f be rotation by an irrational multiple of 2π. The minimal sets are the concentric circles with centre the origin, so they partition the space, in which every point is recurrent.

3·9 EXAMPLE of a case where a non-minimal but recurrent y attacks two inequivalent minimal points. Let $y \in 4^\omega$ attack everything in that space, let a have only 0's and 1's, and let b have only 2's and 3's. By compactness, a and b, if not minimal themselves, will attack minimal points. Neither a nor b can attack y.

3·10 REMARK In a compact space, the set of all points in some minimal set is $\{x \mid \forall y\, x \curvearrowright y \implies y \curvearrowright x\}$, so it is Π^1_1. Hence by a theorem of Burgess either there are at most $\aleph_1$ minimal sets or there is a perfect set of inequivalent minimal points, where points x and y are considered equivalent if $x \curvearrowright y \curvearrowright x$. To see that, let $P(x, f)$ be a Π^1_1 formula saying that x is a minimal point with respect to the continuous function f. Define an equivalence relation $\equiv$ on the whole space by

$$x \equiv y \iff_{\mathrm{df}} \big[(P(x, f) \text{ or } P(y, f)) \implies (x \curvearrowright y \,\&\, y \curvearrowright x)\big] :$$

under this relation, the set of non-minimal points forms a single equivalence class, and two minimal points are equivalent iff they attack each other. From its definition, $\equiv$ is $\Sigma^1_1(f)$ and is defined on the whole space, and so Burgess' theorem applies.

3·11 PROPOSITION *Suppose no image of x is recurrent. Then $\omega_f(x)$ is nowhere dense.*

Proof : Each point of $\omega_f(x)$ is a limit of points (namely the $f^k(x)$, for $k \in \omega$) not in that closed set, hence its interior is empty. ⊣ (3·11)

3·12 COROLLARY *Let r be Cohen generic over $L[f, x]$ and suppose that no image of x is recurrent. Then r is not attacked by x.*

3·13 LEMMA *Suppose that $C = \omega_f(b)$ is compact. Then $f \restriction C$ maps C onto C.*

Proof : C is closed under f. Let $u \in C$. Then $b \curvearrowright u$, so let $(k_i)_i$ be an increasing sequence of positive integers such that $\lim_{i\to\infty} f^{k_i}(b) = u$. Let $(m_j)_j$ be a subsequence of the sequence $(k_i - 1)_i$ such that the sequence $f^{m_j}(b)$ is convergent, to w say. $w \in C$, as C is closed, and $f(w) = u$. ⊣ (3·13)

3·14 PROPOSITION *Let $\mathcal{X}$ be connected. Suppose that f is 1-1 and that C is a compact minimal set with non-empty interior. Then $C = \mathcal{X}$.*

Proof : Deny, let D be the closure of $\mathcal{X} \setminus C$, let $t \in C \cap D$, which will be non-empty by the connectedness of $\mathcal{X}$, and let v be in the interior of C. Pick $\varepsilon > 0$ so that all points distant less than 2ε from v is a subset of C.

$t \curvearrowright v$, by the minimal character of C: let $k > 0$ be such that $d(f^k(t), u) < \varepsilon$. Choose $\delta > 0$ so that $d(f^k(t), (f^k(v)) < \varepsilon$ whenever $d(t, v) < \delta$, and choose $w \notin C$ with $d(t, w) < \delta$. Then $f^k(w) \in C$, so for some i with $0 \leqslant i < k$, $x =_{\mathrm{df}} f^i(w) \notin C$ and $f(x) \in C$. But by the lemma, $f(x) = f(y)$ for some $y \in C$, contradicting the 1-1 character of f. ⊣ (3·14)

Characterising the abode by recurrent points

The following result shows that provided not every point in $\omega_f(a)$ escapes, recurrent points exist. We emphasize that the space is not assumed to be compact. The apparent use of the Axiom of Choice is avoidable.

3·15 THEOREM *Let $\mathcal{X}$ be a complete separable metric space, $f : \mathcal{X} \longrightarrow \mathcal{X}$ a continuous map, and a, x arbitrary points in $\mathcal{X}$. Then*

$$x \in A(a,f) \iff \exists b\; a \curvearrowright b \curvearrowright b \curvearrowright x.$$

Proof : the characterisation given in Lemma 2·1 shows immediately that if $b \curvearrowright b \curvearrowright x$, the point x is in $A(a,f)$, as we could take $x_0 = x$ and $x_i = b$ for $i > 0$. In particular every recurrent point is in $A(a,f)$.

Suppose therefore that for each $i < \omega$, $a \curvearrowright x_{i+1} \curvearrowright x_i$ and $x_0 = x$. Our task is to build a recurrent b with $a \curvearrowright b \curvearrowright x_0$.

We shall define a sequence of points y_i starting with $y_0 = x_0$ and converging to a point b, such that for each i, $a \curvearrowright y_i$ and $b \curvearrowright y_i$. Since $\omega_f(a)$ and $\omega_f(b)$ are closed, that will give $a \curvearrowright b$ and $b \curvearrowright b$, so b is indeed a recurrent point with $a \curvearrowright b \curvearrowright x_0$. The points y_i will each be of the form $f^{m_i}(x_i)$: and we shall derive helpful properties of the sequence (y_i) from those of the sequence (x_i) by using, without further comment, the general lemma that if $c \curvearrowright d$ then $f(c) \curvearrowright d$ and $c \curvearrowright f(d)$. The numbers m_i are strictly increasing and will be chosen to make the sequence y_i a Cauchy sequence.

In defining the sequence y_i we shall define various sequences of positive reals tending monotonically to 0, and we shall define various strictly increasing sequences of positive integers.

More specifically, for each $i < \omega$ we shall define a sequence $(\varepsilon^i_k)_{k<\omega}$ of positive reals tending monotonically to 0, and for $0 < i < \omega$ a strictly increasing sequence $(\ell^i_k)_{k<\omega}$ of natural numbers. Further we shall define a decreasing sequence $(\eta_i)_{i<\omega}$ of positive reals tending to 0, and we shall define a strictly increasing sequence of positive integers $(m_i)_{1\leqslant i<\omega}$.

Our definition takes place in infinitely many rounds. In Round 0, we shall define the point y_0, the sequence (ε^0_k) and the positive real η_0. In Round 1, we shall define m_1, y_1, the sequences (ℓ^1_k) and (ε^1_k) and the positive real η_1. For $n > 1$, we shall by the end of Round $n-1$ have defined m_{n-1}, y_{n-1}, ℓ^{n-1}_k and ε^{n-1}_k for each k, and η_{n-1}, and in Round n we shall define m_n, y_n, ℓ^n_k, ε^n_k and η_n.

Let $\Psi(i,n,\gamma,k)$ be the statement that

$$|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k+\ell^{n-1}_k+\ldots+\ell^i_k}(\gamma) - y_{i-1}| < \varepsilon^{i-1}_k$$

We shall verify in Round n, for $n \geqslant 1$, that

$$\forall k \forall\gamma\forall i\big((k \in \omega \,\&\, \gamma \in \mathcal{X} \,\&\, 1 \leqslant i \leqslant n) \implies \Psi(i,n,\gamma,k)\big).$$

In fact, for each γ and k, $\Psi(n,n,\gamma,k)$ will follow from our choice of ε^n_k and ℓ^n_k; and then the other cases will be covered by the following

3·16 Lemma *If ε^n_k and ℓ^n_k have been defined, then for $i < n$,*

$$\big(\Psi(n,n,\gamma,k) \,\&\, \Psi(i,n-1,f^{\ell^n_k}(\gamma),k)\big) \implies \Psi(i,n,\gamma,k).$$

Proof : Let $|\gamma - y_n| < \varepsilon^n_k$. By $\Psi(n,n,\gamma,k)$, $|f^{\ell^n_k}(\gamma) - y_{n-1}| < \varepsilon^{n-1}_k$: so we may apply $\Psi(i,n-1,f^{\ell^n_k}(\gamma),k)$ and use the fact that

$$f^{\ell^{n-1}_k+\ldots+\ell^i_k}\big(f^{\ell^n_k}(\gamma)\big) = f^{\ell^n_k+\ell^{n-1}_k+\ldots+\ell^i_k}(\gamma) \qquad \dashv (3\cdot16)$$

We are now ready to begin our construction.

Round 0. Put $y_0 = x_0$, choose an arbitrary sequence ε^0_k of positive reals tending monotonically to 0 as $k \longrightarrow \infty$, and set $\eta_0 = \frac{1}{4}\varepsilon^0_0$.

Round 1. Pick m_1 such that $|f^{m_1}(x_1) - y_0| < \eta_0$: that is possible as $x_1 \curvearrowright y_0 = x_0$. Put $y_1 = f^{m_1}(x_1)$. Choose a sequence $\ell^1_0 < \ell^1_1 < \ell^1_2 < \ldots$ such that $\forall k\, |f^{\ell^1_k}(y_1) - y_0| < \frac{1}{2}\varepsilon^0_k$: that can be done as $y_1 \curvearrowright y_0$.

Choose a sequence ε^1_k tending to 0 monotonically from above such that

$$\forall k \forall \gamma{:}{\in} \mathcal{X}\ \big(|\gamma - y_1| < \varepsilon^1_k \implies |f^{\ell^1_k}(\gamma) - f^{\ell^1_k}(y_1)| < \tfrac{1}{2}\varepsilon^0_k\big):$$

that can be done as $f^{\ell^1_k}$ is continuous at y_1.

That implies that for all $k \in \omega$ and all $\gamma \in \mathcal{X}$,

$$|\gamma - y_1| < \varepsilon^1_k \implies |f^{\ell^1_k}(\gamma) - y_0| < \varepsilon^0_k$$

which is the statement $\Psi(1,1,\gamma,k)$. Set $\eta_1 = \min(\frac{1}{8}\varepsilon^0_0, \frac{1}{4}\varepsilon^1_1) = \min(\frac{1}{2}\eta_0, \frac{1}{4}\varepsilon^1_1)$.

Round 2. Pick $m_2 > m_1$ such that $|f^{m_2}(x_2) - y_1| < \eta_1$ — possible as $x_2 \curvearrowright y_1$ — and put $y_2 = f^{m_2}(x_2)$. Choose a sequence $\ell^2_0 < \ell^2_1 < \ell^2_2 < \ldots$ such that $\forall k \; |f^{\ell^2_k}(y_2) - y_1| < \frac{1}{2}\varepsilon^1_k$: that can be done as $y_2 \curvearrowright y_1$.

Choose a sequence ε^2_k tending to 0 monotonically from above such that

$$\forall k \forall \gamma{:}{\in}\mathcal{X} \left(|\gamma - y_2| < \varepsilon^2_k \implies |f^{\ell^2_k}(\gamma) - f^{\ell^2_k}(y_2)| < \tfrac{1}{2}\varepsilon^1_k\right) :$$

that can be done as $f^{\ell^2_k}$ is continuous at y_2.

That implies that for all $k \in \omega$ and for all $\gamma \in \mathcal{X}$,

$$|\gamma - y_2| < \varepsilon^2_k \implies |f^{\ell^2_k}(\gamma) - y_1| < \varepsilon^1_k$$

and therefore

$$|\gamma - y_2| < \varepsilon^2_k \implies |f^{\ell^2_k + \ell^1_k}(\gamma) - y_0| < \varepsilon^0_k$$

which are the statements $\Psi(2,2,\gamma,k)$ and $\Psi(1,2,\gamma,k)$ respectively. Set $\eta_2 = \min(\frac{1}{2}\eta_1, \frac{1}{4}\varepsilon^2_2)$ and continue to the next round.

Round n, for $n > 2$. Pick $m_n > m_{n-1}$ such that $|f^{m_n}(x_n) - y_{n-1}| < \eta_{n-1}$, and put $y_n = f^{m_n}(x_n)$. Choose a sequence $\ell^n_1 < \ell^n_2 < \ldots$ such that $\forall k \; |f^{\ell^n_k}(y_n) - y_{n-1}| < \frac{1}{2}\varepsilon^{n-1}_k$: that can be done as $y_n \curvearrowright y_{n-1}$.

Choose a sequence ε^n_k tending to 0 monotonically from above such that

$$\forall k \forall \gamma{:}{\in}\mathcal{X} \left(|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k}(\gamma) - f^{\ell^n_k}(y_n)| < \tfrac{1}{2}\varepsilon^{n-1}_k\right) :$$

that can be done as $f^{\ell^n_k}$ is continuous at y_n.

That implies that for all $k \in \omega$ and for all $\gamma \in \mathcal{X}$

$$|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k}(\gamma) - y_{n-1}| < \varepsilon^{n-1}_k$$

which is the statement $\Psi(n,n,\gamma,k)$; we have seen that it follows from statements established in previous rounds that for $n > i \geqslant 1$,

$$|\gamma - y_n| < \varepsilon^n_k \implies |f^{\ell^n_k + \ell^{n-1}_k + \ldots + \ell^i_k}(\gamma) - y_{i-1}| < \varepsilon^{i-1}_k$$

which is $\Psi(i,n,\gamma,k)$. Set $\eta_n = \min(\frac{1}{2}\eta_{n-1}, \frac{1}{4}\varepsilon^n_n)$.

Once all the rounds have been completed, we shall have defined a sequence y_i such that for each i, $|y_{i+1} - y_i| < \eta_i$. By definition $\eta_{i+1} = \min(\frac{1}{2}\eta_i, \frac{1}{4}\varepsilon^{i+1}_{i+1})$, so in particular $\eta_{i+1} \leqslant \frac{1}{2}\eta_i$, and so $\sum_{i<\omega} \eta_i$ is convergent. Hence (y_i) is a Cauchy sequence, and hence by the completeness of the space $\mathcal{X}$ is convergent. Let b be its limit.

3·17 LEMMA *For each k, $|b - y_k| < \varepsilon^k_k$.*

Proof : For each k, $\eta_k \leqslant \frac{1}{4}\varepsilon^k_k$, $\eta_{k+1} \leqslant \frac{1}{2}\eta_k \leqslant \frac{1}{8}\varepsilon^k_k$, and so for each $j \geqslant k$, $\eta_j \leqslant 2^{k-2-j}\varepsilon^k_k$; we know that for each i, $|y_{i+1} - y_i| < \eta_i$, and hence for $k < j$, $|y_j - y_k| < \eta_k + \ldots + \eta_{j-1}$; thus

$$|b - y_k| \leqslant \sum_{j \geqslant k} \eta_j \leqslant (\tfrac{1}{4} + \tfrac{1}{8} + \ldots)\,\varepsilon^k_k = \tfrac{1}{2}\varepsilon^k_k. \qquad \dashv (3{\cdot}17)$$

Fix i. We assert that $b \curvearrowright y_i$. Thus, we must show that

$$\forall \varepsilon > 0 \exists n \, |f^n(b) - y_i| < \varepsilon;$$

moreover that n may be chosen arbitrarily large.

Fix $\varepsilon > 0$. Pick $k > i$ such that $\varepsilon_k^i < \varepsilon$. By the lemma, $|b - y_k| < \varepsilon_k^k$, and so applying $\Psi(i{+}1, k, b, k)$,

$$|f^{\ell_k^k + \ell_k^{k-1} + \ldots + \ell_k^{i+1}}(b) - y_i| < \varepsilon_k^i < \varepsilon,$$

as required.

Note finally that as k can be chosen arbitrarily large, the power of f applied to b, which is at least ℓ_k^{i+1}, can also be made arbitrarily large.

Our theorem is proved. $\dashv$ (3·15)

Maximal recurrent points.

In fact our proof of 3·15 has established the following statement:

3·18 Proposition *Given $\mathcal{X}$, f, and a, suppose that for all i $a \curvearrowright z_{i+1} \curvearrowright z_i \curvearrowright \ldots \curvearrowright z_0$. Then there are natural numbers $m_0 < m_1 < \ldots$ such that setting $y_i = f^{m_i}(z_i)$, the sequence (y_i) is convergent with limit b, say, and $b \curvearrowright y_i$ for each i. It follows that b is recurrent, and that for all i, $a \curvearrowright b \curvearrowright z_i$ and $\omega_f(z_i) = \omega_f(y_i)$.*

3·19 Remark Note that if the points z_i' form a second set satisfying the hypothesis of the Proposition, with $\forall i \; z_i \curvearrowright z_i' \curvearrowright z_i$, and y_i', b' are the outcome of repeating the argument, then

$$\forall i \; b \curvearrowright z_{i+1} \curvearrowright z_{i+1}' \curvearrowright y_i' \;\&\; b' \curvearrowright z_{i+1}' \curvearrowright z_{i+1} \curvearrowright y_i$$

and so $b \curvearrowright b' \curvearrowright b$.

3·20 Proposition *In these circumstances, $\omega_f(b)$ is the closure of $\bigcup_i \omega_f(z_i)$.*

Proof: write C_b for $\omega_f(b)$, C_i for $\omega_f(z_i)$, and C for the closure of $\bigcup_i C_i$. Each C_i is closed topologically and also under the action of f, hence so is C. C_b is a closed set containing $\bigcup_i C_i$, and therefore $C_b \supseteq C$. But $b \in C$, being the limit of the sequence y_i, so each $f^k(b)$ is in C, and therefore each point of C_b is in C. $\dashv$ (3·20)

3·21 Definition Call a point b *maximal recurrent in* $\omega_f(a)$ if $a \curvearrowright b \curvearrowright b$ and whenever $a \curvearrowright c \curvearrowright c \curvearrowright b$, then $b \curvearrowright c$.

With the help of the axiom of choice the above proposition yields the following

3·22 Corollary (AC) *If d is a recurrent point in $\omega_f(a)$, then there is a point b which is maximal recurrent in $\omega_f(a)$ with $a \curvearrowright b \curvearrowright d$.*

Proof: set $d_0 = d$. If d_0 is not maximal in $\omega_f(a)$, pick d_1 with $a \curvearrowright d_1 \curvearrowright d_1 \curvearrowright d_0 \not\curvearrowright d_1$; if d_1 is not maximal, continue. Proposition 3·18 tells us that our construction can be continued at countable limit ordinals. If we never encounter a maximal recurrent point, then our construction will yield for every countable ordinal ν a recurrent point d_ν with $a \curvearrowright d_\zeta \curvearrowright d_\nu \not\curvearrowright d_\zeta$ for $\nu < \zeta < \omega_1$. But then the sequence $\langle \omega_f(d_\nu) \mid \nu < \omega_1 \rangle$ will form a strictly increasing sequence of closed sets of order type ω_1, contradicting Lemma 3·4. $\dashv$ (3·22)

3·23 Remark Again, that use of AC could be reduced to an application of DC by working in $L[a, f]$ and appealing to Shoenfield's absoluteness theorem.

3·24 Remark We could also formulate the notion of a *maximal recurrent* point in the space $\mathcal{X}$ as a whole, without reference to a particular point a; the same argument will prove that if recurrent points exist, so do maximal ones. In a case such as the shift function acting on Baire space, the maximal recurrent points will be simply be those whose orbit is dense in the whole space.

3·25 Remark Note that it follows from Theorem 3·15 that each point in the abode A is in the closure of the set of recurrent points. The converse need not hold, as we shall see later; and thus the abode is not necessarily identical with the set of *non-wandering points* studied by earlier writers such as Birkhoff, which exactly equals that closure.

4: Long delays in Baire space

Explicit construction of well-founded trees.

For any ordinal η we can uniformly build a tree of height that ordinal:

4·0 DEFINITION For an arbitrary ordinal η let T_η be the set of all strictly descending sequences of ordinals less than η.

T_η is, naturally, a well-founded tree. We include the empty sequence $\emptyset$ in each T_η as its topmost point.

4·1 PROPOSITION *For each η, $\varrho_{T_\eta}(\emptyset) = \eta$.*

Proof : by induction on η. $T_\emptyset = \{\emptyset\}$, and so $\varrho_{T_\emptyset}(\emptyset) = 0$. $\eta = \beta + 1$: the sequences with first element less than β will all lie in T_β, and have rank $< \beta$ accordingly. The sequences with first element β form a tree isomorphic to T_β (on removing β from each sequence) so $\varrho_{T_\eta}(\langle\beta\rangle) = \beta$ and thus $\varrho_{T_\eta}(\emptyset) = \beta + 1 = \eta$.

Now let η be a limit ordinal: all the non-empty sequences starting below η will have rank less than η, so the empty sequence will have rank η, all ranks here being computed in T_η. ⊣ (4·1)

Evidently when η is countable T_η will be isomorphic to a tree $U \subseteq {}^{<\omega}\omega$; it will be convenient instead to find a tree, isomorphic to T_η, that is a subset of the set $\mathcal{S}$ we now define.

4·2 DEFINITION Let $\mathcal{S}$ be the set of finite strictly increasing sequences of odd prime numbers (excluding 1). We count $\emptyset$, the empty sequence, as a member of $\mathcal{S}$.

4·3 With an eye to applications in §7, we show, more generally, that given a countable *linear* ordering $(X, <)$, we may uniformly define a map ψ, from the set of decreasing finite sequences of members of X to the set of increasing finite sequences of odd primes, which preserves the end-extension relation. Note that $(X, <)$ need not be a well-ordering; it might for example be the set of rationals under the Euclidean order.

Let $h : X \xrightarrow{1-1} \omega$. Using h we can assign to each $x \in X$ an bijection (usually not order-preserving !) g_x of $\{y \in X \mid y < x\}$ and either some finite n or ω.

We set $\psi(\emptyset) = \emptyset$. We map sequences of length 1 to sequences of odd primes of length 1, using h composed with an enumeration p_i of odd primes: $\psi(\langle x\rangle) = \langle p_{h(x)}\rangle$.

Now suppose we have already defined $\psi(s)$, where $s \neq \emptyset$. Let x be the least element of s, and let p_j be the largest element of $\psi(s)$. If $t = s^\frown\langle y\rangle$ where $y < x$, set $\psi(t) = \psi(s)^\frown\langle p_{j+1+g_x(y)}\rangle$.

The plan of attack

We explore and exploit the possibility of embedding countable well-founded trees into the relation $\curvearrowright$.

4·4 LEMMA *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous, and let T be a non-empty well-founded tree with top-most point $\emptyset$. Suppose that we have points x_T and x_s (for $s \in T$) in the space $\mathcal{X}$ such that for all s and t in T, $x_T \curvearrowright x_s$ and if $s \prec t$ then $x_s \curvearrowright x_t$. Then for each $s \in T$, $x_s \in A^{\varrho_T(s)}(x_T, f)$.*

Proof : following 1·6, let $r \prec s \Longrightarrow x_r \in A^{\varrho_T(r)}(x_T, f)$. Then $x_s \in \omega_f(x_T) \cap \bigcap\{A^{\varrho_T(r)+1}(x_T, f) \mid r \prec s\}$ which by 1·7 and 0·2 equals $A^{\varrho_T(s)}(x_T, f)$. ⊣ (4·4)

We would like to have $\forall s{:}\in T\ x_s \notin A^{\varrho_T(s)+1}$, so that $\alpha < \beta \leqslant \varrho_T(\emptyset) + 1 \implies A^\alpha \supsetneq A^\beta$ and we should then have $\theta(x_T, f) > \varrho_T(\emptyset)$. The next lemma gives further conditions on our points x_T, x_s which will make that happen. We present this argument in an abstract setting in terms of a *nearness relation* between points, which, to emphasize its possibly asymmetric character, we write as $b \vartriangleright_f x$, or, more conveniently, as $b \vartriangleright x$, which may be read as "*b is near to x*".

4·5 EXAMPLE In our first application we shall take $b \triangleright x$ to mean that for some $n \geqslant 0$, $f^n(x) = b$; plainly that is liable to be asymmetric. In our second application we shall take $b \triangleright x$ to have the plainly symmetrical meaning that for some non-negative n, m, $f^m(b) = f^n(x)$. In both we shall have $b \triangleright b$ for every b.

4·6 LEMMA *Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous, and let T be a non-empty well-founded tree with top-most point $\emptyset$. Suppose that we have points x_T and x_s (for $s \in T$) in the space $\mathcal{X}$ and a relation $b \triangleright x$ between points such that whenever $s \in T$ & $x_T \curvearrowright c \curvearrowright b \triangleright x_s$, then for some $r \in T$ with $r \prec s$, $c \triangleright x_r$. Then for $b \in \omega_f(x_T)$ and $s \in T$,*

$$x_T \curvearrowright b \triangleright x_s \implies b \notin A^{\varrho(s)+1}(x_T, f).$$

Proof: write $\Phi(s)$ for "$x_T \curvearrowright b \triangleright x_s \implies b \notin A^{\varrho(s)+1}(x_T, f)$". We suppose inductively that $\forall r \prec s\ \Phi(r)$ and prove $\Phi(s)$. So suppose $x_T \curvearrowright c \curvearrowright b \triangleright x_s$. By assumption, $\exists r :\in T\ [r \prec s\ \&\ c \triangleright x_r]$; by $\Phi(r)$, $c \notin A^{\varrho_T(r)+1}$, so $c \notin A^{\varrho_T(s)}$. As c was arbitrary, $b \notin A^{\varrho_T(s)+1}$, and we have proved that $\Phi(s)$ holds. ⊣ (4·6)

These lemmata lead to the following general result:

4·7 THEOREM *Let $\mathcal{X}$ be a complete separable metric space, let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous, and let T be a non-empty well-founded tree with top-most point $\emptyset$. Suppose that we have points x_T and x_s (for $s \in T$) in the space $\mathcal{X}$ and a relation $b \triangleright_f x$ between points of $\omega_f(x_T)$ such that for all s, t in T, writing $\curvearrowright$ for $\curvearrowright_f$ and $\triangleright$ for $\triangleright_f$,*

(4·7·0) $x_T \curvearrowright x_s$;
(4·7·1) $s \prec t \implies x_s \curvearrowright x_t$;
(4·7·2) $x_s \triangleright x_s$;
(4·7·3) $x_T \curvearrowright c \curvearrowright b \triangleright x_s \implies c \triangleright x_r$ *for some* $r \in T$ *with* $r \prec s$.

Then $\theta(x_T, f) > \varrho_T(\emptyset)$.

Proof: our lemmata show that in the above circumstances, $\forall s :\in T\ [x_s \in A^{\varrho_T(s)}(x_T, f) \smallsetminus A^{\varrho_T(s)+1}(x_T, f)]$. In particular, taking $s = \emptyset$, $A^{\varrho_T(\emptyset)}(x_T, f) \neq A^{\varrho_T(\emptyset)+1}(x_T, f)$, and so $\theta(x_T, f) > \varrho_T(\emptyset)$, as required. ⊣ (4·7)

In the present section of the paper we shall construct for particular well-founded trees T of arbitrary countable rank points x_s and x_T in Baire space satisfying the hypotheses of the above theorem, and shall find that $\theta(x_T, \mathfrak{s}) = \varrho_T(\emptyset) + 1$: we shall then modify our examples to obtain in each space points z_T with $\theta(z_T, f) = \varrho_T(\emptyset)$. In the fifth section, we shall construct points in certain compact spaces to which the theorem applies, though the corresponding modification will prove troublesome. In the sixth section we shall essay applications to certain ill-founded trees.

Examples of long delays in the Baire space

4·8 DEFINITION *Baire space*, $\mathcal{N}$, is $\{b \mid b : \omega \longrightarrow \omega\}$; topologically it is the product of $\aleph_0$ copies of ω, each with the discrete topology.

4·9 DEFINITION The *shift function* $\mathfrak{s} : \mathcal{N} \longrightarrow \mathcal{N}$ is defined by $\mathfrak{s}(b)(n) = b(n+1)$ for $b : \omega \longrightarrow \omega$.
Some call $\mathfrak{s}$ the *backward* shift: it does indeed lose information.

The construction below together with the remarks at the end of the section will prove the following:

4·10 THEOREM *Let $\mathcal{N}$ be Baire space ω^ω, and $\mathfrak{s}$ the shift operation. Then for each countable ordinal ζ there is a point $a \in \mathcal{N}$ such that $\theta(a, \mathfrak{s}) = \zeta$.*

Our plan is as follows: to each $s \in \mathcal{S}$ we shall define a point $x_s \in \mathcal{N}$; we shall **for the rest of this section** write $b \triangleright x$ to mean that b is *a finite shift of* x in the sense that $b = \mathfrak{s}^n(x)$ for some $n \geqslant 0$ and $x \in \mathcal{N}$; then for each well-founded $T \subseteq \mathcal{S}$ we shall define a point x_T so that the points x_T and x_s for $s \in T$ together with the relations $\curvearrowright = \curvearrowright_{\mathfrak{s}}$ and $\triangleright$ satisfy the hypotheses of Theorem 4·7.

4·11 DEFINITION We write $u \sqsubset x$ to mean that the non-empty finite sequence u occurs as a segment of the infinite sequence x.

4·12 LEMMA *If $x \curvearrowright y$ and $u \sqsubset y$ then u occurs infinitely often as a segment of x.*

4·13 DEFINITION If $u \sqsubset y$, define the *lag* $\ell(y, u)$ to be the least k such that $\forall p :< \ell h(u)\ y(k + p) = u(p)$: thus u is an initial segment of y if and only if the lag is 0. For $a \in \mathcal{N}$ and $n \in \omega$, we shall write $\ell(y, a, n)$ for $\ell(y, a \restriction n)$, which will be defined only when $a \restriction n \sqsubset y$. If $\ell(y, a, n)$ is defined for each n, we shall have $\ell(y, a, n) \leqslant \ell(y, a, n + 1)$ for each n, and hence $\lim_{n\to\infty} \ell(y, a, n)$ will exist, but may be finite or infinite.

4·14 LEMMA *Suppose that each initial segment of a is a segment of z.*
(i) If $\lim_{n\to\infty} \ell(z, a, n) = \infty$, then $z \curvearrowright a$.
(ii) If $\lim_{n\to\infty} \ell(z, a, n) = m < \infty$, then $a = \mathfrak{s}^m(z)$.

The construction

Define
$$x_\emptyset = 0, 4, 8, \ldots$$
so that $x_\emptyset$ is definitely non-recurrent and attacks nothing.

Suppose now that for some $s \in \mathcal{S}$, we have defined x_s, and that t is an immediate extension of s, so that $t = s^\frown\langle k\rangle$, where k is an odd prime exceeding all those occurring in s. Write π_t for the product of all the primes occurring in t, so that π_t is a square-free number of which k is a factor. For each natural number n we write $\pi_{t,n}$ for the number $(\pi_t)^{n+1}$. We shall use the numbers $\pi_{t,0}, \pi_{t,1}, \ldots$ in defining x_t.

4·15 LEMMA *For non-empty r_1 and r_2 in T, π_{r_1} divides π_{r_2} iff $r_2 \preccurlyeq r_1$.*

Our definition is this:

$$x_t =_{\text{df}} \langle\pi_{t,0}\rangle^\frown(x_s \restriction n_{t,0})^\frown\langle\pi_{t,1}\rangle^\frown(x_s \restriction n_{t,1})^\frown\langle\pi_{t,2}\rangle^\frown(x_s \restriction n_{t,2})^\frown \ldots$$

where the positive integers $n_{t,i}$ are chosen strictly increasing and such that the predecessors of each occurrence of a power of π_t in x_t (after the first) form a strictly increasing sequence of even numbers: to enable that choice to be made we maintain inductively the property that the x_s have infinitely many even numbers in their range. To be more explicit, we might say that $n_{t,0}$ is to be the least n with $x_s(n) = 0$, and $n_{t,i+1}$ is to be the least integer greater than $n_{t,i}$ such that $x_s(n_{t,i+1} - 1) = 4(i + 1)$: this would make n_t actually a function of s, independent of k, and would make our construction of the points x_s recursive.

4·16 REMARK by Lemma 4·12, for no s with $b \triangleright x_s$ can $x_s \curvearrowright b$: hence the dichotomy suggested in Lemma 4·14 will be exact in many cases.

4·17 LEMMA *The only odd numbers occurring in x_s are powers of π_t for t a non-empty initial segment of s (allowing $t = s$).*

Proof by induction on length of s: $\emptyset$ has no non-empty initial segments, and no odd numbers occur in $x_\emptyset$. The odd numbers occurring in $x_{s^\frown\langle k\rangle}$ are the powers of $\pi_{s^\frown\langle k\rangle}$ and the odd numbers occurring in x_s, which by the induction hypothesis are the powers of $\pi_{s'}$ for $\emptyset \neq s' \succcurlyeq s$. $\dashv$ (4·17)

4·18 LEMMA *If $y \sqsubset x_t$ where $t = s^\frown\langle k\rangle \in \mathcal{S}$, and no power of π_t occurs in y then $y \sqsubset x_s$.*

The above is evident from the definition of x_t. These two lemmas immediately yield, by a further induction on the length of s, 4·19 LEMMA *Suppose that $u \sqsubset x_r$ and that s is the longest initial segment t of r such that a power of π_t occurs in u. Then $u \sqsubset x_s$. In particular, a segment of x_r containing no odd numbers is a segment of $x_\emptyset$.*

4·20 LEMMA *Suppose that a is such that whenever $u \sqsubset a$, then for some $r \in \mathcal{S}$, $u \sqsubset x_r$. Suppose $s \in \mathcal{S} \smallsetminus \{\emptyset\}$ is such that a power of π_s occurs in a, but that for no $t \prec s$ does π_t occur in a. Then a is a finite shift of x_s.*

Proof : let $a(\bar{n})$ be the first occurrence of a power of π_s in a. Let u be an initial segment of a of length $> \bar{n}$. For some r, $u \sqsubset x_r$. Since a power of π_s occurs in u, and therefore in x_r, we know by Lemma 4·17 that $r \preccurlyeq s$. But then Lemma 4·19 tells us that $u \sqsubset x_s$. Since $a(\bar{n})$ has exactly one occurrence in x_s, $\lim_{n\to\infty} \ell(x_s, a, n)$ will be finite and so a is a finite shift of x_s. ⊣ (4·20)

4·21 LEMMA *If $t = s^\frown\langle k\rangle \in \mathcal{S}$, $a \in \mathcal{N}$, and $x_t \curvearrowright a$ then either $x_s \curvearrowright a$ or a is a finite shift of x_s.*

Proof : As each power of π_t occurs only once in x_t, they can have no occurrence in a, by Lemma 4·12. Thus each initial segment, $a \restriction n$, of a lies in some interval of x_t strictly between two successive occurrences of powers of π_t, and therefore is a segment of x_s. We may now apply Lemma 4·14. ⊣ (4·21)

Note that all the lemmata so far in this paragraph would hold if we had taken $x_\emptyset$ to be another member of ${}^{<\omega}\{0,4\}$, such as 4^∞ or $(04)^\infty$, provided the definition of the sequences $n_{t,i}$ were altered suitably.

Now let T be a well-founded tree, $T \subseteq \mathcal{S}$ and T closed under shortening. We have seen how define x_s for $s \in T$, and we wish to define a point x_T with $x_T \curvearrowright x_s$ for every $s \in T$. The definition we give is perhaps not in the present context the most natural, but it will permit us to extend our definitions to the case of ill-founded trees whilst ensuring that x_T is recursive in T.

List all members of $\mathcal{S}$ recursively as $\langle s_i \mid i \in \omega\rangle$ so that each occurs infinitely often, and then define

$$x_T = (x_{t_0} \restriction n_{T,0})^\frown\langle 2\rangle^\frown(x_{t_1} \restriction n_{T,1})^\frown\langle 6\rangle^\frown(x_{t_2} \restriction n_{T,2})^\frown\langle 10\rangle \ldots$$

where the integers $n_{T,i}$ are chosen strictly increasing, and such that the immediate predecessors of the occurrences of numbers $n \equiv 2 \pmod 4$ are distinct positive multiples of 4, and t_i is the first s_j in sequence after previous t_k's to be a member of T: so, in effect, we always check to see whether $s_j \in T$, and if it is not, we do nothing at that stage but proceed to the next.

Thus if T is recursive so is the point x_T and the sequence $\langle x_s \mid s \in T\rangle$.

The first three numbered conditions of Theorem 4·7 are easily verified: for (4·7·1), note that $x_{t^\frown\langle k\rangle} \curvearrowright x_t$, and deduce inductively by Proposition 0·0(ii) that $r \prec s \implies x_r \curvearrowright x_s$. It remains to check (4·7·3).

4·22 PROPOSITION *if $x_T \curvearrowright c \curvearrowright b \triangleright x_s$ then $c \triangleright x_r$ for some $r \prec s$.*

Proof : no $n \equiv 2 \pmod 4$ can occur in c by Lemma 4·12, taking $u = \langle n\rangle$. So each initial segment of c is a segment of some x_t. [If T were finite and non-empty, we could then conclude quickly that for some t_i each initial segment of c is a segment of x_{t_i}, and then apply 4·21 and 4·14.]

First suppose that no powers of any π_t occur in c. Then every segment of c is a segment of $x_\emptyset$. With our choice of $x_\emptyset$, that implies that $c \triangleright x_\emptyset$, but then by 4·16 it will be impossible to have $c \curvearrowright b \triangleright x_\emptyset$.

So powers of some π_t occur in c: by the well-foundedness of T we may choose a minimal r such that a power of π_r occurs in c. We assert that $u \sqsubset c \implies u \sqsubset x_r$, and that $r \prec s$. To see that, let u be an initial segment of c; by lengthening u if necessary, we may assume that a power of π_r occurs in u. Moreover, if $s \neq \emptyset$, some power $\pi_{s,q}$, say, of π_s will occur in b since there are infinitely many occurrences of powers of π_s in x_s and b is a finite shift of x_s; then that power $\pi_{s,q}$ will occur infinitely often in c, since $c \curvearrowright b$. Thus by a further lengthening of u we may assume that some power of π_s occurs at least twice in u.

Now $u \sqsubset x_t$ for some t. By Lemma 4·17, r and s will both be initial segments of t. By the T-minimality of r, $r \preccurlyeq s$. Hence we may apply Lemma 4·20 to deduce that $c \triangleright x_r$. But then we shall have $r \neq s$, since there are two distinct occurrences of some $\pi_{s,q}$ in c, which could not happen if $c \triangleright x_s$. ⊣ (4·22)

Refinement at limit ordinals

We have just shown that there are points x in Baire space $\theta(x, \mathfrak{s})$ larger than any given countable ordinal, η, say. Actually our choice of the point $x_\emptyset$ was such that for a countable well-founded tree T with $\varrho_T(\emptyset) = \eta$, $\theta(x_T, \mathfrak{s})$ is exactly $\eta + 1$, and the abode of x_T is empty. To see that, note that by similar reasoning to the last proof, we have, for any choice of $x_\emptyset$ that makes our foregoing lemmata work,

4·23 PROPOSITION *If $x_T \curvearrowright a$, then either $a \triangleright x_s$ for some uniquely determined s or $x_\emptyset \curvearrowright a$.*

With our present choice of $x_\emptyset$, the case $x_\emptyset \curvearrowright a$ never holds, and hence each point a attacked by x_T is near to some x_s with $s \in T$; further, by Proposition 4·22, if $x_T \curvearrowright a \triangleright x_s$ and $x_T \curvearrowright b \triangleright x_t$, then $b \curvearrowright a$ if and only if $t \prec s$. The calculation that $A^{\varrho_T(\emptyset)} = \{\mathfrak{s}^n(x_T) \mid n \in \omega\}$, $A^{\varrho_T(\emptyset)+1} = \varnothing$, and $\theta(x_\emptyset) = \varrho_T(\emptyset) + 1$ is now immediate.

In Baire space $\mathcal{N}$ it is easy to find points x for which $\omega_{\mathfrak{s}}(x) = \mathcal{N}$; for such points, $\theta(x, \mathfrak{s}) = 0$.

Thus it only remains to show how to obtain points for which $\theta(a, \mathfrak{s})$ is a prescribed countable limit ordinal λ.

There are two possible methods: one is to take as starting point a point $z_\emptyset$ so that $z_\emptyset \curvearrowright z_\emptyset$; for example we could take $z_\emptyset = 4^\infty$ or $z_\emptyset = (04)^\infty$. Some adjustments to the argument will be necessary: one has already been mentioned; the definition of the sequence $n_{T,i}$ will need alteration, and in place of Theorem 4·7, we must use this variant:

4·24 THEOREM *Let $\mathcal{X}$ be a complete separable metric space, let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous, and let T be a non-empty well-founded tree with top-most point $\emptyset$. Suppose that we have points x_T and x_s (for $s \in T$) in the space $\mathcal{X}$ and a relation $b \triangleright_f x$ between points of $\omega_f(x_T)$ such that for all s, t in T, writing $\curvearrowright$ for $\curvearrowright_f$ and $\triangleright$ for $\triangleright_f$,*

(4·24·0) $x_T \curvearrowright x_s$;

(4·24·1) $s \prec t \implies x_s \curvearrowright x_t$;

(4·24·2) $x_s \triangleright x_s$;

(4·24·3) for $s \neq \emptyset$, $x_T \curvearrowright c \curvearrowright b \triangleright x_s \implies c \triangleright x_r$ for some $r \in T$ with $r \prec s$.

Then $\theta(x_T, f) \geqslant \varrho_T(\emptyset)$.

With the choices of $z_\emptyset$ suggested above, we shall have

$$x_T \curvearrowright c \implies c \triangleright x_r \text{ for some } r \in T$$

as before, and hence, if we have taken $z_\emptyset = 4^\infty$, we shall have that for each T, $\theta(z_T, f) = \varrho_T(\emptyset)$, and the abode of z_T will be $\{z_\emptyset\}$; if instead we took $z_\emptyset = (04)^\infty$, the abode of z_T would be $\{(04)^\infty, (40)^\infty\}$. but still $\theta(z_T, f) = \varrho_T(\emptyset)$.

We leave the details of this suggestion to the reader, as a similar argument will be given in full in the next section.

The other method would be to partition the set of odd primes into infinitely many infinite pieces A_n, and to pick trees T_n of rank α_n with a topmost point $\langle p_n \rangle$, where the α_n's are an increasing sequence of ordinals with limit λ, and the tree T_n is formed of strictly increasing non-empty finite sequences from A_n with first element always equal to $p_n = \min A_n$.

Pick different starting points $z_{\langle p_n \rangle}$ for each tree. We choose them to be strictly increasing sequences of multiples of 4, no one multiple of 4 occurring in two different points, so that each $z_{\langle p_n \rangle}$ attacks nothing.

Then we put our trees together, with all the top-most points $\langle p_n \rangle$ immediately under a global top-most point $\emptyset$, to form a global tree T. We shall not define $z_\emptyset$ in this approach. As we have only countably many countable trees T_n under consideration at any time, we may define a point z_T as before. We ensure that for $s \in T_n$ and $t \in T_m$ with $n \neq m$, $z_s \not\curvearrowright z_t$, so that the vanishing of the reals attached to the tree T under repeated action of $\curvearrowright$ is simply the independent local vanishing of the reals on each T_n. Hence each point $z_{\langle p_n \rangle}$ disappears at stage $\alpha_n + 1$, and global stability is reached at stage λ. Here the abode $A^\lambda = \varnothing$ as we chose our initial points so that $\neg\exists x\, z_{\langle p_n \rangle} \curvearrowright x$. $\dashv$ (4·10)

A non-wandering escapist

Birkhoff in Chapter VII, section 2 of his book [3] defines the notion of a non-wandering point and forms a transfinite descending sequence of *closed* sets, which therefore terminates after countably many steps. We show that his concepts do not coïncide with ours.

4·25 DEFINITION We define the Birkhoff operator $B(X)$, where $X \subseteq \mathcal{X}$, by

$$B(X) = X \setminus \bigcup\{U \mid U \text{ open } \ \& \ \exists n{:}{>}0\ f^n[U] \cap U \cap X = \varnothing\}$$

Thus if X is closed, $B(X)$ is a closed subset of X. Accordingly the transfinite sequence given by $X_0 = \mathcal{X}$, $X_{\nu+1} = B(X_\nu)$, $X_\lambda = \bigcap_{\nu<\lambda} X_\nu$ for limit λ stops in countable time at a closed set X_∞. X_∞ is the set of *non-wandering points*, and may be proved to equal the closure of the set of recurrent points.

We have remarked in 3·25 above that the abode is contained in that closure. Here is an example in a compact space of an escaping point that is non-wandering.

We work in ${}^\omega 4$ with the shift function. Let

$$\begin{aligned} x &= 0202202220\ldots \\ x_1 &= (021)^\infty \\ x_2 &= (020221)^\infty \\ x_3 &= (0202202221)^\infty \\ \ldots & \quad \ldots \\ a &= (y_1 \restriction n_1)^\frown 3^\frown (y_2 \restriction n_2)^\frown 3^\frown \ldots \end{aligned}$$

where the y_i list the x_j, each occurring infinitely often, and the n_i are chosen strictly increasing and such that each segment is a whole number of cycles of the relevant x_i.

4·26 LEMMA *(i)* $02^k0 \sqsubset x_\ell$ *iff* $k < \ell$*; (ii)* $02^j1 \sqsubset x_\ell$ *iff* $j = \ell$.

Proof : by inspection. $\dashv$ (4·26)

Plainly $a \curvearrowright x_k \curvearrowright x_k$ for each k, and $\lim_k x_k = x$, and so x is a non-wandering point. We show that there is no b with $a \curvearrowright b \curvearrowright b \curvearrowright x$.

4·27 LEMMA *Suppose that* $a \curvearrowright b \curvearrowright b \curvearrowright x$. *Then*

(i) there is no occurrence of 3 in b;

(ii) whenever $u \sqsubset b$, $u \sqsubset x_i$ *for some* i.

Proof : (i) occurrences of 3 in b are rendered impossible by the spacing of its occurrences in a. (ii) follows from (i). $\dashv$ (4·27)

Now $020 \sqsubset x$ so for some non-empty finite sequence w, setting $u = 020^\frown w^\frown 020$, there is some $i > 1$ such that $u \sqsubset x_i$ and $u \sqsubset b$. 1 must occur in w, since 020 occurs at most once in any period of x_i, so examining the predecessors of the first occurrence of 1 in w we see that for some $j > 1$, $02^j1 \sqsubset 0^\frown w$. Since $b \curvearrowright b$ and $0^\frown w \sqsubset b$, 02^j1 must occur infinitely often as a segment of b. Let $k > j$. 02^k0 occurs as a segment of x and therefore infinitely often as a segment of b. Thus there are non-empty finite sequences r and s such that setting $t = 02^j1^\frown r^\frown 02^k0^\frown s^\frown 02^j1$, $t \sqsubset b$. Suppose $t \sqsubset x_\ell$. Then parts (i) and (ii) of Lemma 4·24 show that $j = \ell$ and $k < \ell$, contradicting $k > j$.

5: The compact case

5·0 THEOREM *Let $\mathcal{X}$ be the compact space ${}^\omega 7$, and $\mathfrak{s}$ the shift function. For each countable ordinal δ there is a point a with $\theta(a, \mathfrak{s}) = \delta$.*

It proves unexpectedly difficult to find a point a in this compact space with $\theta(a, \mathfrak{s})$ exactly a given countable limit ordinal: the argument sketched in the third section of [11] is, as noticed by the referee, incomplete; and irreparably so, for as we shall see, such an a must attack infinitely many recurrent points, a condition not met by the constructions of [11]. Hence in the first part of the proof we must work with an extra parameter ℓ.

5·1 DEFINITION **In this section** we write $b \triangleright x$ to mean that for some non-negative m, n, $f^m(b) = f^n(x)$.

We fix a positive integer ℓ. In this section 2^3 will mean a sequence of three 2's and $(45^2)^\infty$ the periodic infinite sequence $455455455\ldots$.

Define

$$y^\ell_\emptyset = (02^\ell)^\infty$$

then $y^\ell_\emptyset \curvearrowright y^\ell_\emptyset$.

We use the same trees as before; for each strictly increasing sequence s of odd prime numbers we shall define π_s as in the last section, and an infinite sequence y^ℓ_s in the space ${}^\omega\{0, 1, 2, 4, 5\}$.

Perhaps our definition is best introduced by example:

$$\begin{aligned}
y^\ell_3 &=_{\mathrm{df}} 1(45^\ell)^3 02^\ell 02^\ell \ldots 02^\ell (45^\ell)^3 02^\ell 02^\ell \ldots 02^\ell (45^\ell)^3 \ldots \\
y^\ell_{3,5} &=_{\mathrm{df}} 1(45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^3 02^\ell \ldots 02^\ell {}^\frown \\
&\quad {}^\frown (45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^3 02^\ell \ldots 02^\ell \ldots \\
y^\ell_{3,5,7} &=_{\mathrm{df}} 1(45^\ell)^{3\cdot 5\cdot 7} 1(45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^{3\cdot 5\cdot 7} 1(45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots 02^\ell (45^\ell)^{3\cdot 5} 1(45^\ell)^3 02^\ell \ldots
\end{aligned}$$

Now for the formal definition. Let $t = s^\frown\langle k\rangle$. Suppose that we have defined y^ℓ_s. Define

$$y^\ell_t =_{\mathrm{df}} 1(45^\ell)^{\pi_t}(y^\ell_s \restriction m^\ell_{t,0})(45^\ell)^{\pi_t}(y^\ell_s \restriction m^\ell_{t,1})(45^\ell)^{\pi_t}(y^\ell_s \restriction m^\ell_{t,2}) \ldots$$

where the positive numbers $m^\ell_{t,i}$ are chosen to be strictly increasing and such that the last ℓ entries of each finite sequence $y^\ell_s \restriction m^\ell_{t,i}$ are all 2's.

5·2 LEMMA *$y^\ell_{s^\frown\langle k\rangle} \curvearrowright y^\ell_s$; hence $r \prec s \implies y^\ell_r \curvearrowright y^\ell_s$.*

5·3 DEFINITION We define the number χ^ℓ_t for $t \in \mathcal{S}$ by recursion on the length of t, thus:

$$\begin{aligned}
\chi^\ell_{\langle k\rangle} &=_{\mathrm{df}} (1+\ell)k && \text{for } k \text{ an odd prime;} \\
\chi^\ell_{s^\frown\langle k\rangle} &=_{\mathrm{df}} (1+\ell)\pi_{s^\frown\langle k\rangle} + 1 + \chi^\ell_s.
\end{aligned}$$

Thus if $t = \langle p_0, p_1, \ldots p_{n-1}\rangle$, $\chi^\ell_t = (1+\ell)\pi_t + 1 + (1+\ell)\pi_{t\restriction(n-1)} + 1 + \ldots + (1+\ell)p_1 \cdot p_0 + 1 + (1+\ell)p_0$.

5·4 DEFINITION By a $\langle$*list*$\rangle$-*segment* of a finite or infinite sequence we mean a segment in which every digit occurring is among those in the list. A maximal $\langle$*list*$\rangle$-segment of a sequence will be called an $\langle$*list*$\rangle$-*interval* of that sequence.

We will be chiefly concerned with 145-segments and 02-segments. Note that a 45-segment is also a 145 segment.

5·5 LEMMA *Let u be a 145-interval of y^ℓ_t. Then the length of u is $1 + \chi^\ell_t$ if u is an initial segment of y^ℓ_t and is χ^ℓ_t otherwise.*

5·6 COROLLARY $y^\ell_s \not\curvearrowright y^\ell_s$.

5·7 REMARK Each y^ℓ_s divides into intervals of two kinds. There are 02-intervals, in which the only digits occurring are 0 and 2; there are 145-intervals in which the only digits occurring are 1, 4 and 5. Every 02-interval begins with a 0 and ends with a 2; the initial 145-interval of y^ℓ_s begins with a 1 and ends with a 5; every other 145-interval of y^ℓ_s begins with a 4 and ends with a 5. These facts are readily checked by induction on the length of s.

5·8 LEMMA *$(45^\ell)^{\pi_t}$ occurs infinitely often as a segment of y^ℓ_t, and no longer 45-segment occurs at all.*

5·9 LEMMA *If u is a segment of y^ℓ_t of the form $1(45^\ell)^i 1(45^\ell)^j$, or of the form $2(45^\ell)^i 1(45^\ell)^j$, then $i > j$.*

Proof : by induction on the length of t. ⊣ (5·9)

5·10 LEMMA *The first occurrence of 4 in y^ℓ_t following a given occurrence of 2 is immediately preceded by a 2.*

Proof : there are no 4's in $y^\ell_\emptyset$, so the assertion is vacuously true in this case. Suppose $t = s^\frown\langle k\rangle$, and that the lemma is true for y^ℓ_s. Each occurrence of 2 in y^ℓ_t is in a segment u of the form $y^\ell_s \restriction m$. If there is no 4 in u after the given occurrence of 2, then the first occurrence of 4 is the immediately succeeding first entry of $(45^\ell)^{\pi_t}$, which is immediately preceded by a 2, by choice of $m^\ell_{t,i}$'s. If there *is* a 4 in u after that 2, the induction hypothesis applies to tell us that its predecessor is a 2. ⊣ (5·10)

5·11 LEMMA *2 is never immediately followed by 1 in y^ℓ_t.*

Proof : a similar but easier induction on the length of t. ⊣ (5·11)

5·12 LEMMA *5 is never immediately followed by 2 in y^ℓ_t.*

Hence we are led to scrutinise the occurrences of the following finite sequences as segments of a and of y^ℓ_s, which again we shall introduce by example:

$$\begin{aligned} v_3 &=_{\rm df} 2(45^\ell)^3 0 \\ v_{3,5} &=_{\rm df} 2(45^\ell)^{3\cdot 5} 1(45^\ell)^3 0 \\ v_{3,5,7} &=_{\rm df} 2(45^\ell)^{3\cdot 5\cdot 7} 1(45^\ell)^{3\cdot 5} 1(45^\ell)^3 0 \end{aligned}$$

More formally, define inductively

$$\begin{aligned} v'_{\langle k\rangle} &=_{\rm df} (45^\ell)^k && \text{for } k \text{ an odd prime} \\ v'_t &=_{\rm df} (45^\ell)^{\pi_t} 1^\frown v'_s && \text{for } t = s^\frown\langle k\rangle \end{aligned}$$

and then set

$$v_s =_{\rm df} 2^\frown v'_s{}^\frown 0$$

5·13 REMARK v_t occurs infinitely often as a segment of y^ℓ_t; no two occurrences overlap; the gap between successive occurrences increases strictly, and so by 4·12 if v_t occurs twice in a, $y^\ell_t \not\curvearrowright a$.

5·14 LEMMA *let u be an initial segment of y^ℓ_t of the form $1v0$ where v is a 145-segment. Then $v = v'_t$.*

Proof : let w be the tail of u after discarding $1 + (1+\ell)\pi_t + 1$ terms. Let $t = s^\frown k$. If $s = \emptyset$, w is empty, as the first term of y^ℓ_s will be 0; $t = \langle k\rangle$, and $v = (45^\ell)^k = v'_t$.

Suppose $s \prec \emptyset$. There is a zero in $y^\ell_s \restriction m^\ell_{t,0}$, so $1w$ is an initial segment of $y^\ell_s \restriction m^\ell_{t,0}$, and hence of y^ℓ_s. By induction, $w = v'_s 0$, and so $v = v'_t$. ⊣ (5·14)

5·15 LEMMA *Let $t \prec \emptyset$, let u be a (non-initial) segment of y^ℓ_t of the form $2v0$ where v is a non-empty 145-segment. Then $v = v'_s$ and $u = v_s$ for some $s \succcurlyeq t$ with $s \neq \emptyset$.*

Proof : it might be that the first 2 of u is the last entry of some $y^\ell_s \restriction m^\ell_{t,i}$, when v begins with $(45^\ell)^{\pi_t}$ and continues into the next initial segment of y^ℓ_s, but no further as that contains a 0; let w be the tail forming that initial segment. By Lemma 5·14, $w = 1v'_s0$ and so $v = v'_t$.

If not, that 2 is an earlier entry in some $y^\ell_s \restriction m$; then u must be a segment of y^ℓ_s (since u contains no other 2 and the last entry of $y^\ell_s \restriction m$ is a 2) so $s \neq \emptyset$ and the inductive hypothesis applies. ⊣ (5·15)

5·16 COROLLARY *If v_s occurs as a segment of y^ℓ_t then $t \preccurlyeq s$.*

5·17 LEMMA *Let $t = s^\frown k$. Let $u \sqsubset y^\ell_t$, let u start with a non-(4 or 5), contain some 4's and 5's, and end with a non-(4 or 5). Suppose that $(45^\ell)^{\pi_t}$ has no occurrence in u then $u \sqsubset y^\ell_s$.*

Proof : if u is not contained in one $y^\ell_s \restriction m^\ell_{t,i}$, it must in these circumstances contain the whole of some $(45^\ell)^{\pi_t}$.

⊣ (5·17)

5·18 LEMMA *Let r be such that v_r occurs in $u \sqsubset y^\ell_t$ but no $(45^\ell)^{\pi_s}$ occurs in u with $s \prec r$, and suppose that u starts with a non-(4 or 5) and ends with a non-(4 or 5): then $u \sqsubset y^\ell_r$.*

Proof : by 5·16 we must have $t \preccurlyeq r$. Now iterate the previous lemma. ⊣ (5·18)

5·19 LEMMA *Let $t \neq \emptyset$. If the finite sequence u starts with a 2 and ends with a 0, and if $(45^\ell)^{\pi_t} \sqsubset u \sqsubset y^\ell_t$, then $v_t \sqsubset u$.*

Proof : write $t = s^\frown\langle k\rangle$. Let u^* be the first occurrence of $(45^\ell)^{\pi_t}$ as a segment of u. The lag $\ell(y^\ell_t, u) > 0$ since u begins with a 2. Hence, by 5·10, the term of u immediately before the start of u^* will be a 2, and we may write $u = u_0^\frown 2^\frown u^{*\frown} v$. v will be an initial segment of y^ℓ_s. If $s = \emptyset$, then v starts with a 0, and $2^\frown u^{*\frown}0$ is the desired occurrence of v_t. If $s \neq \emptyset$, v will start with a 1; it contains 0's, for it ends in 0; let m be least with $v(m) = 0$; then $v \restriction m = 1^\frown v'_s$, by Lemma 5·14, and so $v_t = 2^\frown(45^\ell)^{\pi_t\frown}1^\frown v'^\frown_s 0 \sqsubset u$. ⊣ (5·19)

Now let $T \subseteq \mathcal{S}$ be a well-founded tree. As before we enumerate $\{y^\ell_s \mid s \in T\}$ as $\langle z^\ell_i \mid i < \omega\rangle$, with each y listed infinitely often. We define

$$y^\ell_T \;=_{\mathrm{df}}\; 3^\frown(z^\ell_0 \restriction m^{T,\ell}_0)^\frown 3^\frown(z^\ell_1 \restriction m^{T,\ell}_1)^\frown\ldots$$

where we choose the integers $m^{T,\ell}_i$ to be strictly increasing, and such that each segment $z^\ell_i \restriction m^{T,\ell}_i$ ends with ℓ 2's.

The first three of the numbered hypotheses of Theorem 4·24 are easily checked; we must prove that (4·24·3) holds. We begin by investigating the consequences of $y^\ell_T \curvearrowright a$.

5·20 REMARK If T is infinite, $\{(45^\ell)^\infty, 1(45^\ell)^\infty, 31(45^\ell)^\infty, (02^\ell)^\infty, 45^\ell(02^\ell)^\infty\} \subseteq \omega_f(y^\ell_T)$.

5·21 LEMMA *If $y^\ell_T \curvearrowright a$ then 3 occurs at most once in a. Furthermore, each occurrence of 0 in a is immediately followed by ℓ 2's, each occurrence of 4 in a is immediately followed by ℓ 5's, and 1^2, $2^{\ell+1}$ and $5^{\ell+1}$ have no occurrence in a.*

Proof : The first statement holds by 4·12 since the intervals at which 3 occurs in y^ℓ_T are ever increasing. The remaining statements are inherited from the corresponding statements about the y^ℓ_s's, which can be proved by induction on the length of s.

5·22 LEMMA *If $y^\ell_T \curvearrowright a$ and no 3 occurs in a then every segment of a is a segment of some y^ℓ_t.*

5·23 LEMMA *If $y^\ell_T \curvearrowright a$ and a contains only finitely many 2's then $a \triangleright (45^\ell)^\infty$.*

Proof : a will have finitely many 0's, since each 0 is immediately followed by a 2. Then a finite shift will bring us to an a' composed only of 1's, 4's and 5's. But as 1^2 is never found, a' would end either in $(45^\ell)^\infty$, which *is* possible, or else in a sequence $1(45^\ell)^h 1(45^\ell)^i 1(45^\ell)^j \ldots$ where by Lemma 5·9 $h > i > j \ldots$, which is impossible. Indeed our numbers χ^ℓ_t give a bound for the length of such an interval. ⊣ (5·23)

5·24 LEMMA *If* $y^\ell_T \curvearrowright a$ *and* a *consists solely of 0's and 2's, then* a *is in the orbit of* $y^\ell_\emptyset$.

Proof : As every 0 is immediately followed by 2^ℓ, and $2^{\ell+1}$ does not occur, $a = 2^{\ell'} {}^\frown y^\ell_\emptyset$ for some ℓ' with $0 \leqslant \ell' \leqslant \ell$. ⊣ (5·24)

5·25 LEMMA *If* $y^\ell_T \curvearrowright a$ *and* a *has only finitely many 4's,* $a \rhd y^\ell_\emptyset$.

Proof : a will have only finitely many 1's since each 1 is followed by a 4, and only finitely many 5's, since each 5^ℓ is preceded by a 4, so after a finite shift, we may assume that a consists solely of 0s and 2's. The result now follows from the previous lemma, bearing in mind the meaning given to $\rhd$ in this section. ⊣ (5·25)

5·26 LEMMA *Suppose that* $y^\ell_T \curvearrowright a$ *and that* a *has no 3's, infinitely many 2's and infinitely many 4's.*

(i) a *will contain infinitely many 0's and infinitely many segments of the form* v_t *with* $t \neq \emptyset$.

(ii) Suppose further that $a(0) = 2$ *and that* r *is* T*-minimal such that* $v_r \sqsubset a$*: then* $a \rhd y^\ell_r$ *or* $y^\ell_r \curvearrowright a$.

(iii) There is a non-empty $t \in T$ *with* $a \rhd y^\ell_t$.

Proof of (i): starting at an arbitrary term of a, go to the next 2 and then to the end of its 02-interval (which will be finite, as another 4 must occur sometime). That digit will be a 2. A 145-interval now starts (with a 4). Continue till it ends, which it will as another 2 must occur eventually. The next digit will be a 0. We have now found a segment of the form $2v0$ with v a 145-segment, the whole will be a segment of some y^ℓ_t, and we may apply Lemma 5·15 to establish part (i).

Proof of (ii): let u be any initial segment of a containing an occurrence of v_r and ending with 0. We assert that $u \sqsubset y^\ell_r$. We know that $u \sqsubset y^\ell_t$ for some t, where by Lemma 5·16, $t \preccurlyeq r$. By 5·19, the hypotheses of Lemma 5·18 hold, and so we indeed have $u \sqsubset y^\ell_r$. Now Lemma 4·14 tells us that either $y^\ell_r \curvearrowright a$ or $a \rhd y^\ell_r$.

Proof of (iii): by part (ii) we may shift our original a to an a_1 with starts with a 2, and thus find an r_1 T-minimal with $v_{r_1} \sqsubset a_1$; Remark 5·13 tells us that if v_{r_1} has two occurrences in a_1, then $y^\ell_{r_1} \curvearrowright a_1$ is impossible, and so we have $a_1 \rhd y^\ell_{r_1}$, and hence $a \rhd y^\ell_{r_1}$, as desired. If v_{r_1} has only one occurrence in a_1, shift to a_2 where it has no occurrence. a_2 will have infinitely many 2's and infinitely many 4's, so we may find r_2 T-minimal with $v_{r_2} \sqsubset a_2$. The argument above shows that $r_1 \prec r_2$: hence there are only finitely many possibilities for r_2. If v_{r_2} has only 1 occurrence in a_2, repeat this last step. Within finitely many steps we find either that $a \rhd y^\ell_\emptyset$ which is impossible as $y^\ell_\emptyset$ contains no 4's, or that we have reached a $t \neq \emptyset$ for which v_t has at least two occurrences in (some shift of) a and that $a \rhd y^\ell_t$. ⊣ (5·26)

Putting our lemmata together, we see that we have proved the following

5·27 PROPOSITION *If* $y^\ell_T \curvearrowright a$ *then* **either** $a \rhd (45^\ell)^\infty$ **or** $a \rhd y^\ell_t$ *for some* $t \in T$.

Proof : Each such a must have either infinitely many 2's or infinitely many 4's, and possibly both. The first alternative holds if a has only finitely many 2's; $a \rhd y^\ell_\emptyset$ if a has only finitely many 4's; and $a \rhd y^\ell_t$ with $t \neq \emptyset$ if if a has both infinitely many 2's and infinitely many 4's. ⊣ (5·27)

5·28 REMARK $y^\ell_\emptyset$ being periodic, the only points it attacks are the finitely many points in its orbit, and each of those attack it.

Now we are in a position to verify (4·24·3).

5·29 PROPOSITION *Suppose that* $s \neq \emptyset$ *and that* $y^\ell_T \curvearrowright c \curvearrowright b \rhd y^\ell_s$. *Then* $c \rhd y^\ell_r$ *for some* $r \prec s$.

Proof : c will contain at most one 3 as the 3's in y^ℓ_T are at ever widening intervals. Apply a shift to remove that 3. Hence no segment of c can cross the divide between different segments of y^ℓ_T of the form $y^\ell_t \upharpoonright m$, and so each segment of c will be a segment of some y^ℓ_t with $t \in T$.

Choose r T-minimal such that v_r occurs at least twice as a segment of c: the argument of lemma 5·26 shows how to do that. Then we shall find as before that $c \rhd y^\ell_r$. Corollary 5·16 shows that $r \preccurlyeq s$, as v_s occurs infinitely often in y^ℓ_s and therefore at least twice in any $b \rhd y^\ell_s$. But then since $c \curvearrowright b$, the spacing of the occurrences of v_s in c precludes the possibility of $c \rhd y^\ell_s$. Thus $r \prec s$ as required. ⊣ (5·29)

5·30 PROPOSITION *If* $y^\ell_T \curvearrowright c \curvearrowright b \rhd y^\ell_\emptyset$*, then either* c *has only finitely many 4's and is near* $y^\ell_\emptyset$*, or* c *has infinitely many 4's and is near some* y^ℓ_t *with* $t \neq \emptyset$.

Proof : c must have infinitely many 2's, as $y_\emptyset$ does. If it has only finitely many 4's, then by Lemma 5·25, $c \triangleright y_\emptyset$; if c has infinitely many 4's, then (possibly after a finite shift to remove at most one occurrence of 3) Lemma 5·26 applies to show that for some $t \prec \emptyset$, $c \triangleright y_t$. ⊣ (5·30)

We may now compute $\theta(y_T, \mathfrak{s})$, which by 4·24 is at least $\eta =_{\rm df} \varrho_T(\emptyset)$. From 5·27,

$$y_T^\ell \curvearrowright a \implies a \triangleright (45^\ell)^\infty \text{ or } \exists t{:}{\in} T\ a \triangleright y_t^\ell$$

$(45^\ell)^\infty$ is periodic so abides and attacks nothing save its own orbit. Anything near to it will have gone at time 1, because there is nothing to sustain it: each y_t^ℓ has a bound on the length of 45-intervals. Anything near to some y_t^ℓ with $t \neq \emptyset$ will vanish before time η. The things near to $y_\emptyset^\ell$ but not in its orbit go at moment η, as their support has disappeared; the points in the orbit of $y_\emptyset^\ell$ are all periodic, and therefore abide; thus stability comes at time $\eta + 1$. Hence we have proved the following:

5·31 PROPOSITION *For each ℓ and T, $\theta(y_T^\ell, \mathfrak{s}) = \varrho_T(\emptyset) + 1$.*

5·32 REMARK In [11], we started from $y_\emptyset = 0202202220\ldots$, which does not attack itself. Then there are no c, b, with $y_T \curvearrowright c \curvearrowright b \triangleright y_\varnothing$ and $c \triangleright y_\varnothing$, so all the clauses of 4·7 hold, and thus $\theta(y_T, \mathfrak{s}) > \rho_T(\emptyset)$. However, not only does $y_\emptyset$ attack the fixed point 2^∞ but also what we shall call *ghosts* of that fixed point, namely points such as 02^∞ and 202^∞. These ghosts cause a further delay; with T, η as above, $y_\emptyset$ will escape at time η, the ghosts of 2^∞ will go one moment later, at time $\eta + 1$, and stability will be reached at time $\eta + 2$.

In the case of Baire space, the problem of ghosts did not arise, as we had infinitely many symbols at our disposal and could make much cleaner definitions. Before completing our proof of Theorem 5·0, we illustrate the problem caused by ghosts when trying to score a limit ordinal in a compact space: in all the examples so far, x_T attacks only finitely many recurrent points, and we shall see that the score of such points in a compact space cannot be a countable limit ordinal.

Ghosts and limits

First a general point:

5·33 PROPOSITION *If for given $\mathcal{X}$, f and a, there are only finitely many points r with $a \curvearrowright_f r \curvearrowright_f r$, then each such r is periodic.*

Proof : from the remark that $x \curvearrowright y \implies (x \curvearrowright f(y)$ & $f(x) \curvearrowright y$, we know that $r \curvearrowright r \implies f(r) \curvearrowright f(r)$. Hence there are $k < \ell$ with $f^k(r) = f^\ell(r)$, and so $f^k(r)$ is periodic. But by the same remark, $f^k(r) \curvearrowright r$, which is impossible unless r is in the finite orbit of $f^k(r)$ and is thus itself periodic. ⊣ (5·33)

In a compact space, to each point a there is by 3·2 at least one recurrent point r with $a \curvearrowright r$. In the present space with the shift function, there will also be non-recurrent points near to r which are also attacked by a.

5·34 LEMMA *Let Σ be a finite set of symbols, and consider the space ${}^\omega\Sigma$ with the shift function $\mathfrak{s}$. Suppose that x is not periodic, and that $x \curvearrowright r$ where r is periodic. Then there is a symbol σ and a periodic point s (in the orbit of r) such that $\sigma{}^\frown s$ is not periodic and either $x \curvearrowright \sigma{}^\frown s$ or $\sigma{}^\frown s$ is a finite shift of x.*

Proof : let u be the shortest finite sequence such that $r = u^\infty$. Let $p = \ell h(u)$. Suppose no finite shift of x equals r. Then for each k with $x(k) = r(0)$, there will be an $m > k$ such that $x(m+1) \neq r(m+1-k)$: let $\ell(k)$ be the least such.

Similarly we may consider an occurrence of some u^j as a segment of x, starting above some $\ell(k)$, and now try to extend it backwards. That is, suppose the occurrence starts at $x(m)$. Ask if $x(m-1){}^\frown u^j \sqsubset r$: if yes, go on to $x(m-2)$ ...

Thus we may consider maximal segments of x that are segments of r and are of length at least p. There may be some overlap; for example suppose $r = (322422)^\infty$, and suppose x starts 3224223223224222, then two such segments are 322422322 and 22322422, overlapping by two places. But the overlap cannot be as much as a whole period.

Since each $u^k \sqsubset x$ we may choose an infinite sequence u_i of such maximal segments of x with $\ell h(u_i) \geqslant (i+1)p$, non-overlapping; indeed with a gap of at least $2p$ between the end of one and the beginning of the next.

For each such sequence u_i, let ℓ_i be the lag $\ell(u_i, r)$ as defined in 4·13. By periodicity, that lag will be less than p, and so by discarding various u_i's if necessary, we may suppose that there is an integer $\bar{\ell} \in [0, p)$ with each $\ell(u_i, r) = \bar{\ell}$.

Examine the symbol σ_i immediately preceding the occurrence of u_i as a segment of x. That symbol is a member of the finite set Σ: therefore some symbol, call it $\bar{\sigma}$, must be that preceding symbol for infinitely many i. Discard all u_j for $\sigma_j \neq \bar{\sigma}$. Now take $\sigma = \bar{\sigma}$ and $s = \mathfrak{s}^{\bar{\ell}}(r)$. Then $x \curvearrowright \sigma\frown s$.

If some finite shift of x equals r, let k be minimal such that $\mathfrak{s}^k(x)$ is in the orbit of r, and take $s = \mathfrak{s}^k(x)$. Then s is periodic; $k \neq 0$ as x is not periodic, so take $\sigma = x(k-1)$. Then $\sigma\frown s = \mathfrak{s}^{k-1}(x)$. ⊣ (5·34)

The function β used below is that defined in 0·4.

5·35 PROPOSITION *Let Σ be a finite set of symbols, and suppose that in the space ${}^{\omega}\Sigma$ with the shift function, a attacks only finitely many recurrent points. Then for all non-recurrent x, if $a \curvearrowright x$ then there is a non-periodic point of the form $\sigma\frown s$ with s periodic such that $\beta(\sigma\frown s, a, \mathfrak{s}) \geqslant \beta(x, a, \mathfrak{s})$.*

Proof : let r be a periodic point attacked by x, and take σ, r as in the lemma. Write $y = \sigma\frown s$. y being non-periodic is in the escape set, since the only points attacked by a periodic point are those in its orbit, so in our context a non-periodic point cannot be attacked by a recurrent point.

If $x \curvearrowright y$, then actually $\beta(y, a, \mathfrak{s}) > \beta(x, a, \mathfrak{s})$. If y is a finite shift of x, then whenever $z \curvearrowright x$, $z \curvearrowright y$, so $\beta(y, a, \mathfrak{s}) \geqslant \beta(x, a, \mathfrak{s})$. ⊣ (5·35)

5·36 THEOREM *Let Σ be a finite set of symbols, and let a be a point of the space ${}^{\omega}\Sigma$, and $\mathfrak{s}$ the shift function acting on that space. Suppose that $\theta(a, \mathfrak{s}) \geqslant \lambda$, where λ is a countable limit ordinal. Suppose further that only finitely many recurrent points are attacked by a. Then $\theta(a, \mathfrak{s}) > \lambda$.*

Proof : were a periodic, $\theta(a, \mathfrak{s}) = 0$; so we know that a is not periodic. By Theorem 3·2, we know that a will attack recurrent points, and by Proposition 5·33 we know that each recurrent point attacked by a is periodic. Let λ_n be a strictly increasing sequence of ordinals with supremum λ and x_n a sequence of points in $E(a, \mathfrak{s})$ with $\beta(x_n, a, \mathfrak{s}) = \lambda_n$. As we are in a compact space, we know that each x_n attacks some recurrent (and therefore, in our context, periodic) point. By the last proposition, we may find a recurrent point s_n and a symbol σ_n such that $z_n =_{\rm df} \sigma_n\frown s_n$ is not periodic and $\beta(z_n, a, \mathfrak{s}) \geqslant \beta(x_n, a, \mathfrak{s})$.

By assumption there are only finitely many symbols and only finitely many possible recurrent points, so for some symbol σ, some recurrent point s and infinitely many n, $z_n = \sigma\frown s$ and $\sigma\frown s$ is not periodic.

For each such n, $\beta(\sigma\frown s, a, \mathfrak{s}) \geqslant \lambda_n$, and therefore $\beta(\sigma\frown s, a, \mathfrak{s}) \geqslant \lambda$; and hence $\theta(a, \mathfrak{s}) > \lambda$. ⊣ (5·36)

So to score exactly a countable limit ordinal, we must have infinitely many recurrent points, and we shall get those from our earlier construction by varying ℓ.

Attacking infinitely many periodic points

5·37 We take a new symbol 6. Let λ be a countable limit ordinal, and we take a strictly increasing sequence λ_n ($(1 \leqslant n < \omega)$ of ordinals with limit λ. We take a tree T_n with $\rho_{T_n}(\emptyset) = \lambda_n$ and (taking $\ell = n$ for the $n^{\rm th}$ tree), the point $y_n =_{\rm df} y^n_{T_n}$ with $\theta(y_n, \mathfrak{s}) = \lambda_n + 1$, starting from $y^n_\emptyset = (02^n)^\infty$. We now list each y_n infinitely often in the list z_i, and for a strictly increasing sequence p_i of positive integers set

$$z = 6\frown(z_0 \restriction p_0)\frown 6\frown(z_1 \restriction p_1)\frown 6\frown \ldots$$

We assert that $\theta(z, \mathfrak{s}) = \lambda$.

Proof : suppose that $z \curvearrowright a$. At most one 6 can occur in a in view of the spacing of 6's in z; so by making a finite shift if necessary, we may suppose that there are no 6's in a. Then each segment u of a is a segment of some y_n, with n possibly dependent on u.

By inspection, the recurrent points attacked by z are these:

$$5^\infty,\ 2^\infty,\ \text{ and for } n \geqslant 1, (45^n)^\infty \text{ and } (02^n)^\infty,\ \text{ and points in their orbits.}$$

5·38 Let us first check a for lengths of 5-intervals. In a given y_n, 5^{n+1} has no occurrence. Immediately after a 5-interval there is either a single 1 followed by a new but shorter 45-interval, so after finite time (computable from the function χ) that option is no longer available; or, which must happen eventually, a 0 signalling the start of a 02-interval.

Hence we see three possibilities for a:

(5·38·0) there are only finitely many 5's in a;

(5·38·1) there are infinitely many 5's, but each 5-interval is finite;

(5·38·2) there is an infinite 5-interval, so $a \triangleright 5^\infty$.

5·39 Now a check for 2 intervals. In any given y_m, 2^{m+1} has no occurrence. A 2-interval is followed either by a 0 and another 2-interval or by a 4, signalling the start of a 45-interval. However, the 02-intervals are of unbounded length.

Again we see three possibilities for a:

(5·39·0) there are only finitely many 2's in a;

(5·39·1) there are infinitely many 2's, but each 2-interval is finite;

(5·39·2) $a \triangleright 2^\infty$.

Thus *prima facie* there are nine combinations to be discussed, but (5·38·2) implies (5·39·0), (5·39·2) implies (5·38·0), and the conjunction ((5·38·0) and (5·39·0)) cannot happen, for there are no 6's in a; some digit must occur infinitely often; each 3 is followed by a 1, each 1 by a 4, and each 4 by a 5; and each 0 is followed by a 2.

So of the nine cases, we can deal easily with six:

(5·38·0) and (5·39·0): impossible

(5·38·0) and (5·39·2): $a \triangleright 2^\infty$

(5·38·1) and (5·39·2): impossible

(5·38·2) and (5·39·0): $a \triangleright 5^\infty$.

(5·38·2) and (5·39·1): impossible

(5·38·2) and (5·39·2): impossible

5·40 If (5·38·1) holds, there must be a fixed n such that 5^ℓ occurs infinitely often as a 5-interval of a iff $\ell = n$, for a segment u of a containing two 5-intervals ending before the end of u cannot be a segment of one y_m unless the intervals are of the same length.

Similarly if (5·39·1) holds, there must be a fixed m such that 2^ℓ occurs infinitely often as a 2-interval of a iff $\ell = m$. Further, if both (5·38·1) and (5·39·1) hold, the fixed n and fixed m must be equal.

Hence if either (5·38·1) or (5·39·1) holds, there is a fixed n such that each segment of a is a segment of y_n: by Lemma 4·14, either $a \triangleright y_n$, or $y_n \curvearrowright a$. We already know, by 5·27, that if $y_n = y^n_{T_n} \curvearrowright a$, then either $a \triangleright y^n_s$ for some $s \in T$, or $a \triangleright (45^n)^\infty$.

With these remarks in mind, we may complete our classification as follows.

(5·38·0) and (5·39·1): $\exists n\ a \triangleright (02^n)^\infty = y^n_\emptyset$

(5·38·1) and (5·39·0): $\exists n\ a \triangleright (45^n)^\infty$

(5·38·1) and (5·39·1): $\exists n\ a \triangleright y_n$ or $a \triangleright y^n_s$, for some $s \neq \emptyset$.

We now have a list of all points attacked by z, and proceed to compute their moment of disappearance:

those near y_n or near some y^n_s with $s \neq \emptyset$ go before time λ_n;
those near $(02^n)^\infty$ go at time λ_n;
those near $(45^\ell)^\infty$ go at time 2;
those near 5^∞ go at time 1;
and those near 2^∞ go at time 1.

Hence $\theta(z, \mathfrak{s}) = \lambda$. $\dashv$ (5·0)

6: Recursive ill-founded trees

In §4 we gave a method of placing points at the nodes of a well-founded subtree of $\mathcal{S}$ which permits the construction of a point a with $\theta(a, f)$ equalling the rank of that tree, which might be any pre-assigned countable ordinal. But the method of assignment of points was independent of the tree under consideration, and in particular did not rely on the trees being well-founded; and in the present section we apply it to ill-founded trees and obtain this result:

6·0 THEOREM *There is a recursive point a in Baire space $\mathcal{N}$ such that $\theta(a, \mathfrak{s}) = \omega_1^{CK}$, the first non-recursive ordinal.*

Our proof is inspired by the Lorenzen–Kreisel analysis ([8], and see also its review [16] by Moschovakis) of the Cantor–Bendixson theorem.

Our starting point is the existence of a recursive linear ordering which is not a well-ordering but which has no descending chains which are *hyperarithmetic* in the sense of Kleene. Such linear orderings were studied by Harrison in his Stanford thesis of 1966, (published as [5]; see also the review [15]) where they are termed *pseudo-well-orderings*. They have the property that their maximal initial well-founded segment is of length exactly ω_1^{CK}.

6·1 REMARK The reader who is willing to take on trust the existence of pseudo-well-orderings as just described will not need great familiarity with the concept of hyperarithmetic (or HYP for short) to follow the arguments of this section. It may aid the reader's intuition to be told that the HYP sets are those constructible in the sense of Gödel at a recursive stage in the generation of the constructible hierarchy. They form a natural model of the weak system of set theory known as KP, for Kripke–Platek.

Fuller information on these concepts will be found in the books [1], [9], [17] and [18], the paper [2], and the preprint [12] and its published version [13].

From a pseudo-well-ordering we may, as discussed in 4·3, build a recursive tree $T \subseteq \mathcal{S}$ which will be ill-founded, but will have for each recursive ordinal η a node $s \in T$ such that the tree below s, $\{t \in T \mid t \preccurlyeq s\}$ will be well-founded, so that a rank function is definable on this portion of the tree and we shall have $\rho_T(s) = \eta$. We shall call such nodes s a *well-founded* node of rank η. No well-founded node of T will have rank $\geqslant \omega_1^{CK}$, from the way T is constructed from a pseudo-well-ordering.

Exactly as in §4 we ascribe points x_s to each node s of T, starting from $x_\emptyset = 048\ldots$, and define a "global" point x_T. The effective nature of our definitions ensures that this point will be recursive, and from the arguments of §4 we know that if s is a well-founded node of rank a recursive ordinal η, $\theta(x_s, \mathfrak{s}) = \eta + 1$, and hence that $\theta(x_T, \mathfrak{s}) \geqslant \omega_1^{CK}$.

To complete the proof of the theorem, therefore, we must examine the behaviour of our construction below ill-founded nodes s. Corresponding to descending chains in the pseudo-well-ordering will be infinite paths through the tree T.

Points at the end of paths

We consider the shift function $\mathfrak{s}$ acting on Baire space, as in §4. We have a tree $T \subseteq \mathcal{S}$ which is closed under shortening and we have defined points x_s for $s \in T$, and a point x_T.

6·2 First, suppose that we have an infinite path p through T.

Set $a_i = x_{p\restriction i}$. Then by construction $a_{i+1} \curvearrowright a_i$; as in the proof of 3·15 we may find integers m_i such that if we set $y_i = f^{m_i}(a_i)$, the sequence (y_i) will be convergent to a point we shall call x_p. There is likely to be freedom in our choice of integers m_i, so that we do not know that x_p is uniquely determined by the path p. However, by 3·19, $\omega_f(x_p)$ is uniquely determined, and we know that for each i,

$$x_T \curvearrowright x_p \curvearrowright x_p \curvearrowright x_{p\restriction i}$$

and that for any point z,

$$\text{if } \forall i\ x_T \curvearrowright z \curvearrowright x_{p\restriction i} \text{ then } \forall i\ z \curvearrowright y_i \text{ and hence } z \curvearrowright x_p.$$

6·3 PROPOSITION *We may adjust our choices of the various integers employed so that x_p is always recursive in p.*

Proof : by inspection of the proofs of 3·15 and 3·18. ⊣ (6·3)

6·4 PROPOSITION *If $x_T \curvearrowright \beta$ then* **either** *there is a uniquely determined infinite path p through T such that $x_T \curvearrowright x_p \curvearrowright x_p \curvearrowright \beta$,* **or** *there is an $s \in T$ with $\beta \rhd x_s$.*

Proof : Suppose that $x_T \curvearrowright \beta$. Any odd number that occurs in β is of the form $\pi_{t,n}$. Suppose that $\beta(i) = \pi_{t,n}$ and $\beta(j) = \pi_{s,m}$ are odd numbers occurring in β, then, taking $k > \max\{i, j\}$, and applying 4·17 to a $v \in T$ with $\beta \restriction k \sqsubset x_v$, we see that both s and t must be initial segments of v. Hence the t's such that a power of π_t occurs in β define a path through T which may be empty, or finite, or infinite.

If that path is empty, or, in other words, if no odd number occurs in β, then $\beta \rhd x_\emptyset$, by 4·19 and the fact that no number occurs twice in $x_\emptyset$. If that path is non-empty but finite, then there is a longest s such that some power of π_s occurs, and then by 4·20 $\beta \rhd x_s$. If the path is infinite, let us call it p_β. Then $x_{p_\beta} \curvearrowright \beta$, since given k there is an ℓ such that $\beta \restriction k \sqsubset x_{p\restriction \ell}$, and $x_p \curvearrowright x_{p\restriction \ell}$. ⊣ (6·4)

6·5 REMARK Thus, in this context, if β defines p_β, $x_p \leqslant_{\text{Turing}} p_\beta \leqslant_{\text{Turing}} \beta$.

The above proposition, coupled with some facts about hyperarithmetic sets, is enough for the proof of Theorem 6·0. We pause to prove a refinement.

6·6 PROPOSITION *Given any subtree T of $\mathcal{S}$ closed under shortening, the numbers $n_{t,k}$ (for $t \in T$) may be chosen so that whenever $x_T \curvearrowright \beta$ and β defines an infinite path p_β through T, $\beta \curvearrowright x_s$ for each $s \in p_\beta$, and therefore $\beta \curvearrowright x_{p_\beta}$ and, since $x_{p_\beta} \curvearrowright \beta$, β is recurrent.*

Proof : we ensure that whenever $t = s^\frown\langle k\rangle$, the integers $n_{t,\ell}$ are chosen so that for each $s' \succcurlyeq s$, $x_{s'} \restriction \ell h(t) \sqsubset x_s \restriction n_{t,\ell}$, that the numbers $x_s(n_{t,\ell} - 1)$ immediately preceding each power of π_t in x_t should form a strictly increasing sequence of multiples of 4, and that $x_s(n_{t,\ell})$ will always be a power of π_s.

If we have done that, then we may show that for $\bar{s} \in p_\beta$, $\beta \curvearrowright x_{\bar{s}}$. For let N be given, and pick t of length at least $\max(N, \ell h(\bar{s}) + 1)$ for which some power of π_t equals $\beta(a)$, where $a > N$. Let c be the least integer exceeding a such that $\beta(c)$ is a power of π_u for some u with $u \prec t$. Let b be the largest number less than c for which $\beta(b)$ is a power of π_t, so that $a \leqslant b < c$. Let $s = t \restriction (\ell h(t) - 1)$. Then $\beta \restriction c + 1 \sqsubset x_w$ say, and so the segment $\beta \restriction [b+1, c)$ equals $x_s \restriction n_{t,m}$ for some m, and hence $x_{\bar{s}} \restriction N$ is a segment of $\beta \restriction [b+1, c)$ and thus is a segment of β starting after stage N. ⊣ (6·6)

6·7 REMARK Generally, by Theorem 3·15,

$$A(x_T, \mathfrak{s}) = \{\beta \mid \exists \gamma\ x_T \curvearrowright \gamma \curvearrowright \gamma \curvearrowright \beta\}.$$

In our context, for given β not near or attacked by any x_s, the recurrent $\gamma = x_{p_\beta}$ that attacks β is recursive in β. This gives us an easy way of showing that $\theta(x_T, \mathfrak{s})$ is countable.

Let $\mathfrak{A}$ be the set of nodes s such that the tree below s is ill-founded. $\mathfrak{A}$ is of course a countable set of finite sequences, and therefore codable by a single real, $\mathfrak{a}$. Thus we have

$$A(x_T, \mathfrak{s}) = \{\beta \mid \exists s{:}{\in}\mathfrak{A}\ \beta \rhd x_s\} \cup \{\beta \mid \exists \gamma {\leqslant_{\text{Turing}}} \beta\ x_T \curvearrowright \gamma \curvearrowright \gamma \curvearrowright \beta\}$$

which is arithmetical in $\mathfrak{a}$. $A(x_T, \mathfrak{s})$ is thus Borel and the ordinal $\theta(x_T, \mathfrak{s})$ countable.

Proof of our main result.

Now suppose $x_T \curvearrowright \beta$. If $\beta \triangleright x_s$ where $s \in T$, β will be HYP, indeed recursive, as each x_s is recursive. Given our choice of $x_\emptyset$, there will be no points attacked by some x_s that are not near some x_t with shorter t. If the tree below s is ill-founded, β will abide, otherwise, if the tree below s is well-founded, then β will escape, and the ordinal at which it escapes will be recursive.

So the only case remaining to be discussed is when β defines an infinite path p_β through the tree: in this case, since p_β is recursive in β, and, by choice of T, p_β cannot be HYP, β cannot be HYP. By §4, $x_{p_\beta} \curvearrowright \beta$: since x_{p_β} is recurrent, $\beta \in A^\infty$.

So every point that escapes does so at a recursive ordinal, yielding $\theta(x_T, \mathfrak{s}) \leqslant \omega_1^{CK}$. Hence $\theta(x_T, \mathfrak{s}) = \omega_1^{CK}$. $\dashv$ (6·0)

6·8 Remark If we make the more refined choice of $n_{t,i}$'s sketched in 6·6, we shall have the following exact picture of $\omega_\mathfrak{s}(x_T)$: the points that escape are those near to, that is, are finite shifts of, the x_s with s in the well-founded part of the tree. All such points are recursive. The points that abide are those near to x_s with s in the ill-founded part, — those points again, individually, are recursive — and the recurrent points, which are exactly the points equivalent to, in the sense that they attack and are attacked by, the points x_p placed at the end of each infinite path p. All recurrent points are non-HYP. There are no minimal points, and all recurrent points are maximal.

6·9 Remark Note that in these examples, if $\gamma \triangleright x_s$ where s is in the ill-founded part of the tree, there is a sequence of points attacking γ, with each point of the sequence being recursive, but the sequence itself not even hyperarithmetic.

6·10 Remark In [12] and [13] it is shown that recurrent points must exist for any iteration with recursive data which does not stabilise at a recursive ordinal.

7: Two complete Σ^1_1 sets and a strange subspace

7·0 The uniformity of our construction shows that in Baire space the Σ^1_1 set

$$P =_{\rm df} \{\alpha \in \mathcal{N} \mid \exists \rho\ \alpha \curvearrowright_\mathfrak{s} \rho \curvearrowright_\mathfrak{s} \rho\}$$

is a complete Σ^1_1 set, that is, that every $\mathbf{\Sigma^1_1}$ subset of $\mathcal{N}$ is reducible to it by a continuous function.

We shall use the familiar fact (Theorem 4A.3 on page 193 of [14]) that to any $\mathbf{\Pi^1_1}$ set $Q \subseteq \mathcal{N}$ there is a continuous map $\pi : \mathcal{N} \to LORD$ such that $\forall \alpha[\alpha \in Q \iff \pi(\alpha) \in WORD]$, where $LORD$ is the set of those $\beta \in \mathcal{N}$ that code linear orderings and $WORD \subseteq LORD$ is the set of those that indeed code well-orderings. It then suffices to reduce $WORD$ to $\mathcal{N} \setminus P$ by a recursive function.

But our construction yields that: we have seen how to associate to each countable linear ordering a tree of finite descending sequences in that linear ordering, which may be copied to a subset T of $\mathcal{S}$, and from there to build a point x_T recursive in T. The procedure works whether the trees are well-founded or not, and is recursive in the starting datum of a linear ordering.

If the linear ordering is a well-ordering, the tree is well-founded and the abode is empty, so that no recurrent points are attacked by x_T. If the linear ordering has a descending chain, the tree has infinite paths, which generate recurrent points attacked by x_T.

7·1 P is the set of points whose abode under $\mathfrak{s}$ is non-empty. In a compact space no abode is empty, so to obtain a complete analytic set we must modify the definition. In the space $\mathcal{Z} =_{\rm df} {}^\omega 7$, define four sets:

$$\begin{aligned} Q &=_{\rm df} \{x \in \mathcal{Z} \mid \text{ the abode of } x \text{ under } \mathfrak{s} \text{ is infinite}\}; \\ F &=_{\rm df} \{(02)^\infty, (20)^\infty, (45)^\infty, (54)^\infty\}; \\ A &=_{\rm df} \{x \in \mathcal{Z} \mid A(x, \mathfrak{s}) \subseteq F\}; \\ B &=_{\rm df} \{x \in \mathcal{Z} \mid A(x, \mathfrak{s}) \text{ is uncountable}\}. \end{aligned}$$

Then Q has these three properties: it is Σ^1_1, it contains B and it is disjoint from A. We shall see that any such set is a complete Σ^1_1 set.

Proposition 3·18, on which §6 rests, holds for all Polish spaces and continuous functions. Hence the remarks of §6 can be adapted to the present compact space $\mathcal{Z}$, and they, together with the discussion of section 5 show that there is a recursive way of assigning to every $\alpha \in \mathit{LORD}$ a tree $T(\alpha) \subseteq \mathcal{S}$ and a point $y^1_{T(\alpha)}$ in $\mathcal{Z}$ the abode of which will, if $\alpha \in \mathit{WORD}$, be contained in F. If $\alpha \notin \mathit{WORD}$, then there will be $2^{\aleph_0}$ infinite paths through the associated tree — think of a perfect set of pairwise almost disjoint subsets of a given descending sequence in the linear ordering. Distinct paths yield distinct recurrent points y_p attacked by $y^1_{T(\alpha)}$; hence $A(y^1_T(\alpha), \mathfrak{s})$ will be uncountable. Our association $\alpha \mapsto y^1_{T(\alpha)}$ defines a continuous map sending WORD to A and $\mathit{LORD} \setminus \mathit{WORD}$ to B. Hence any Σ^1_1 set containing B and disjoint from A will be complete. Such is the set Q; and such are the other examples mentioned in §0. If we consider $y^0_{T(\alpha)}$ instead of $y^1_{T(\alpha)}$, we can sharpen F to a set of size two, and contain our example in the space ${}^\omega 4$.

7·2 Consider now the set $R = \mathcal{Z} \setminus Q$, which we have just seen to be a complete Π^1_1 set in the compact Polish space $\mathcal{Z} = {}^\omega 7$; therefore not a G_δ subset of $\mathcal{Z}$; therefore not itself a Polish space. From our discussion of ghosts and limits, we see that this space has the following curious combination of properties: it is closed under $\mathfrak{s}$; if x is in it and $x \curvearrowright y$ then y is in it; so that our general iteration question makes sense restricted to this space; no point scores ω_1, as the abode is always finite and therefore Borel; every countable successor ordinal is the score of some point in the space R, as is the ordinal 0; but no countable limit ordinal is the score of any point in the space R.

One might wonder whether there are yet stranger subspaces in which the set of possible scores is an arbitrary prescribed subset of ω_1.

7·3 REMARK Given f, a and b with $a \curvearrowright_f b$, the argument of section 2 shows how to embed the tree $T^a_b(f)$ in a countable tree $S^a_b(f)$. The argument of sections 4, 5 and 6 shows how to embed a given countable tree in a tree of the form $T^a_b(\mathfrak{s})$. Combining the two suggests a reduction of some kind of $\curvearrowright_f$ to $\curvearrowright_{\mathfrak{s}}$.

8: Some open problems

It would be natural to ask whether 7 symbols can be reduced to 2 in my discussion of compact spaces under shift; in recent work Christian Delhommé has shown how to accomplish that.

The present paper leaves open this question:

8·0 PROBLEM Can there be $\mathcal{X}$, f and a with $\theta(a, f) = \omega_1$?

A negative answer would follow from the following conjecture:

8·1 CONJECTURE In all cases $\theta(a, f)$ is at most the least ordinal not recursive in a and (a code of) f.

A discussion and related open problems with a recursion-theoretic flavour may be found in [12] and [13].

8·2 REMARK An example is given in 7·4 of [13] of a point a such that $A^1(a, \mathfrak{s})$ is not closed. That may be combined with the techniques of §4 to yield for any countable successor ordinal ν a point a_ν with $A^\nu(a_\nu, \mathfrak{s})$ not closed: given a countable ordinal η, let T be a countable well-founded tree with $\varrho_T(\emptyset) = \eta$, and build x_T starting from $x_\emptyset = a$ as above; with care, we shall have $\theta(x_T, \mathfrak{s}) = \eta + 1 + \theta(a, \mathfrak{s})$, $A^\eta(x_T, \mathfrak{s}) = \{a\} \cup \omega_{\mathfrak{s}}(a)$ and $A^{\eta+1}(x_T, \mathfrak{s}) = A^1(a, \mathfrak{s})$.

8·3 PROBLEM Is there an example where A^1 is strictly Σ^1_1 ?

8·4 PROBLEM In [13] a set is constructed of points in Baire space, all recurrent under $\mathfrak{s}$, which is strictly linearly ordered by the relation $\curvearrowright_{\mathfrak{s}}$ in the order type of the real line with every rational point doubled. Todorčević asks if such a set can be found with order type that of the real line with *every* point doubled.

If every abiding point is recurrent, the abode will be a G_δ set. The abode is also G_δ in every example constructed in this paper.

8·5 Problem Is $A(a, f)$ always a G_δ set ?

The next problem is inspired by an attempt to generalise the situation described in 6·8.

8·6 Problem Given a and f, call a point δ of $A(a, f)$ *bad* if it is not attacked by a recurrent point hyperarithmetic in δ. Is the set of bad points countable ? Equivalently, since that set is analytic, is every bad point hyperarithmetic in a and f ?

Positive answers to either 8·5 or 8·6 would imply a negative answer to Question 8·0.

The last two questions arises out of an attempt to use analytic determinacy to answer the first. Consider Baire space with the shift function. We know that the set P is a complete Σ^1_1 set. Suppose there is a point α which scores ω_1 under shift. Then its abode A is an analytic but not Borel set, so if we assume analytic determinacy, it too is a complete Σ^1_1 set and there is a Borel isomorphism $\psi : \mathcal{N} \leftrightarrow \mathcal{N}$ which takes A to P and $\mathcal{N} \setminus A$ to $\mathcal{N} \setminus P$ Thus for each δ,

$$\exists\rho\ \alpha \curvearrowright \rho \curvearrowright \rho \curvearrowright \delta \Longleftrightarrow \exists\sigma\ \psi(\delta) \curvearrowright \sigma \curvearrowright \sigma,$$

and so the function ψ reverses the $\curvearrowright$ relation.

8·7 Problem Prove that that is impossible.

Finally here are two attempts to define a game that will answer the main question: fix α as above with non-Borel abode. If the second player has a winning strategy in either of the two following games, we shall have a contradiction. In both games player I builds δ and player II builds ε, and player I loses unless $\exists\rho\alpha \curvearrowright \rho \curvearrowright \rho \curvearrowright \delta \not\curvearrowright \delta$; then

Game 1: player II wins if $\alpha \curvearrowright \varepsilon \curvearrowright \varepsilon \curvearrowright \delta$;

Game 2: player II wins if there is a ρ hyperarithmetic in ε with $\alpha \curvearrowright \rho \curvearrowright \rho \curvearrowright \delta$.

8·8 Problem Prove, assuming projective determinacy, that the second player has a winning strategy in Game 2.

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