SOLUTION OF PROBLEMS OF CHOQUET AND PURITZ

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DEFINITIONS. A *filter* on $\omega = \{0,1,2,\ldots\}$ is a collection F of subsets of ω with the properties that $x \in F$ & $y \in F \rightarrow x \cap y \in F$ and that $\omega \supseteq y \supseteq x$ & $x \in F \rightarrow y \in F$. If $0 \in F$, F is *improper*; otherwise F is *proper*. F is *principal* or *fixed* if $F = \{y \mid \omega \supseteq y \supseteq x\}$ for some $x \subseteq \omega$; otherwise F is *non-principal* or *free*. If $\forall x{:}\subseteq\omega\ (x \in F \text{ or } \omega - x \in F)$, then F is an *ultrafilter*; that is equivalent to being a maximal proper filter. For $A \subseteq \mathcal{P}(\omega)$, write $\tilde{A} =_{df} \{x \mid x \subseteq \omega \ \& \ \omega - x \in A\}$. I is an *ideal* if $\tilde{I}$ is a filter, and I is further described as *principal*, *proper*, etc. accordingly. I is *prime* if $\tilde{I}$ is an ultrafilter.

Let $A \subseteq \mathcal{P}(\omega)$ and $f : \omega \longrightarrow \omega$. Write $f_*A =_{df}$ $\{x \mid x \subseteq \omega \ \& \ f^{-1}``x \in A\}$. Then if A is an ideal, a prime ideal, a filter or an ultrafilter, so is $f_*A$. Further, write $f^{-1}A =_{df}$ $\{x \mid \exists y \in A \ x \subseteq f^{-1}``y\}$. If I is an ideal, so is $f^{-1}I$, but it will only be prime if I is and if $\{n \mid f^{-1}``\{n\} \text{ has more than one element}\} \in I$.

A filter F is *rare* if F contains all cofinite sets and given any partition of ω into non-empty finite sets s_i $(i < \omega)$ there is an $x \in F$ such that for all i, $\overline{\overline{x \cap s_i}} = 1$. An ultrafilter F on ω is a *p-point* if it is free and given any family $\{x_i\}_{i<\omega}$ of elements of F, there is a $y \in F$ such that $\forall i{:}<\omega\ y - x_i$ is finite. An ultrafilter is *Ramsey* if it is both a p-point and rare.

The term p-point arises from the following topological considerations: let βN be the set of all ultrafilters on ω, and take as a basis for a topology all sets of the form $\{F \mid x \in F\}$ where $x \subseteq \omega$. Then $F \in \beta N$ is a p-point in the sense defined above iff $\{F\}$ is not open and the intersection of countably many neighbourhoods of F is again a neighbourhood. Another formulation is that a free ultrafilter is a p-point iff given any partition of ω into non-empty pieces S_i $(i < \omega)$ there is an $x \in F$ such that for all but finitely many i $x \cap S_i$ is finite.

If $F \in \beta N$ and $f : \omega \longrightarrow \omega$, then $f_* F \in \beta N$. The Rudin-Keisler ordering of βN is defined by writing $G \preccurlyeq F$ iff $\exists f\ G = f_* F$. $F \preccurlyeq G$ & $G \preccurlyeq F$ iff there is a permutation h of ω with $F = h_* G$. If F is a p-point then $f_* F$ is fixed or a p-point; if F is Ramsey, $f_* F$ is fixed or Ramsey; but if $2^{\aleph_0} = \aleph_1$, then $\forall F{:}\in\beta N\ \exists G{:}\in\beta N$ G rare and $F \preccurlyeq G$. Kunen [5] has shown that $\exists F,G{:}\in\beta N\ \forall f\ f_* F \neq G$ & $f_* G \neq F$. Rudin [8] contains more information.

Choquet [2a, page 48] asked whether for every free ultrafilter F there is an f with $f_* F$ a p-point. He uses the term "ultrafiltre absolument 1-simple" for p-points. (Cf. [2b], where he also discusses Ramsey ultrafilters.) The present paper answers Choquet's question, subject to the continuum hypothesis, by proving the following

THEOREM. If $2^{\aleph_0} = \aleph_1$ then there is a free ultrafilter F such that for no f is $f_* F$ a p-point.

The proof may not be intelligible to persons unfamiliar with the foundational approach to the projective hierarchy, and it is hoped to publish a more lucid version in [6].

A subset A of $\mathcal{P}(\omega)$ is Σ^1_1 if there is an $a \subseteq \omega$ such that for all $x \subseteq \omega$,

$$x \in A \longleftrightarrow \exists y{:}\subseteq\omega\ R(a,x,y)$$

where R is arithmetical, that is, built up from a recursive matrix by quantifiers binding variables ranging over ω. By notorious tricks [9, page 174], if $R(a,x,y,z)$ and $S(a,x,y,n)$ are arithmetical, then $\{x \mid \exists y{:}\subseteq\omega\ \exists z{:}\subseteq\omega\ R(a,x,y,z)\}$ and $\{x \mid \forall n{:}\in\omega\ \exists y{:}\subseteq\omega\ S(a,x,y,n)\}$ are Σ^1_1; furthermore "there is a sequence y_0, y_1, ... of subsets of ω" can be expressed in Σ^1_1 form by remembering that a sequence y_i can be coded by the single set $\{2^m 3^i \mid m \in y_i\}$.

Examples 1. For $g : \omega \longrightarrow \omega$ define $I_g =$ $\{x \mid \exists k{:}\in\omega\ \forall i{:}>k\ x \cap g^{-1}\{i\}$ is finite$\}$ and $I^r_g =$ $\{x \mid \exists k{:}\in\omega\ \exists \ell{:}\in\omega\ \forall i{:}>k\ x \cap g^{-1}\{i\} < \ell\}$. I_g and I^r_g are both Σ^1_1 (indeed, arithmetical) sets and are possibly improper free ideals. A free ultrafilter $\mathcal{U}$ is Ramsey iff $\forall g\ \mathcal{U} \cap I^r_g \neq 0$, and is a p-point iff $\forall g\ \mathcal{U} \cap I_g \neq 0$.

2. If A is Σ^1_1 and $f : \omega \longrightarrow \omega$ then $f_* A$ and $f^{-1}A$ are Σ^1_1.

3. If α_i $(i < \omega)$ is a strictly decreasing divergent series of positive real numbers with limit 0, then $\{x \mid \Sigma_{i \in x}\ \alpha_i < \infty\}$ is a Σ^1_1 ideal.

4. If $\pi : [\omega]^2 \longrightarrow 2$, then $\{x \mid \exists y_0 \ldots \exists y_{k-1}\ \forall i{:}<k\ y_i$ is homogeneous for π and $x \subseteq y_0 \cup \ldots \cup y_{k-1}\}$ is a (possibly improper) Σ^1_1 ideal R_π. (Here $[A]^2 = \{\{i,j\} \mid i \in A,\ j \in A,\ i \neq j\}$; y is homogeneous for π if π is constant on $[y]^2$.) The term "Ramsey" stems from the fact (proved in [1]) that a free ultrafilter F is Ramsey iff $\forall \pi\ R_\pi \cap F \neq 0$: Ramsey's theorem asserts that each π possesses an infinite homogeneous y.

5. Put, for α a complex number of modulus 1 and ϵ a positive real number, $K(\alpha,\epsilon) = \{n \mid |1-\alpha^n| < \epsilon\}$. Then $K =_{df}$ $\{x \mid \exists \alpha_1 \ldots \alpha_n\ \exists \epsilon_1 \ldots \epsilon_n\ x \supseteq K(\alpha_1,\epsilon_1) \cap \ldots \cap K(\alpha_n,\epsilon_n)\}$ is a Σ^1_1 proper filter, by a theorem of Dirichlet. [4, Theorem 201.]

6. If A is Σ^1_1 then $\{y \mid \{x \mid x \in A\ \&\ x \cap y$ is infinite$\}$ is infinite$\}$ is Σ^1_1, and its complement in $\mathcal{P}(\omega)$ is an ideal which is proper

iff A contains infinitely many infinite subsets of ω.

DEFINITION. An ideal is gaunt if it is proper, contains all finite sets and is Σ^1_1. A filter is gaunt if its dual ideal is.

LEMMA 1. (Sierpiński) No gaunt ideal is prime.

Proof. A non-principal prime ideal would have to have Lebesgue outer measure 1 and inner measure 0, but every Σ^1_1 set is Lebesgue measurable. Aliter, the statement that a given Σ^1_1 set forms a free prime ideal is Π^1_2 and is therefore false by Shoenfield's absoluteness lemma and the fact, due to Feferman [3], that there is a Boolean extension of the universe in which the statement that there are no free prime ideals on ω has Boolean truth value $\underset{\sim}{1}$.

LEMMA 2. Let I be gaunt. There are subsets x_i $(i < \omega)$ of ω such that no $x_i \in I$, $i \neq j \longrightarrow x_i \cap x_j = 0$ and $\bigcup_{i<\omega} x_i = \omega$.

Proof. By repeated application of Lemma 1. Say x is undecided by an ideal J if neither x nor $\omega - x$ is in J; and for $A \subseteq \mathcal{P}(\omega)$, write id(J,A) for the ideal generated by J and A, that is to say,

$$\mathrm{id}(J,A) = \{x \mid \exists y{:}\in J\ \exists z_0 \ldots z_k{:}\in A\ x \subseteq y \cup z_0 \cup \ldots \cup z_k\}.$$

If A is also an ideal, then $\mathrm{id}(J,A) = \{x \mid \exists y{:}\in J\ \exists z{:}\in A\ x = y \cup z\}$. id(J,A) is Σ^1_1 if both J and A are. Now for Lemma 2: by Lemma 1, there is an x_0 undecided by I. $\mathrm{id}(I,\{x_0\})$ is gaunt, and so some x_1' is undecided by it: put $x_1 = (x_1' \cup \{0\}) - x_0$. Then x_1 too is undecided by $\mathrm{id}(I,\{x_0\})$; $\mathrm{id}(I,\{x_0,x_1\})$ is gaunt and so fails to decide some x_2': put $x_2 = (x_2' \cup \{1\}) - (x_0 \cup x_1)$ ... Continuing in this manner we obtain a sequence x_i with $\bigcup_{i<\omega} x_i = \omega$ as required.

$f : \omega \longrightarrow \omega$ is I-infinite if $\forall i{:}<\omega\ f^{-1}\text{``}\{i\} \in I$.

LEMMA 3. If I is gaunt, then there is a gaunt $I' \supseteq I$ and a $\psi : \omega \longrightarrow \omega$ such that $I_\psi \subseteq I'$; in particular ψ is I'-infinite and for each $h : \omega \longrightarrow \omega$, $\{n \mid h(\psi(n)) \geq n\} \in I'$.

Proof. Let $\{x_i\}_{i\in\omega}$ be a sequence as in Lemma 2. Define $\psi(n) = i$ for $n \in x_i$. Put $I' = id(I, I_\psi)$. I' is Σ^1_1 and contains all finite sets. Suppose that $\omega = x \cup y$ where $x \in I$ and $y \in I_\psi$. Let i be such that $x_i \cap y$ is finite: then $x_i - x$ is finite and so $x_i \in I$. ※ Hence I' is proper. Put $X = \{n \mid h(\psi(n)) \geq n\}$. $\psi(n) = i \longrightarrow n \leq h(i)$ and so for each i, $X \cap \psi^{-1}\{i\}$ is finite, and so $X \in I_\psi$.

The last clause shows that, in the terminology introduced by Puritz [7], ψ will be in a lower sky than the identity with respect to any ultrafilter extending $\tilde{I}'$.

LEMMA 4. Let I be gaunt and f I-infinite. Then there is a gaunt $I' \supseteq I$ and a $\psi : \omega \longrightarrow \omega$ with $I_\psi \subseteq f_* I'$: in particular, for every $h : \omega \longrightarrow \omega$, $\{n \mid h(g(n)) \geq f(n)\} \in I'$, where g is the composition of f and ψ, viz. $\lambda n\ \psi(f(n))$.

Proof. $f_* I$ is gaunt: let ψ be as in Lemma 3, and put $I' = id(I, f^{-1} id(f_* I, I_\psi))$: in fact $I' = id(I, f^{-1} I_\psi)$.
$\{n \mid h(\psi(f(n))) \geq f(n)\} = f^{-1}``\{k \mid h(\psi(k)) \geq k\}$, whence the last part, as $\{k \mid h(\psi(k)) \geq k\} \in I_\psi$ by Lemma 3.

LEMMA 5. Let I_i $(i < \omega)$ be a sequence of gaunt ideals with $I_i \subseteq I_{i+1}$ for all i. Then $\bigcup_{i<\omega} I_i$ is gaunt.

Proof. $\omega \notin \bigcup_{i<\omega} I_i$; that $\bigcup_{i<\omega} I_i$ is an ideal containing all finite sets is trivial; that it is Σ^1_1 is immediate from the classical result that the union of countably many Σ^1_1 sets is Σ^1_1.

It is now easy using Lemmata 4 and 5 and the continuum hypothesis to construct a free prime ideal I such that

for each I-infinite f there is an I-infinite g with
$\{n \mid h(g(n)) \geq f(n)\} \in I$ for all $h : \omega \longrightarrow \omega$

and such that

for all $f : \omega \longrightarrow \omega$ there is a $\psi : \omega \longrightarrow \omega$ with $I_\psi \subseteq f_* I$.

From the second property, $\tilde{I}$ is a free ultrafilter such that $f_*\tilde{I}$ is never a p-point; from the first, there is no lowest sky in the ultrapower $\omega^\omega/\tilde{I}$, which answers a question of Puritz [7]. There is a connection between the two problems, for if f is in the lowest sky of $\omega^\omega/\mathcal{U}$, then $f_*\mathcal{U}$ is a p-point.

The theorem has been improved by Mr. R. A. Pitt of Leicester University, who has shown that

if $2^{\aleph_0} = \aleph_1$, there is a free ultrafilter F such that for no f is f_*F either rare or a p-point, and there is a p-point $\mathcal{U}$ such that for no f is $f_*\mathcal{U}$ Ramsey.

His proofs, which are presumably more "elementary" in that they do not use the notion of a Σ_1^1 ideal, will appear in his doctoral dissertation. The present author has proved both parts of Pitt's theorem using Σ_1^1 ideals (the first part after and the second part before hearing of Pitt's proofs); the key step in the proof of the first part being the following

THEOREM. No gaunt filter is rare.

The existent proof of that uses forcing: a direct proof would be welcome. It is intended that [6] shall contain a discussion of the properties of gaunt ideals and filters. Let us say that a filter F is

<u>tall</u> if there is no infinite $x \subseteq \omega$ such that $\forall y{:}\in F$ $x-y$ is finite. There are tall gaunt filters which can, assuming $2^{\aleph_0} = \aleph_1$, be extended to p-points, for instance, that dual to the ideal in Example 3, and there are tall gaunt filters which can be extended to rare filters, for example $\tilde{I}_g$ where $\forall i\ g^{-1}\{i\}$ is infinite, but, and this is the essential fact in the author's proof of the second part of Pitt's theorem, no tall gaunt filter can be extended to a Ramsey ultrafilter; and indeed a free ultrafilter $\mathcal{U}$ is Ramsey iff it contains no tall gaunt filter. That is a corollary of the following theorem, which will be proved in [6]:

THEOREM. A free ultrafilter $\mathcal{U}$ is Ramsey iff for every Σ^1_1 set $A \subseteq \mathcal{P}(\omega)$ there is an $x \in \mathcal{U}$ such that for every infinite subset y of x,

$$x \in A \leftrightarrow y \in A.$$

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