Analytic sets under attack

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To David Fremlin

Abstract. A point in Baire space is found for which the first derived ω-limit set is not Borel, whilst the second is empty. A second point is found for which the sequence of derived ω-limit sets does not stabilise until the first uncountable ordinal. The two points are recursive.

1. *Introduction*

This paper solves two problems left open in the author's paper [**4**] which will be cited as *Delays*.

We begin by summarising the general background: further details and any unexplained notation will be found in that paper. For a short and informal motivation of this type of problem from the point of view of topological dynamics, the reader may wish to consult [**1**, introduction].

General notation

This paper, as did *Delays*, applies set-theoretic ideas to a problem of analysis, and therefore our notation will draw on that of two mathematical traditions. Thus we usually denote the set $\{0, 1, 2, \ldots\}$ of *natural numbers* by ω, though occasionally by $\mathbb{N}$; this visual distinction allows us to write ω^n for the ordinal power and $\mathbb{N}^n$ for the set of n-tuples of natural numbers.

$\mathbb{N}^+$ is the set $\{1, 2, 3, \ldots\}$ of *positive integers*: in Definition 4·3 the difference between $\mathbb{N}$ and $\mathbb{N}^+$ is important.

Let $\mathcal{X}$ be a Polish space, and $f : \mathcal{X} \longrightarrow \mathcal{X}$ a continuous map. We write $x \curvearrowright_f y$, or sometimes $y \curvearrowleft_f x$, read x *attacks* y, if y is a cluster point of the set of successive images of x under f; and we write $\omega_f(x)$ for $\{y \mid x \curvearrowright_f y\}$, which is a closed set, being the intersection over all i of the closures of the sets $\{f^n(x) \mid n \geqslant i\}$.

We define an operator Γ_f on subsets of $\mathcal{X}$ by

$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\}.$$

Using this operator and starting from a given point $a \in \mathcal{X}$, we define a transfinite sequence of sets:

$$\begin{aligned} A^0(a, f) &= \omega_f(a) \\ A^{\beta+1}(a, f) &= \Gamma_f(A^\beta(a, f)) \\ A^\lambda(a, f) &= \bigcap_{\nu<\lambda} A^\nu(a, f) && \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

$\Gamma_f(X)$ is always $\curvearrowright_f$-closed, and if A is $\curvearrowright_f$-closed, then $\Gamma_f(A) \subseteq A$. Hence $A^0(a, f) \supseteq A^1(a, f)$; $\mathcal{X} \supseteq B \supseteq C \implies \Gamma_f(B) \supseteq \Gamma_f(C)$; thus

$$A^0(a, f) \supseteq A^1(a, f) \supseteq A^2(a, f) \cdots ;$$

as we take intersections at limit ordinals we shall have that for all ordinals α, β,

$$\alpha < \beta \implies A^\alpha(a, f) \supseteq A^\beta(a, f).$$

Definition 1·0. The *escape set* or *boundary* is the union over all ordinals β of the set of those points in $\omega_f(a)$ eliminated at stage β of the iteration:

$$E(a,f) \ =_{\rm df} \ \bigcup_\beta \big(A^\beta(a,f) \smallsetminus A^{\beta+1}(a,f)\big).$$

Here $X \smallsetminus Y$ is the set-theoretic difference $\{x \mid x \in X \text{ and } x \notin Y\}$.

Definition 1·1. For $x \in E(a,f)$, we write $\beta(x,a,f)$ for the unique β with $x \in A^\beta(a,f) \smallsetminus A^{\beta+1}(a,f)$.

In *Delays* it is shown that $\beta(x,a,f)$ is always a countable ordinal, and therefore by the stage of the first uncountable ordinal, ω_1, any point that is to escape has done so. Thus if we make the following

Definition 1·2. $\theta(a,f) \ =_{\rm df}$ the least ordinal θ with $A^\theta(a,f) = A^{\theta+1}(a,f)$,

we know that $\theta(a,f)$ is always well defined and at most ω_1. Further for all $\delta \geqslant \theta$, $A^\delta(a,f) = A^\theta(a,f)$.

Definition 1·3. We write $A(a,f)$ for this final set $A^{\theta(a,f)}(a,f)$. We call $A(a,f)$ the *abode*, and the ordinal $\theta(a,f)$ the *score* of the point a under f.

Thus $E(a,f) = \omega_f(a) \smallsetminus A(a,f)$. We say that points in $A(a,f)$ *abide*, and points in $E(a,f)$ *escape*.

The results of the paper

Question 8·3 of *Delays* asks for an example where A^1 is not a Borel set: we give such an example in the next section, working in Baire space with the shift function.

Question 8·0 of *Delays* asked whether $\theta(a,f)$ is always a countable ordinal. We give a counterexample in Section 3, working in a space $\mathcal{Y}$ of infinite sequences of countably many specially designed symbols, again with the shift function. $\mathcal{Y}$ is of course homeomorphic to Baire space, but the construction is easier to understand if the running information is written into the symbols used.

Of the problems listed in section 8 of *Delays*, only 8·4 survives unchanged by these new results; the others are now answered in their original formulation, though in some cases there are reformulations of interest.

2. *An example with A^1 strictly analytic and A^2 empty*

Let $\mathcal{N}$ be *Baire space*, the space of infinite sequences of natural numbers, often denoted $\mathbb{N}^\mathbb{N}$: for each finite such sequence r we have the basic open set $N_r \ =_{\rm df} \ \{\alpha \mid \alpha \restriction \ell h(r) = r\}$, where $\ell h(r)$ means the length of r and $\alpha \restriction \ell h(r)$ denotes the restriction of α to the set $\{0,1,2,\dots r-1\}$. Here we are following customary set-theoretic practice of treating finite or infinite sequences as functions defined on a (possibly improper) initial segment of ω.

Definition 2·0. $\mathfrak{s} : \mathcal{N} \longrightarrow \mathcal{N}$ is the (backward) *shift function* given by $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$.

Definition 2·1. If r is a finite sequence and x a finite or infinite sequence, we write $r \sqsubseteq x$ to mean $\exists m \forall n < \ell h(r)\ r(n) = x(m+n)$, where $\ell h(r)$ is the length of r; in words, that r is a segment of x.

Let $\mathsf{p}_0 = 2, \mathsf{p}_1 = 3, \mathsf{p}_2 = 5, \dots$ enumerate the rational primes in increasing order. For both finite and infinite sequences of natural numbers, we introduce two variants of the familiar course-of-values functions.

Definition 2·2. Suppose that $v : m \longrightarrow \omega$. Then $\hat{v}$ is by definition the sequence with domain m, satisfying $\hat{v}(0) = \mathsf{p}_0^{v(0)+2}$, $\hat{v}(1) = \mathsf{p}_0^{v(0)+2} \cdot \mathsf{p}_1^{v(1)+1}$, and generally for $1 < k < m$,

$$\hat{v}(k) \ =_{\rm df} \ \mathsf{p}_0^{v(0)+2} \cdot \prod_{1 \leqslant i \leqslant k} \mathsf{p}_i^{v(i)+1}$$

The sequence $\check{v}$ is then defined by setting $\check{v}(k) = 2 + \hat{v}(k)$ for all $k < m$.

Similarly for $\alpha \in \mathcal{N}$ we define $\hat{\alpha}$ in $\mathcal{N}$ by $\alpha(0) = \mathsf{p}_0^{\alpha(0)+2}$, $\hat{\alpha}(1) = \mathsf{p}_0^{\alpha(0)+2} \cdot \mathsf{p}_1^{\alpha(1)+1}$, and generally for $k > 1$,

$$\hat{\alpha}(k) \ =_{\rm df} \ \mathsf{p}_0^{\alpha(0)+2} \cdot \prod_{1 \leqslant i \leqslant k} \mathsf{p}_i^{\alpha(i)+1}$$

and then define $\check{\alpha}$ by $\check{\alpha}(k) = 2 + \hat{\alpha}(k)$.

Remark 2·3. For each α, $\hat{\alpha}$ and $\check{\alpha}$ are strictly monotonic and take only values $\equiv 0 \pmod 4$ and $\equiv 2 \pmod 4$ respectively. Both $\hat{\alpha}$ and $\check{\alpha}$ are recursive in α, and α is uniformly recursive in each $\mathfrak{s}^n(\hat{\alpha})$ and each $\mathfrak{s}^n(\check{\alpha})$.

Preparing a shift-closed strictly analytic set of strictly increasing functions

Definition 2·4. For P any analytic subset of $\mathcal{N}$, set

$$\acute{S}(P) \;=_{\mathrm{df}}\; \{\beta \mid \exists n \exists \alpha[\alpha \in P \;\&\; \beta = \mathfrak{s}^n(\hat{\alpha})]\}$$

PROPOSITION 2·5. *$\acute{S}(P)$ is a shift-closed analytic set such that every β in $\acute{S}(P)$ is strictly increasing and takes only values $\equiv 0 \pmod 4$.*

Remark 2·6. Since each of P and $\acute{S}(P)$ is recursively reducible to the other, they will be of the same Wadge degree; and if P is complete, $\acute{S}(P)$ will be too.

THEOREM 2·7. *Let P be any analytic subset of $\mathcal{N}$. Then there is a point $a_P \in \mathcal{N}$ with $\acute{S}(P) = A^1(a_P, \mathfrak{s})$.*

Remark 2·8. In such a case, $A^2(a_P, \mathfrak{s})$ will be empty, as $\acute{S}(P)$ contains only strictly monotonic functions, which can attack nothing.

A real a with $A^1(a, \mathfrak{s}) = \acute{S}(P)$

LEMMA 2·9. *Given P, we may find a closed subset $C \subseteq \mathcal{N} \times \mathcal{N}$ and a set R of pairs of finite sequences of natural numbers, which, if P is light-face analytic, may be chosen to be recursive, such that:*

(i) *if $(s,\, t) \in R$ then $\ell h(s) = \ell h(t)$ and s is a strictly increasing sequence of natural numbers each congruent to* $0 \pmod 4$.

(ii) *if $(s,\, t) \in R$ and $n < \ell h(s)$, then $(s \restriction n, t \restriction n) \in R$;*

(iii) *$C = \{(\beta,\, \gamma) \mid \forall n\, (\beta \restriction n, \gamma \restriction n) \in R\}$;*

(iv) *$\acute{S}(P) = \{\beta \mid \exists \gamma\, (\beta,\, \gamma) \in C\}$.*

Proof. By familiar representations of analytic sets, such as are discussed in the first two chapters of [**5**] or in [**3**, chapter III].

To prove Theorem 2·7 is easy if P is empty; henceforth we suppose that it is not. For each pair $(\beta,\, \gamma) \in C$, define the infinite sequence

$$\xi_{\beta,\gamma} = \big(\beta(0), \check{\gamma}(0), \beta(0), \beta(1), \check{\gamma}(1), \beta(0), \beta(1), \beta(2), \check{\gamma}(2), \ldots\big).$$

LEMMA 2·10. *If $\xi_{\beta,\gamma} \curvearrowright_{\mathfrak{s}} \eta$ then for some n, $\eta = \mathfrak{s}^n(\beta)$; conversely each $\mathfrak{s}^n(\beta)$ with $\beta \in \acute{S}(P)$ is attacked by each $\xi_{\beta,\gamma}$ with $(\beta,\, \gamma) \in C$.*

Proof. No value of $\check{\gamma}$ occurs more than once in $\xi_{\beta,\gamma}$, so cannot occur in η. Hence for each $k > 0$ there is $n(k)$ such that $\eta \restriction k = (\mathfrak{s}^{n(k)}(\beta)) \restriction k$; but $n(k)$ must be some constant n, as β, being in $\acute{S}(P)$, is strictly increasing, and thus $\eta = \mathfrak{s}^n(\beta)$, as required.

The truth of the converse is plain.

(2·11) As P is assumed not to be empty, R is an infinite set of pairs of finite sequences: list it, recursively if possible, as

$$(s_0,\, v_0), (s_1,\, v_1), (s_2,\, v_2), \ldots.$$

For each $i \in \mathbb{N}$, define the finite sequence w_i thus:

$$w_i \;=_{\mathrm{df}}\; \big(s_i(0), \check{v}_i(0), s_i(0), s_i(1), \check{v}_i(1), \ldots, s_i(0), s_i(1), \ldots s_i(\ell h(s_i) - 1), \check{v}_i(\ell h(s_i) - 1)\big).$$

Write m_n for the sequence $\langle 2n + 1 \rangle$ of length 1. We call such sequences *markers*.

We shall call numbers $\equiv 2 \pmod 4$ *witnesses*, as they help to define some $\check{\gamma}$ where γ attests the membership of some β of $\acute{S}(P)$.

Finally, define the infinite sequence

$$a =_{\mathrm{df}} w_0{}^\frown m_0{}^\frown w_1{}^\frown m_1{}^\frown w_2{}^\frown m_2{}^\frown \ldots$$

where $^\frown$ indicates the concatenation of sequences.

LEMMA 2·12. *If* $(\beta, \gamma) \in C$, $a \curvearrowright_{\mathfrak{s}} \xi_{\beta,\gamma}$.

LEMMA 2·13. *If* $a \curvearrowright_{\mathfrak{s}} \delta \curvearrowright_{\mathfrak{s}} \zeta$, *then* $\zeta \in \acute{S}(P)$.

Proof. No $q \equiv 1 \pmod 2$ occurs more than once in a, so δ contains no markers; therefore to each k there is an $i(k)$ such that $\delta \restriction k \sqsubseteq w_{i(k)}$.

Were δ to contain only finitely many witnesses then after some point it would be strictly increasing, (as each s_i is) and therefore could not attack ζ or anything else.

Hence δ contains infinitely many witnesses. But now the spacing between witnesses is extremely informative: if $\delta(i)$ and $\delta(i+j+1)$ are successive witnesses in δ, then the next witness will be $\delta(i+2j+2)$.

List all the witnesses occurring in δ in order as $p_0, p_1, \ldots, p_k \ldots$ Write u_{k+1} for the segment of δ strictly between the two successive witnesses p_k and p_{k+1}.

Choose $(s_{i_{k+1}}, t_{i_{k+1}})$ in R with u_{k+1} an initial segment of $s_{i_{k+1}}$ and $p_{k+1} = \check{t}_{i_{k+1}}(\ell h(u_{k+1})-1)$. Then u_{k+1} will be a proper initial segment of u_{k+2}, since both are initial segments of $s_{i_{k+2}}$, so there is a well-defined infinite sequence $\bigcup_{k\to\infty} u_{k+1}$: call it β.

Further, however we have made the above choice of $(s_{i_{k+1}}, t_{i_{k+1}})$, the preceding witness p_k will equal $\check{t}_{i_{k+1}}(\ell h(u_{k+1}) - 2)$. Hence the witnesses cohere to define a $\gamma \in \mathcal{N}$ such that for some lag ℓ, the witness $p_k = \check{\gamma}(\ell + k)$: the "missing" initial segment of γ, $\gamma \restriction \ell$, can be recovered from any witness in δ.

By Lemma 2·9 (ii), each $(\beta \restriction n, \gamma \restriction n) \in R$ and hence $(\beta, \gamma) \in C$.

We may now verify that $\delta = \mathfrak{s}^{\ell}(\xi_{\beta,\gamma})$. Since $\delta \curvearrowright_{\mathfrak{s}} \zeta$, Lemma 2·10 shows that ζ is a finite shift of β, and hence is in $\acute{S}(P)$ as $\acute{S}(P)$ is shift-closed.

Remark 2·14. In fact one can show that if $a \curvearrowright_{\mathfrak{s}} \delta$ and δ contains one witness it must contain infinitely many: if $\delta(j) = \hat{v}_i(k)$, say, k will be computable as one less than the number of distinct prime factors of $\delta(j)$, and hence $\delta(j+k+3)$ will be another witness.

Remark 2·15. By the discussion of 7·0 of *Delays*, the set $\{\alpha \mid \exists\rho\ \alpha \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho\}$, there called P, is a complete analytic set, and therefore not Borel. In this case, $\acute{S}(P)$ would also be complete analytic, and R, its enumeration, and the point a_P constructed above, may easily be arranged to be recursive.

The plan of attack

(3·0) Let R, P, M be three pairwise disjoint infinite subsets of ω. In *Delays* we showed how to assign to each node s of a countable tree T of finite sequences a point $x_s^T \in \mathcal{N}$, using $\mathsf{R} \cup \mathsf{P}$ as an alphabet. Using the additional alphabet M, we defined x_T which would attack each x_s^T.

Here we shall do something similar, but for each tree simultaneously. To avoid interference between the points placed at nodes in distinct trees, we shall, in a continuous fashion, assign different alphabets to distinct trees, so that any two alphabets have but finite intersection.

Trees will be coded by members τ of a set $\mathcal{T}$, to be defined, of infinite sequences of natural numbers; the nodes of, or rather the finite paths through, a tree will correspond under this coding to certain finite sequences of positive integers that we shall call τ*-sequences.* Then we shall define, in a certain space, a point ξ_s^τ for each $\tau \in \mathcal{T}$ and s a τ-sequence; but rather than define ξ^τ for each τ we shall then use M to define a recursive point b that attacks every ξ_s^τ. b is the point that will prove to be of score ω_1.

We must work not with the members τ of $\mathcal{T}$ but with their finite initial segments u. Thus for each such u and each u-sequence s, we shall define a finite sequence z_s^u, such that ξ_s^τ will be the limit of the sequences $z_s^{\tau\restriction k}$ as $k \longrightarrow \infty$. The point b will then be defined by concatenating all the z_s^u but using members of M to separate them.

We begin therefore by discussing those finite approximations to trees and paths through them. The intuition behind our definitions will become clearer when we turn to a discussion of infinite sequences.

Finite trees and paths

We write $\ell h(u)$ for the length of a finite sequence u.

Definition 3·1. $\mathcal{F} \;=_{\rm df}\; \{u \mid u$ a non-empty finite sequence $\big(u(1), u(2), \ldots, u(\ell h(u))\big)$ of natural numbers $u(i)$ with $0 \leqslant u(i) < i$ for $1 \leqslant i \leqslant \ell h(u)\}$.

Remark 3·2. Contrary to habitual practice among set theorists, the terms of u are indexed by $1, \ldots, \ell h(u)$ rather than $0, \ldots, \ell h(u) - 1$.

For $1 \leqslant k \leqslant \ell h(u)$ we write $u_{\leqslant k}$ for the sequence $\big(u(1), \ldots, u(k)\big)$; that will be an element of $\mathcal{F}$.

Definition 3·3. If $u = \big(u(1), u(2), \ldots, u(\ell h(u))\big) \in \mathcal{F}$, a *positive u-sequence* is a non-empty finite sequence $s = (p_1, \ldots, p_\ell)$ with $1 \leqslant p_1 < p_2 < \cdots < p_\ell \leqslant \ell h(u)$, so that $\ell = \ell h(s)$ and $p_\ell = \max s$; we further require that $u(p_1) = 0$, and for $1 \leqslant i < \ell h(s)$, $u(p_{i+1}) = p_i$.

The *u-sequences* are the positive u-sequences and the empty sequence, which we write as $\circledcirc$.

As above, we write $s_{\leqslant k}$ for the sequence $(p_1, \ldots, p_k)$, where $1 \leqslant k \leqslant \ell h(s)$; that too will be a positive u-sequence. Further, we interpret $s_{\leqslant 0}$ as the empty sequence, $\circledcirc$.

Example 3·4. If u is the sequence (0,0,2,1,0), the u-sequences are $\circledcirc$, (1), (2), (5), (1,4), and (2,3).

(3·5) We shall build our point in a space of infinite sequences of *symbols*, of which there will be three kinds, *recorders, predictors* and *markers.* Certain symbols will contain information that is either an element u of $\mathcal{F}$—such symbols will be called *recorders*, because they contain information about the recent past of the infinite sequence of symbols under consideration—or else a pair of finite sequences s, u where $u \in \mathcal{F}$ and s is a positive u-sequence—such symbols will be called *predictors* because they contain information about the near future of that infinite sequence. Nothing is required of the third kind of symbol, the *markers*, save that there be a countable infinity of them and that they be all distinct from each other and from all recorders and predictors.

It is extremely important that, from the point of view of the shift function that we shall apply, each symbol is a single object; and, to give visual emphasis to that point, we shall use square brackets [,] to encase each individual symbol, whereas we shall use pointed brackets $\langle, \rangle$, to encase finite or infinite sequences of symbols.

We shall associate to each recorder and each predictor two natural numbers, its *weight* and its *height.*

Definition 3·6. A *recorder* is an object $[u]$ where u is in $\mathcal{F}$. Its *weight* is 0 and its *height* is the length $\ell h(u)$ of u as a member of $\mathcal{F}$.

Definition 3·7. A *predictor* is an object $[s;u]$ where $u \in \mathcal{F}$ and s is a positive u-sequence. s will be called the *path* of the predictor $[s;u]$, and u its *tree*. The predictor's *weight* is the length of its path, and its *height* is the length of its tree.

Remark 3·8. Plainly the weight of $[s;u]$ is not greater than its height.

Definition 3·9. We say that s is *tight* in u, or that u *tightly contains* s, if s is a u-sequence and $\max s = \ell h(u)$. In the contrary case we shall use the words *loose* and *loosely*. We may indeed define the *looseness of* u *over* s as $\ell h(u) - \max s$.

(3·10) For each $u \in \mathcal{F}$ and each u-sequence s we shall define a finite sequence z^u_s of symbols. Our definition will proceed by a mode of induction that will also be used in proving our theorem, which we shall call *double induction*. To spell the method out in greater detail: we first consider the case $s = \circledcirc$. Then we suppose that $m \geqslant 1$ and that we have already treated all pairs u, s with s a u-sequence of length $< m$. On that supposition, we take an s of length m, and consider all $u \in \mathcal{F}$ for which s is a u-sequence, starting with those u for which $\ell h(u) = \max s$, and then progressively treating longer u; thus for given s we proceed by induction on the looseness of u over s.

In using double induction the following convention will be useful.

Definition 3·11. We write s' for the sequence s with its last element removed—so that if s is of length 1, $s' = \circledcirc$—and we write u' for u with its last element removed.

We proceed to our definition of z^u_s by double induction, and first treat the case of $s = \circledcirc$.

Definition 3·12. For $u \in \mathcal{F}$,

$$z^u_{\circledcirc} =_{\mathrm{df}} \big\langle [u_{\leqslant 1}], [u_{\leqslant 2}], \ldots, [u_{\leqslant \ell h(u)-1}], [u] \big\rangle.$$

Remark 3·13. The length of $z^u_{\circledcirc}$ equals that of u.

Example 3·14. $z^{(0,0,2,1,0)}_{\circledcirc} = \big\langle [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle$

Now for $u \in \mathcal{F}$ and s a positive u-sequence we shall define z^u_s.

Definition 3·15.

$$z^u_s =_{\mathrm{df}} \begin{cases} \big\langle [s;u] \big\rangle ^\frown z^u_{s'} & \text{if } \max s = \ell h(u); \\[2ex] z^{u'}_s {}^\frown \big\langle [s;u] \big\rangle ^\frown z^u_{s'} & \text{if } \max s < \ell h(u). \end{cases}$$

The first clause handles the case that u tightly contains s, and the second the cases when $\ell h(u)$ is strictly greater than $\max s$.

Remark 3·16. Note that $[s;u]$ occurs only once in z^u_s; we shall refer to it as the *peak* of z^u_s. It is the only symbol in z^u_s with sum of weight and height equal to $\ell h(s) + \ell h(u)$.

We give several examples to illustrate that definition.

Example 3·17. If s is of length 1, then $z^u_s = \big\langle [s;u] \big\rangle ^\frown z^u_{\circledcirc}$ if $\max s = \ell h(u)$ and $z^u_s = z^{u'}_s {}^\frown \big\langle [s;u] \big\rangle ^\frown z^u_{\circledcirc}$ otherwise.

Example 3·18. If u is the sequence (0,0,2,1,0), then $z^u_{(5)}$ is

$$\big\langle [(5);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle,$$

a sequence of six symbols, whereas $z^u_{(2)}$ is

$$\begin{aligned} \big\langle &[(2);(0,0)], [(0)], [(0,0)], [(2);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], \\ &[(2);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], \\ &[(2);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle, \end{aligned}$$

which has eighteen, of which the heights, in order, are 2, 1, 2; 3, 1, 2, 3; 4, 1, 2, 3, 4; 5, 1, 2, 3, 4, 5.

We compute $z^u_{(1)}$, $z^u_{(1,4)}$ and $z^u_{(2,3)}$ in greater detail:

$$\begin{aligned}
z^{(0)}_{(1)} &= \big\langle [(1);(0)], [(0)] \big\rangle;\\
z^{(0,0)}_{(1)} &= \big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)] \big\rangle;\\
z^{(0,0,2)}_{(1)} &= \big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)] \big\rangle;\\
z^{(0,0,2,1)}_{(1)} &= \big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)],\\
&\qquad [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)] \big\rangle;\\
z^{(0,0,2,1,0)}_{(1)} &= \big\langle [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)], [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)],\\
&\qquad [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)],\\
&\qquad [(1);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle.\\
z^{(0,0,2,1)}_{(1,4)} &= \big\langle [(1,4);(0,0,2,1)] \big\rangle ^\frown z^{(0,0,2,1)}_{(1)}\\
&= \big\langle [(1,4);(0,0,2,1)], [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)],\\
&\qquad [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)] \big\rangle;\\
z^{(0,0,2,1,0)}_{(1,4)} &= z^{(0,0,2,1)}_{(1,4)} {}^\frown \big\langle [(1,4);(0,0,2,1,0)] \big\rangle ^\frown z^{(0,0,2,1,0)}_{(1)}\\
&= \big\langle [(1,4);(0,0,2,1)], [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)],\\
&\qquad [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)],\\
&\quad [(1,4);(0,0,2,1,0)], [(1);(0)], [(0)], [(1);(0,0)], [(0)], [(0,0)],\\
&\qquad [(1);(0,0,2)], [(0)], [(0,0)], [(0,0,2)], [(1);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)],\\
&\qquad [(1);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \big\rangle.
\end{aligned}$$

$$
\begin{aligned}
z_{(2,3)}^{(0,0,2)} &= \langle [(2,3);(0,0,2)] \rangle^\frown z_{(2)}^{(0,0,2)};\\
z_{(2,3)}^{(0,0,2,1)} &= z_{(2,3)}^{(0,0,2)}{}^\frown \langle [(2,3);(0,0,2,1)] \rangle^\frown z_{(2)}^{(0,0,2,1)};\\
z_{(2,3)}^{(0,0,2,1,0)} &= z_{(2,3)}^{(0,0,2,1)}{}^\frown \langle [(2,3);(0,0,2,1,0)] \rangle^\frown z_{(2)}^{(0,0,2,1,0)}\\
&= z_{(2,3)}^{(0,0,2)}{}^\frown \langle [(2,3);(0,0,2,1)] \rangle^\frown z_{(2)}^{(0,0,2,1)}{}^\frown \langle [(2,3);(0,0,2,1,0)] \rangle^\frown z_{(2)}^{(0,0,2,1,0)}\\
&= \langle [(2,3);(0,0,2)],\\
&\qquad\quad [(2);(0,0)], [(0)], [(0,0)],\\
&\qquad\quad [(2);(0,0,2)], [(0)], [(0,0)], [(0,0,2)],\\
&\qquad [(2,3);(0,0,2,1)],\\
&\qquad\quad [(2);(0,0)], [(0)], [(0,0)],\\
&\qquad\quad [(2);(0,0,2)], [(0)], [(0,0)], [(0,0,2)],\\
&\qquad\quad [(2);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)],\\
&\qquad [(2,3);(0,0,2,1,0)],\\
&\qquad\quad [(2);(0,0)], [(0)], [(0,0)],\\
&\qquad\quad [(2);(0,0,2)], [(0)], [(0,0)], [(0,0,2)],\\
&\qquad\quad [(2);(0,0,2,1)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)],\\
&\qquad\quad [(2);(0,0,2,1,0)], [(0)], [(0,0)], [(0,0,2)], [(0,0,2,1)], [(0,0,2,1,0)] \rangle.
\end{aligned}
$$

Example 3·19. Suppose that $3 + \max t = \ell h(v)$. Let $v_i = v_{\leqslant i+\max t}$, so that $v_0 = v_{\leqslant \max t}$ and $v_3 = v$. Then z_t^v is

$$\langle [t;v_0] \rangle^\frown z_{t'}^{v_0}{}^\frown \langle [t;v_1] \rangle^\frown z_{t'}^{v_1}{}^\frown \langle [t;v_2] \rangle^\frown z_{t'}^{v_2}{}^\frown \langle [t;v] \rangle^\frown z_{t'}^{v},$$

which has precisely the four predictors shown of weight equal to the length of t; all other predictors in z_t^v will be of lesser weight.

Here is a first example of proof by double induction:

PROPOSITION 3·20. *If s is not $\circledcirc$, then the first symbol of z_s^u is the predictor* $[s; u_{\leqslant \max s}]$.

Proof. If u tightly contains s, $z_s^u = \langle [s;u] \rangle^\frown z_{s'}^u$ of which the first symbol is $[s;u]$, which equals $[s; u_{\leqslant \max s}]$. Otherwise $z_s^u = z_s^{u'}{}^\frown \langle [s;u] \rangle^\frown z_{s'}^u$, of which the first symbol is that of $z_s^{u'}$, which, by the induction hypothesis, is the predictor $[s; u'_{\leqslant \max s}]$; but that in the context equals $[s; u_{\leqslant \max s}]$.

Notation for finite sequences

We shall follow *Delays* in using the following notation for the extension relation between finite sequences of arbitrary objects.

Definition 3·21. $t \preccurlyeq s \iff_{\mathrm{df}}$ t is an extension of s; $t \prec s \iff_{\mathrm{df}}$ t is an proper extension of s; $s \succcurlyeq t \iff_{\mathrm{df}}$ s is an initial segment of t; $s \succ t \iff_{\mathrm{df}}$ s is a proper initial segment of t.

Remark 3·22. Thus $s \succcurlyeq t \iff t \preccurlyeq s$, and so on. $\circledcirc$ has no proper initial segments, but is itself a proper initial segment of every finite sequence of positive length. Note that longer sequences are lower in this ordering.

Definition 3·23. We shall say that two finite sequences s and t *cohere* if either $s \succcurlyeq t$ or $t \succcurlyeq s$.

Properties of finite sequences

PROPOSITION 3·24. *Let u and v be members of $\mathcal{F}$, and let t be both an u-sequence and a v-sequence.*

(i) $\ell h(u) = \ell h(z_\circledcirc^u)$;

(ii) *for* $\ell \leqslant \ell h(v)$, $z_\circledcirc^v \restriction \ell = z_\circledcirc^{v \restriction \ell}$;

(iii) $v \prec u \implies z_t^v \prec z_t^u$;

(iv) $z_t^v = z_t^u \implies v = u$;

(v) $z_t^v \prec z_t^u \implies v \prec u$.

Proof of 3·24 (iii). If $t = \circledcirc$, use (ii): otherwise use an earlier instance to note that $z_t^v \prec z_t^{v'} \preccurlyeq z_t^u$.

Proof of 3·24 (iv). Compare peaks.

Proof of 3·24 (v). The peak of z_t^v cannot be in z_t^u, for otherwise $u = v$; whence $z_t^u \succcurlyeq z_t^{v'}$, giving, inductively, $v' \preccurlyeq u$.

Definition 3·25. An *m-predictor* is a predictor of weight exactly m. An *m-stretch* is a finite sequence of symbols all of weight at most m.

LEMMA 3·26. *Let $u \in \mathcal{F}$, s a u-sequence of weight $> m$. Let $x \sqsubseteq z_s^u$ be an m-stretch.*

(i) $x \sqsubseteq z_{s'}^u$;

(ii) *in fact* $x \sqsubseteq z_{s_{\leqslant m}}^u$.

Proof of 3·26 (i). Its weight forbids the peak of z_s^u to lie in x.

Case 1: s is tight in u. Then $z_s^u = \langle [s;u] \rangle ^\frown z_{s'}^u$, whence $x \sqsubseteq z_{s'}^u$.

Case 2: otherwise. Then $z_s^u = z_s^{u'} {}^\frown \langle [s;u] \rangle ^\frown z_{s'}^u$, so either $x \sqsubseteq z_s^{u'}$ or $x \sqsubseteq z_{s'}^u$; if the second alternative is false, we may iterate the first, progressively shortening u till it does tightly contain s, and then apply Case 1.

Proof of 3·26 (ii). By iterating Lemma 3·26 (i), progressively shortening s.

Indeed we can sharpen that result:

PROPOSITION 3·27. *Let x be an m-stretch with all symbols of height at most h. Suppose that $x \sqsubseteq z_s^u$. Then $x \sqsubseteq z_{s_{\leqslant m}}^{u_{\leqslant h}}$.*

Proof. For fixed x by double induction on s and u. If the peak of z_s^u occurs in x, then both the height and weight of x equal those of z_s^u, and then the proposition is trivially true. Otherwise $x \sqsubseteq z_s^{u'}$ or $x \sqsubseteq z_{s'}^u$; in the first case the height is less and in the second the weight. In either case we have a reduction to an earlier instance of the induction.

LEMMA 3·28. *The recorders in z_s^u are those in $z_\circledcirc^u$: namely non-empty initial segments of u. Hence any two recorders in z_s^u cohere.*

Proof. By applying Proposition 3·27 to 0-stretches of length 1.

LEMMA 3·29. *If $s \succcurlyeq t$ and t is a u-sequence, then z_s^u is a final segment of z_t^u; if $s \succ t$, that final segment is immediately preceded by the predictor $[s^+;u]$, where $s^+ = t_{\leqslant \ell h(s)+1}$.*

Proof. Write $t_0 = t$, and progressively write $t_{k+1} = t'_k$ till we reach $t_n = s$. If $n = 0$ the Lemma is trivial; if $n > 0$, then we remark that for each k, $z_{t_k}^u$ ends in $z_{t_{k+1}}^u$ which is preceded by $[t_k;u]$; finally note that $t_{n-1} = t_{\leqslant \ell h(s)+1}$.

LEMMA 3·30. *if $u \succcurlyeq v$ and s is a u-sequence, then $z_s^u \succcurlyeq z_s^v$; if $u \succ v$, the term in z_s^v after that occurrence of z_s^u is $[s;u^+]$. where $u^+ = v_{\leqslant \ell h(u)+1}$.*

Proof. The first part is Proposition 3·24 (iii) rephrased; the second part holds if $v' = u$, and stays true for longer v by an easy induction, as then $u \succ v' \succ v$.

LEMMA 3·31. *If $[s;u]$ occurs in z_t^v then $s \succcurlyeq t$ and $u \succcurlyeq v$.*

Proof. By a double induction on t and v. The lemma is true if $[s;u] = [t;v]$. Otherwise $[s;u]$ occurs in $z_{t'}^v$ or, provided t is loose in v, in $z_t^{v'}$; in either case we have a reduction to an earlier instance of the induction, to which we then link either the fact that $t' \succ t$ or that $v' \succ v$.

LEMMA 3·32. *An occurrence of $[s;u]$ in z_t^v is followed by the whole of $z_{s'}^u$.*

Proof. By a similarly structured induction on t and v.

LEMMA 3·33. *In any z_s^u the immediate successor of an m-predictor is a symbol of weight $m-1$.*

Proof. Immediate from the definition if $m = 1$; by Proposition 3·20 otherwise.

LEMMA 3·34. *If s is of length $m+1$, $\langle[s;u]\rangle^\frown x$ is a final segment of z_s^w and x is an m-stretch, then $u = w$ and $x = z_{s'}^u$.*

Proof. $[s;w]$ is the last symbol of weight $m+1$ in z_s^w.

PROPOSITION 3·35. *If s is of length $m+1$, x is an m-stretch, and $y =_{\rm df} \langle[s;u]\rangle^\frown x^\frown\langle[s;v]\rangle \sqsubseteq z_r^w$, then $u = v'$ and $x = z_{s'}^u$.*

Proof by double induction. By Proposition 3·27, we can suppose $r = s$. If $v \neq w$, we have $z_s^w = z_s^{w'}{}^\frown\langle[s;w]\rangle^\frown z_{s'}^w$ and therefore $y \sqsubseteq z_s^{w'}$; thus we may reduce the length of w until $w = v$.

So our proposition is now reduced to the case that $y \sqsubseteq z_s^v$. We then have

$$\langle[s;u]\rangle^\frown x^\frown\langle[s;v]\rangle \sqsubseteq z_s^{v'}{}^\frown\langle[s;v]\rangle^\frown z_{s'}^v;$$

since $[s;v]$ occurs in neither $z_s^{v'}$ nor in $z_{s'}^v$, we may be sure that the last symbol of y occurs as the peak of z_s^v; but then $\langle[s;u]\rangle^\frown x$ forms a final segment of $z_s^{v'}$, so we may apply Lemma 3·34 to infer that $u = v'$ and $x = z_{s'}^u$.

COROLLARY 3·36. *If $y = \langle[s;u_1]\rangle^\frown x_1{}^\frown\langle[s;u_2]\rangle^\frown x_2{}^\frown\langle[s;u_3]\rangle \sqsubseteq z_r^w$, where s is of length $m+1$ and both x_1 and x_2 are m-stretches, then $x_1 \succ x_2$, and $\ell h(u_2) = \ell h(u_1) + 1$.*

Proof. In the circumstances, $x_1 = z_{s'}^{u_1}$, $x_2 = z_{s'}^{u_2}$, and $u_1 = (u_2)'$.

LEMMA 3·37. *If s is of length $m+1$, x is an m-stretch, and $x^\frown\langle[s;v]\rangle \sqsubseteq z_t^w$, then x is a final segment of $z_s^{v'}$.*

Proof. The hypotheses imply, by Proposition 3·27, that $x^\frown\langle[s;v]\rangle \sqsubseteq z_s^v$, in which the only occurrence of $[s;v]$ is the peak; but then x must be a final segment of the preceding sequence, which is $z_s^{v'}$.

LEMMA 3·38. *If the recorder $[e]$, of height at least 2, occurs in z_s^u, its predecessor is $[e_{\leqslant \ell h(e)-1}]$; if of height 1, its predecessor, if any, will be a predictor of weight 1.*

PROPOSITION 3·39. *If $z_s^u(i)$ and $z_s^u(i+1)$ are both recorders then $\ell h(z_s^u(i+1)) = 1 + \ell h(z_s^u(i))$.*

Remark 3·40. The unique longest m-stretch in z_s^u is at the end, namely $z_{s_{\leqslant m}}^u$: for if s is of weight m, z_s^u is itself an m-stretch; and if s is of greater weight, the m-stretches in z_s^u are those of $z_{s'}^u$ and, provided s is loose in u, of $z_s^{u'}$. By induction, the unique longest of those are $z_{s_{\leqslant m}}^u$ and $z_{s_{\leqslant m}}^{u'}$, of which two the first is in any case strictly longer.

PROPOSITION 3·41. *Suppose that $x =_{\rm df} \langle[s;u]\rangle^\frown z_{s'}^u \sqsubseteq z_r^w$ but is not a final segment thereof. Then the first symbol after the segment x of z_r^w is of the form $[t;v]$ where $v' = u$ and $t \preccurlyeq s$, and if $t \prec s$ there will be a later occurrence in z_r^w of a symbol of weight that of s.*

Remark 3·42. $\langle[s;u]\rangle^\frown z_{s'}^u$ is a final segment of z_s^u, properly so if and only if s is loose in u.

Towards the proof of Proposition 3·41, we first prove a Lemma to cover the case $s = r$.

LEMMA 3·43. *$x =_{\rm df} \langle[s;u]\rangle^\frown z_{s'}^u$ is a final segment of z_s^w if and only if $u = w$.*

Proof. One way is covered by Remark 3·42. For the other, since $z_s^w = z_s^{w'}{}^\frown\langle[s;w]\rangle^\frown z_{s'}^w$, the peak of z_s^w is its last symbol of weight $\ell h(s)$ and therefore if x is a final segment of z_s^w, the first symbol of x must be that peak, whence $z_{s'}^u = z_{s'}^w$, whence $u = w$.

Proof of Proposition 3·41. We consider s and u to be fixed and do a double induction on r and w.

As always, we have

$$z_r^w = z_r^{w'}{}^\frown\langle[r;w]\rangle^\frown z_{r'}^w$$

The hypotheses imply that $r \preccurlyeq s$ and, by Lemma 3·43, that $w \prec u$; hence the peak of z_r^w cannot lie in x, and therefore either $x \sqsubseteq z_{r'}^w$ or $x \sqsubseteq z_r^{w'}$.

If $x \sqsubseteq z_{r'}^w$, then x will not be a final segment of $z_{r'}^w$, and so the induction will apply.

If $x \sqsubseteq z_r^{w'}$, either $w' \prec u$, whence by Lemma 3·43 x is not final in $z_r^{w'}$, and the induction will again apply; or $w' = u$, x is final—again by Lemma 3·43—in $z_r^{w'}$ and the next symbol is $[r;w]$, which is of the desired form $[t;v]$ with $v' = u$ and $t \preccurlyeq s$.

The final clause follows from Lemma 3·33.

PROPOSITION 3·44. *In any z^u_s, if the same symbol, of weight m, occurs twice, then between the two occurrences there must be an occurrence of a symbol of weight $m+1$.*

Proof by double induction. The indicated symbol, that which repeats, cannot be the peak of z^u_s, which occurs only once there.

If s is tight in u, the two occurrences must both be in $z^u_{s'}$, and we have reduced to an earlier case.

Otherwise $z^u_s = z^{u'}_s \frown \langle [s;u] \rangle \frown z^u_{s'}$, and there are three possibilities: both occurrences are before the peak, when both lie in $z^{u'}_s$; both lie after, and therefore both lie in $z^u_{s'}$—both times we have a reduction to an earlier case—or one lies before the peak and the other after; but then the proposition is proved, for the peak is of weight greater than m, and, if of weight $> m+1$, will by Lemma 3·33 immediately be followed by symbols of weights declining by 1 at each step, thus reaching a symbol of weight $m+1$ before the second occurrence of the indicated symbol.

4. *Introducing infinite sequences*

We have introduced two of our three kinds of symbol. For the third, the *markers*, we take infinitely many objects $[\mathsf{m}_0], [\mathsf{m}_1], \ldots$ distinct from each other and from all recorders and predictors.

We define $\mathcal{Y}$ to be the space of all sequences of length ω of symbols. Here we return to normal set-theoretic convention by considering the domain of such sequences to be $\omega = \{0,1,2,\ldots\}$.

On $\mathcal{Y}$ we may define the shift function, which we again denote by $\mathfrak{s}$: $\mathfrak{s}(\zeta)(n) = \zeta(n+1)$ for $n \geqslant 0$.

As in section 4 of *Delays* we write $\zeta \rhd \xi$, read ζ *is near to* ξ, if $\zeta = \mathfrak{s}^n(\xi)$ for some $n \geqslant 0$.

Definition 4·0. The *weight* of a point ζ of $\mathcal{Y}$ is the supremum of the weight of its predictors: thus either a natural number or ∞. The *height* of a point $\zeta \in \mathcal{Y}$ is the supremum of the height of its recorders and predictors: again either a natural number or ∞.

Introducing the real b

At last we are in a position to define our point b, which will lie in the space $\mathcal{Y}$.

Definition 4·1. Enumerate all sequences z^u_s where $u \in \mathcal{F}$ and s is a u-sequence, in some recursive fashion as z_i $(i = 0,1,\ldots)$.

Define

$$b =_{\mathrm{df}} z_0 \frown \langle [\mathsf{m}_0] \rangle \frown z_1 \frown \langle [\mathsf{m}_1] \rangle \frown \ldots$$

We now work towards our principal result:

THEOREM 4·2. $\theta(b, \mathfrak{s}) = \omega_1$.

To classify the points of $\mathcal{Y}$ attacked by b, we shall use the infinite trees to which the members of $\mathcal{F}$ are codes of finite approximations.

Introducing infinite trees

Definition 4·3. $\mathcal{T} =_{\mathrm{df}} \{\tau : \mathbb{N}^+ \longrightarrow \mathbb{N} \mid \text{ for all } n \geqslant 1,\ 0 \leqslant \tau(n) < n\}$

Remark 4·4. With the product topology of discrete finite spaces, $\mathcal{T}$ is a compact space.

Remark 4·5. If one regards $\mathcal{F}$ as a tree, $\mathcal{T}$ is the set of all infinite paths through it.

Definition 4·6. For $\tau \in \mathcal{T}$, a (*positive*) τ*-sequence* is a (non-empty) finite sequence of positive integers $p_1 < \cdots < p_k$ with $\tau(p_1) = 0$ and $\tau(p_{n+1}) = p_n$ for each $1 \leqslant n < k$. Thus $\circledcirc$ is a τ-sequence. A τ*-path* is an infinite sequence $\pi = (p_1, p_2, \ldots)$ with $\tau(p_1) = 0$ and $\tau(p_{n+1}) = p_n$ for each $n \geqslant 1$. For such π we write $\pi_{\leqslant k}$ for its initial segment $(p_1, p_2, \ldots, p_k)$, where $k \geqslant 1$.

We speak of τ as *well-founded* if there are no τ-paths: *ill-founded* if there are.

Remark 4·7. We may regard each $\tau \in \mathcal{T}$ as coding a tree, of which the top point is 0 and $m <_\tau n$ if m is not 0 and for some $\ell > 0$, $\tau^\ell(m) = n$.

Remark 4·8. Every countably infinite tree T of finite sequences under end-extension is coded by some $\tau \in \mathcal{T}$. To see that, partition ω into infinitely many infinite sets X_i. List the members of T as $v_0, v_1, v_2, \ldots$

We define a *first assignment* λ of natural numbers to members of T by induction on the length of each member as a finite sequence.

Assign 0 to the top point $\circledcirc$ of T. Once a natural number $\lambda(v_i)$ has been assigned to v_i, assign distinct members of $X_i \setminus \{m \mid m \leqslant \lambda(v_i)\}$ to the immediate extensions of v_i. Let $\mu : \Im(\lambda) \cong \omega$ be the order-preserving bijection of the set of all natural numbers used in the first assignment λ, so that $\mu \circ \lambda$ is a bijection between T and ω, which is the *final assignment*; let χ be its inverse.

Now set $\tau(n)$ to be the m such that $\chi(m) = \chi(n)'$. Then $\tau \in \mathcal{T}$, and $(\omega, <_\tau) \cong (T, \prec)$.

Properties of infinite sequences attacked by b

LEMMA 4·9. *If the recorder $[e]$, of height at least 2, occurs in some ζ attacked by b, its predecessor in ζ is $[e_{\leqslant \ell h(e)-1}]$; if of height 1, its predecessor, if any, in ζ will be a predictor of weight 1.*

Proof. By Lemma 3·38.

PROPOSITION 4·10. *If $\zeta(i)$ is a recorder then $\zeta(i+1)$, if a recorder, is of height one more than $\zeta(i)$.*

LEMMA 4·11. *If $b \curvearrowright_{\mathbf{s}} \xi$, ξ contains no markers: hence to each ℓ there are u and s with $\xi \restriction \ell \sqsubseteq z^u_s$.*

Proof. No marker occurs twice in b.

LEMMA 4·12. *Any two recorders, d and e, in ξ cohere.*

Proof. Pick ℓ with both d and e occurring in $\xi \restriction \ell$, and let $\xi \restriction \ell \sqsubseteq z^v_t$. Then by Lemma 3·28 both d and e are initial segments of v.

LEMMA 4·13. *If $b \curvearrowright \zeta$ and an m-predictor occurs in ζ, then m-predictors occur infinitely often in ζ.*

Proof. By Lemma 3·32 and Proposition 3·41.

PROPOSITION 4·14. *If $b \curvearrowright_{\mathbf{s}} \zeta$ then the height of ζ is ∞.*

Proof by cases, according to the weight of ζ. If ζ is of weight 0, then we use Proposition 4·10.

If on the other hand ζ is of positive finite weight, m, we consider the sequence of m-predictors in ζ. By Proposition 3·41, their height increases by one each time. Hence the $z^u_{\circledcirc}$'s that ζ contains are of unbounded length.

Finally, if ζ is of infinite weight, then by Remark 3·8 it must also be of infinite height. Hence it contains recorders of every height.

(4·15) Thus if $b \curvearrowright_{\mathbf{s}} \zeta$, ζ has recorders of unbounded height; they cohere to define a tree, which we shall call τ_ζ, in $\mathcal{T}$. This tree is uniquely determined by ζ; by the coherence property, Lemma 4·12, no $u \in \mathcal{F}$ other than the initial segments of τ_ζ may occur in ζ.

Points of finite weight attacked by b

We proceed to give an exact description of the points of finite weight attacked by b.

Definition 4·16. For $\tau \in \mathcal{T}$ and s a τ-sequence, set

$$\xi^\tau_s \;=_{\mathrm{df}} \bigcup_{k \geqslant \max s} z^{\tau \restriction k}_s$$

which will be a member of our symbol space $\mathcal{Y}$.

Example 4·17. $\xi^\tau_{\circledcirc} = \big\langle [\tau_{\leqslant 1}], [\tau_{\leqslant 2}], [\tau_{\leqslant 3}], \ldots, [\tau_{\leqslant k-1}], [\tau_{\leqslant k}], [\tau_{\leqslant k+1}], \ldots \big\rangle$, which has no predictors.

Example 4·18. Suppose that s is a positive τ-sequence with $\max s = 5$. Then

$$\xi^\tau_s = \big\langle [s; \tau_{\leqslant 5}] \big\rangle^\frown z^{\tau_{\leqslant 5}}_{s'}{}^\frown \big\langle [s; \tau_{\leqslant 6}] \big\rangle^\frown z^{\tau_{\leqslant 6}}_{s'}{}^\frown \big\langle [s; \tau_{\leqslant 7}] \big\rangle^\frown z^{\tau_{\leqslant 7}}_{s'}{}^\frown \ldots {}^\frown \big\langle [s; \tau_{\leqslant k}] \big\rangle^\frown z^{\tau_{\leqslant k}}_{s'}{}^\frown \ldots$$

which has infinitely many predictors of weight 5 but none of weight 6 or more.

Remark 4·19. The tree defined by ξ^τ_s equals τ.

Points of weight nought attacked by b

PROPOSITION 4·20. *For each $\tau \in \mathcal{T}$, $b \curvearrowright_{\mathfrak{s}} \xi^{\tau}_{\circledcirc}$; each $\xi^{\tau}_{\circledcirc}$ is of weight nought; no γ near $\xi^{\tau}_{\circledcirc}$ attacks itself.*

Proof. The first part holds since each $z^{u}_{\circledcirc}$ occurs infinitely often as a segment of b; the second is plain; and the third holds because no recorder occurs twice in any $\xi^{\tau}_{\circledcirc}$.

PROPOSITION 4·21. *If $b \curvearrowright_{\mathfrak{s}} \zeta$ and ζ is of weight 0, then $\zeta \triangleright \xi^{\tau_\zeta}_{\circledcirc}$.*

Proof. By Proposition 4·10, if $\zeta(0)$ is of height k then $\zeta = \mathfrak{s}^{k-1}(\xi^{\tau_\zeta}_{\circledcirc})$.

Points of positive finite weight attacked by b

PROPOSITION 4·22. *For each positive τ-sequence s, $b \curvearrowright_{\mathfrak{s}} \xi^{\tau}_{s}$; each ξ^{τ}_{s} is of finite weight equal to $\ell h(s)$; and no γ near ξ^{τ}_{s} attacks itself.*

Proof of the last part. No predictor of weight $\ell h(s)$ occurs twice in ξ^{τ}_{s}.

LEMMA 4·23. *If the weight of ζ, attacked by b, is bounded, let m be the largest weight of a predictor occurring in ζ. Then:*

(i) *ζ has infinitely many predictors of weight m;*

(ii) *there is a unique sequence s_ζ of length m such that every predictor of weight m occurring in ζ is of the form $[s_\zeta; v]$ for some $v \in \mathcal{F}$ with v an initial segment of τ_ζ and s_ζ a v-sequence;*

(iii) *to each ℓ there are $u \succ \tau_\zeta$ and $t \succcurlyeq s_\zeta$ with $\zeta \restriction \ell \sqsubseteq z^{u}_{t}$ and the two stretches $\zeta \restriction \ell$ and z^{u}_{t} having the same height and weight.*

Proof. The first part is just Lemma 4·13. The second part is a consequence of the principle of coherence. The third follows from Proposition 3·27.

PROPOSITION 4·24. *If $b \curvearrowright_{\mathfrak{s}} \zeta$ and ζ is of finite weight $m > 0$, then there is a unique τ_ζ-sequence s_ζ, of length m, such that $\zeta \triangleright \xi^{\tau_\zeta}_{s_\zeta}$.*

Proof. By comparing Lemma 4·23 with Example 4·18; in each case a segment $\big\langle [s;u] \big\rangle ^\frown z^{u}_{s'}$ is promptly followed by a segment $\big\langle [s;v] \big\rangle ^\frown z^{v}_{s'}$ where $v' = u$ and $s = s_\zeta$. The "missing" initial segment determines the shift required.

LEMMA 4·25. *If t and s are τ-sequences with $s = t'$, then $\xi^{\tau}_{t} \curvearrowright_{\mathfrak{s}} \xi^{\tau}_{s}$.*

Proof. By examination of Example 4·18.

PROPOSITION 4·26. *If t and s are τ-sequences with $t \prec s$, then $\xi^{\tau}_{t} \curvearrowright_{\mathfrak{s}} \xi^{\tau}_{s}$.*

Points at the end of a path

Before discussing the points of infinite weight attacked by b it will be helpful to review some material from section 3 of *Delays*.

We showed there, in the general context of a continuous map f of a Polish space $\mathcal{X}$ into itself, that if we have an infinite sequence of points b_i, with $b_0 \curvearrowleft_f b_1 \curvearrowleft_f b_2 \ldots \curvearrowleft_f b$, then we can choose integers n_i, (increasing if we wish), such that putting $y_i = f^{n_i}(b_i)$, the y_i form a Cauchy sequence converging to a point y with $b \curvearrowright_f y \curvearrowright_f y \curvearrowright_f b_i$ for each i, for in these circumstances $f^n(b_j) \curvearrowright b_i$ for $j > i$ and arbitrary n.

That lends interest to the following definition:

Definition 4·27. Let $b_0 \curvearrowleft_f b_1 \curvearrowleft_f b_2 \ldots$ be an infinite path descending in the relation $\curvearrowright_f$. We say that a point y *lies at the end of the path* if it satisfies two conditions:

(i) there are numbers n_i such that $y = \lim_{i\to\infty} f^{n_i}(b_i)$;

(ii) for each i, $y \curvearrowright_f b_i$.

PROPOSITION 4·28. *If both y and z are at the end of the same path, then $y \curvearrowright_f z \curvearrowright_f y$; in particular all points at the end of a given path are recurrent and attack each other.*

Proof. True because z attacks each b_i, hence attacks each $f^{n_i}(b_i)$; hence attacks y; and the situation is symmetric.

Remark 4·29. When, as here, $\mathcal{X} = \mathcal{Y}$ and $f = \mathfrak{s}$, the first condition will follow if one proves that to each ℓ there is a large i and an n_i with $y \restriction \ell \sqsubseteq \mathfrak{s}^{n_i}(b_i)$.

Points of infinite weight attacked by b

Suppose that ζ, attacked by b, is of infinite weight. We know that if $[s;u]$ and $[t;v]$ occur in ζ then as they both lie in some z_r^w, u and v cohere, both being initial segments of w, and s and t cohere, both being initial segments of r. The union of the trees of the predictors in ζ will be the tree τ_ζ. The union of the paths of the predictors in ζ will be a τ_ζ-path that we shall call π_ζ. π_ζ is infinitely long because ζ is of infinite weight; hence τ_ζ is ill-founded.

Denote by s_k the τ_ζ-sequence $(\pi_\zeta)_{\leqslant k}$ and by γ_k the point $\xi_{s_k}^{\tau_\zeta}$ of $\mathcal{Y}$.

Plainly $\gamma_{k+1} \curvearrowright_{\mathfrak{s}} \gamma_k$ for each k. We wish to show that ζ lies at the end of the path $\gamma_0 \curvearrowleft_{\mathfrak{s}} \gamma_1 \curvearrowleft_{\mathfrak{s}} \ldots$ There are two things to be verified: that ζ is the limit of well-chosen finite shifts of the γ_k's, and that ζ attacks each γ_k.

First, given $\ell \in \omega$, let m be the greatest weight of any symbol occurring in the initial segment $\zeta \restriction \ell$, so that segment is an m-stretch. Then by Proposition 3·27 there are w and s such that $\zeta \restriction \ell \sqsubseteq z_s^w$ with w an initial segment of τ_ζ and s an initial segment of π_ζ, and therefore

$$\zeta \restriction \ell \sqsubseteq z_{s_{\leqslant m}}^w \sqsubseteq \gamma_m.$$

Thus $\zeta \restriction \ell$ will be an initial segment of an appropriate shift of γ_m.

Secondly, for given k and ℓ pick initial segments w and s of τ_ζ and π_ζ so that $\gamma_k \restriction \ell \sqsubseteq z_s^w$. Let $[r;u]$ be a predictor occurring as late in ζ as desired and of weight strictly exceeding the height of w. Then $u \prec w$ and $r \prec s$. Therefore

$$\gamma_k \restriction \ell \sqsubseteq z_s^w \sqsubseteq z_{r'}^u \sqsubseteq \zeta,$$

since $z_{r'}^u$ occurs as a segment of ζ immediately after the given occurrence of $[r;u]$.

Thus, remembering Proposition 4·28, we have proved:

Proposition 4·30. *If $b \curvearrowright_{\mathfrak{s}} \zeta$ and ζ is of infinite weight, then there are unique τ_ζ and π_ζ defined by ζ; τ_ζ is ill-founded, and ζ lies at the end of the τ_ζ-path π_ζ and is therefore recurrent.*

So we have shown that all points of infinite weight attacked by b are recurrent. We now prove the converse.

Proposition 4·31. *If $b \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho$, then ρ contains recorders and also contains predictors of every positive weight.*

Proof. ρ has no markers; hence for each ℓ there are u and s such that $\rho \restriction \ell \sqsubseteq z_s^u$. The immediate successor of a predictor of weight $m > 1$ will be a predictor of weight $m-1$; the immediate successor of a predictor of weight 1 will be a recorder. Hence ρ must contain recorders.

Since ρ is recurrent, any symbol in it recurs infinitely often. We complete the proof by remarking, following Proposition 3·44, that between two occurrences of the same recorder, there must occur a predictor of weight one; and between two occurrences of the same predictor of weight m there must occur a predictor of weight $m+1$.

Putting those two propositions together, we have this characterisation:

Proposition 4·32. *If $b \curvearrowright_{\mathfrak{s}} \rho$, ρ is recurrent if and only if it is of infinite weight.*

Example 4·33. We illustrate the way in which recurrent points arise. Suppose that $\tau \in \mathcal{T}$ is ill-founded, and that π is an infinite τ-path. We choose strictly increasing integers n_k such that $\pi_{\leqslant k}$ is a $\tau_{\leqslant n_k - 1}$ path, so that $\pi(k) < n_k$ and $\pi_{\leqslant k}$ is not tight in $\tau_{\leqslant n_k}$.

The most "efficient" choice might be to set $n_k = \pi(k) + 1$, but other choices are of course possible.

Fix k, and suppose for the sake of example that n_{k+1} equals $n_k + 3$. Write s for $\pi_{\leqslant k}$, and t for $\pi_{\leqslant k+1}$, so that $t' = s$. Write u for $\tau_{\leqslant n_k}$, u^+ for $\tau_{\leqslant n_k+1}$, u^{++} for $\tau_{\leqslant n_k+2}$, and v for $\tau_{\leqslant n_{k+1}}$, so that $v' = u^{++}$.

Consider the following string of symbols:

$$\underbrace{\underbrace{\underbrace{\underbrace{z_s^{u'}{}^\frown\langle[s;u]\rangle^\frown z_{s'}^{u}}_{z_s^u}{}^\frown\langle[s;u^+]\rangle^\frown z_{s'}^{u^+}}_{z_s^{u^+}}{}^\frown\langle[s;u^{++}]\rangle^\frown z_{s'}^{u^{++}}}_{z_s^{u^{++}}}{}^\frown\langle[t;v]\rangle^\frown z_{t'}^{v}}_{\mathfrak{s}^\ell(z_t^v)}$$

To see that the entire string is expressible as a shift of z_t^v, note that $z_s^{u^{++}} = z_{t'}^{v'}$, which is an end-segment of $z_t^{v'}$, so we must choose ℓ to be 1 if t is tight in v', and to be $1 + \ell h(z_t^{v''})$ otherwise.

Remark 4·34. For further variety in the construction of recurrent points, reflect that if s is loose in u and has successive extensions s^+, s^{++}, say, which are also u-sequences, the string $\langle [s;u] \rangle ^\frown z_{s'}^u$ is an end-segment of z_s^u but also of $z_{s^+}^u$ and of $z_{s^{++}}^u$, and hence can be followed by $\langle [s;u^+] \rangle ^\frown z_{s'}^{u^+}$, $\langle [s^+;u^+] \rangle ^\frown z_s^{u^+}$ or $\langle [s^{++};u^+] \rangle ^\frown z_{s^+}^{u^+}$, to yield, respectively, end-segments of $z_s^{u^+}$, $z_{s^+}^{u^+}$ or $z_{s^{++}}^{u^+}$.

Proof of the main result

LEMMA 4·35. *If $b \curvearrowright_{\mathfrak{s}} \xi$, $\tau_\xi \neq \sigma \in \mathcal{T}$, and s is a σ-sequence, then $\xi \not\curvearrowright_{\mathfrak{s}} \xi_s^\sigma$.*

Proof. Let e be an initial segment of σ that is not one of τ_ξ. Then $[e]$ occurs in ξ_s^σ but not in any $\xi_{\pi_\xi \upharpoonright k}^{\tau_\xi}$, hence not in ξ.

LEMMA 4·36. *If $b \curvearrowright_{\mathfrak{s}} \zeta \curvearrowright_{\mathfrak{s}} \gamma \triangleright \xi_s^\sigma$ and σ is well-founded, then ζ is of finite weight.*

Proof. If ζ were of infinite weight, τ_ζ would be ill-founded. But $\tau_\zeta = \sigma$.

PROPOSITION 4·37. *Let $\sigma \in \mathcal{T}$ be well-founded. If $b \curvearrowright_{\mathfrak{s}} \zeta \curvearrowright_{\mathfrak{s}} \gamma \triangleright \xi_s^\sigma$ then there is a $t \prec s$ such that ζ is near ξ_t^σ.*

Proof. Take $t = s_\zeta$, as in Proposition 4·24. $t \preccurlyeq s$ since $\zeta \curvearrowright_{\mathfrak{s}} \gamma$; since $\gamma \not\curvearrowright_{\mathfrak{s}} \gamma$, $t \prec s$.

COROLLARY 4·38. *For σ well-founded, $\beta(\xi_{\circledcirc}^\sigma, b, \mathfrak{s}) = \varrho_\sigma(0)$.*

Here β is as in Definition 1·1, and ϱ_σ is the rank function defined on the nodes of σ, as in section 1 of *Delays*. The number 0 is the top node in the tree relation $<_\sigma$ defined in Remark 4·7 above.

Proof. By lemmata 4·4 and 4·6 of *Delays*, taking T to be the tree coded by σ, x_T to be b and, for s a σ-sequence, x_s to be ξ_s^σ. Proposition 4·37 above shows that b plays the rôle required of x_T in lemma 4·6 of *Delays*.

Proof of Theorem 4·2. Let η be any countable ordinal and let $\sigma \in \mathcal{T}$ be well-founded with $\varrho_\sigma(0) = \eta$: such σ may be constructed following Remark 4·8 and *Delays*, proposition 4·1. Theorem 4·7 of *Delays* may now be applied, to show that $\theta(b, \mathfrak{s}) > \eta$. Since η was arbitrary, $\theta(b, \mathfrak{s}) \geqslant \omega_1$; by *Delays*, corollary 2·5, $\theta(b, \mathfrak{s}) \leqslant \omega_1$; thus $\theta(b, \mathfrak{s}) = \omega_1$.

(4·39) Thus we arrive at the following attractive picture: the recurrent points attacked by b are all at the ends of paths through ill-founded trees, and they are all *maximal recurrent in* b in the sense of definition 3·21 of *Delays*; all other points attacked by b are near to some ξ_s^τ for uniquely determined τ and s; the points that escape are those near to ξ_s^τ with τ well-founded below s.

Remark 4·40. The abode $A(b, \mathfrak{s})$ is a complete analytic set, since the assignment $\tau \mapsto \xi_{\circledcirc}^\tau$ is continuous, and τ is ill-founded if and only if $\xi_{\circledcirc}^\tau \in A(b, \mathfrak{s})$. Similarly $E(b, \mathfrak{s})$ is a complete co-analytic set.

Remark 4·41. Our methods confirm a conjecture of Martin Goldstern: let

$$G =_{\mathrm{df}} \{\alpha \in \mathcal{N} \mid \overline{\overline{\omega_{\mathfrak{s}}(\alpha)}} \leqslant \aleph_0\}.$$

G is co-analytic since

$$\alpha \in G \iff \forall \beta (\alpha \curvearrowright_{\mathfrak{s}} \beta \implies \beta \text{ is hyperarithmetic in } \alpha).$$

We shall show that G is complete by exhibiting a continuous reduction of the collection of well-founded trees to it.

For $\tau \in \mathcal{T}$, define ξ^τ by modifying Definition 4·1: let $(w_i^\tau)_i$ list all z_s^u where $u \succ \tau$ and s is a u-sequence—plainly such a list may be found uniformly recursive in τ by deleting all z_s^u with $u \not\succ \tau$ from the recursive list $(z_i)_i$—and then set

$$\xi^\tau =_{\mathrm{df}} w_0^\tau {}^\frown \langle [\mathsf{m}_0] \rangle ^\frown w_1^\tau {}^\frown \langle [\mathsf{m}_1] \rangle ^\frown w_2^\tau {}^\frown \langle [\mathsf{m}_2] \rangle ^\frown \cdots$$

If τ is well-founded, ξ^τ will be in G, since it attacks only points near to ξ_s^τ for some τ-sequence s.

If τ is ill-founded, then ξ^τ will attack some recurrent point at the end of a τ-path. The variety of construction of recurrent points indicated in Example 4·33 and Remark 4·34 may readily be exploited to prove that the set of recurrent points at the end of a given path is uncountable, and indeed contains a perfect set.

Thus if τ is ill-founded, ξ^τ will not be in G.

Since the association $\tau \mapsto \xi^\tau$ is continuous, indeed recursive, we have reduced a known complete co-analytic set to G, which must, therefore, itself be complete co-analytic.

5. *Acknowledgments*

My interest in this area was aroused by my encounters with the group in topological dynamics working at the Universitat Autònoma de Barcelona under the leadership of Professor Ll. Alsedà, concerning whom further information may be obtained from the Web page `http://mat.uab.es/sisdin/people/lalseda.html`

I was inspired to seek the construction of Section 2 by a counterexample of David Fremlin which showed that A^1 need not be a G_δ set, and thus answered in the negative problem 8·5 of *Delays*.

Here, with his permission, are his examples:

Definition 5·0. For $a \in \mathbb{N}^{\mathbb{N}}$ write

$$\begin{aligned} A_1(a) &= \{x : a \curvearrowright_{\mathfrak{s}} x\}, \\ A_2(a) &= \{x : \exists\, y \in A_1(a),\ y \curvearrowright_{\mathfrak{s}} x\}, \\ B_1(a) &= \{x : x \curvearrowright_{\mathfrak{s}} a\}, \\ B_2(a) &= \{x : \exists\, y \in B_1(a),\ x \curvearrowright_{\mathfrak{s}} y\}. \end{aligned}$$

Observe that because $\curvearrowright_{\mathfrak{s}}$ is a G_δ subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, $A_1(a)$ and $B_1(a)$ are G_δ (in fact, $A_1(a)$ is closed), and $A_2(a)$, $B_2(a)$ are analytic.

Example 5·1 (Fremlin). Let W be the set of all finite sequences $w \in \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ such that $w(i)$ is even for every $i < \ell h(w)$ and if $i+1$, $j < \ell h(w)$ and $w(i) < w(j)$ then $w(i) \leqslant w(i+1)$. (The idea is that w can move down only from a maximal value, and the rest of the time is non-decreasing.) Observe that if $w \in W$ and $m + n \leqslant \ell h(w)$, then $\langle w(m+i)\rangle_{i<n} \in W$. Enumerate W as $\langle w_n\rangle_{n\in\mathbb{N}}$. Let $a \in \mathbb{N}^{\mathbb{N}}$ be the sequence

$$w_0{}^\frown 1^\frown w_1{}^\frown 3^\frown w_2{}^\frown 5^\frown w_4{}^\frown 7^\frown \ldots.$$

Now each odd number appears only once in a. So if $a \curvearrowright_{\mathfrak{s}} x$, $x(n)$ must be even for every n, and $x \restriction n \in W$ for every n; conversely, of course, any such sequence belongs to $A_1(a)$. Now a member of $A_1(a)$ must be either bounded or monotonic. If x is an unbounded monotonic sequence, then $x \not\curvearrowright_{\mathfrak{s}} y$ for any y; so $A_2(a)$ consists only of bounded sequences, and it is easy to check that $A_2(a)$ is precisely the set of bounded sequences x such that $x \restriction n \in W$ for every n.

If we now look at the set M of bounded non-decreasing sequences taking even values only, then M is a relatively closed countable subset of $A_2(a)$ with no isolated points. So $A_2(a)$ is not a G_δ set in $\mathbb{N}^{\mathbb{N}}$. (It is, of course, F_σ.)

Example 5·2 (Fremlin). Let $a \in \mathbb{N}^{\mathbb{N}}$ be the sequence $(0,1,2,3,\ldots)$. This time, let $M \subseteq \mathbb{N}^{\mathbb{N}}$ be the set of non-decreasing sequences. For $x \in M$, define $x' \in \mathbb{N}^{\mathbb{N}}$ by setting

$x'(n) = x(n) + s$ whenever m, n, r, $s \in \mathbb{N}$, $n = m + r^2 + s$, $s \leqslant 2r$, $x(m) = x(n)$ and either $m = 0$ or $x(m) > x(m-1)$.

Next, for $k \in \mathbb{N}$ let $u_k \in \mathbb{N}^{k+1}$ be the finite sequence $(0,1,\ldots,k)$, and for $x \in M$ let $z_x \in \mathbb{N}^{\mathbb{N}}$ be the sequence

$$u_{x'(0)}{}^\frown u_{x'(1)}{}^\frown u_{x'(2)}{}^\frown u_{x'(3)}{}^\frown u_{x'(4)}{}^\frown \ldots.$$

Set $M_0 = \{x : x \in M$ is eventually constant$\}$, and $F = \{z_x : x \in M_0\}$. Observe that if $x \in M_0$, so that there is some k such that $x(n) \leqslant k$ for every n, then $z_x \curvearrowright_{\mathfrak{s}} w$, where w is the sequence

$$(0,1,\ldots,k,0,1,\ldots,k+1,0,1,\ldots,k+2,\ldots)$$

and $w \curvearrowright_{\mathfrak{s}} a$, so $z_x \in B_2(a)$.

If $x \in M$ and $x(n) > k$ for some n, then $x'(n) > k$ for all n large enough, so there are only finitely many m such that $z_x(m) = k$ and $z_x(m+1)$ is not $k+1$. This means that if $x \in M \setminus M_0$ then the only

y such that $z_x \curvearrowright_{\mathfrak{s}} y$ are of the form $s^m(a)$ for some m, and do not belong to B_1; thus $z_x \notin B_2(a)$ for any $x \in M \setminus M_0$.

The functions $x \mapsto x'$, $x \mapsto z_x$ are continuous, and if x, $y \in M_0$ are eventually different then $z_x \neq z_y$; so F has no isolated points. Of course F is countable. If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in M_0 such that $\langle z_{x_n} \rangle_{n\in\mathbb{N}}$ is convergent to $z \in \mathbb{N}^{\mathbb{N}}$, then

either $\langle x_n \rangle_{n\in\mathbb{N}}$ has a subsequence convergent to some $x \in M_0$, so that $z = z_x \in F$;

or $\langle x_n \rangle_{n\in\mathbb{N}}$ has a subsequence convergent to some $x \in M \setminus M_0$, so that $z = z_x \notin B_2(a)$;

or $\langle x_n \rangle_{n\in\mathbb{N}}$ has a subsequence $\langle y_n \rangle_{n\in\mathbb{N}}$ where, for some m, $y_n(i) = y_0(i)$ for every $n \in \mathbb{N}$, $i < m$ while $y_n(m) \geqslant n$; in which case $z(n) \to \infty$ as $n \to \infty$ so $z \not\curvearrowright_{\mathfrak{s}} w$ for any w, and $z \notin B_2(a)$.

Thus F is relatively closed in $B_2(a)$. Consequently $B_2(a)$ cannot be G_δ.

(5·4) Reading Fremlin's paper caused me to discard the mindset in which I had proposed the problems of section 8 of *Delays*, and to seek counterexamples. With the time gained by my refusal to attend yet another meeting of a sadly familiar type, the discussion of research its only purpose and the disruption thereof its only effect, I was led first to the construction of Section 2, and then, encouraged by that success, to seek a point of uncountable score; but, ironically, the construction above of b makes no use of the insights from Section 2 that enabled that construction to be found !

The reason is that our presentation incorporates several simplifications, due to Christian Delhommé, to our original construction. In particular he suggested the direct use of trees, where I had followed the more roundabout method given in *Delays* of starting from a linear ordering and then associating to that a tree of finite sequences. The elegant definition of $\mathcal{T}$ is also his suggestion. Further, he noticed that it is unnecessary to incorporate points ξ^τ, similar to the points x_T of *Delays*, into the construction of b. Our original procedure was to show that $\{\xi^\tau \mid \tau \in \mathcal{T}\}$ is a closed set, C_0 say, and then to define a point c attacking each member of C_0 using the method of §2 of this paper. The point c, too, proves to have score ω_1, but the verifications were more delicate. The point b that we have constructed will not attack the points ξ^τ and hence b and c are different.

Remark 5·5. Our examples have been given in spaces of infinite sequences of infinitely many symbols. The referee asks whether they might be, in some suitable sense, universal. One might hope for a positive answer since corresponding examples in the *Cantor space* of infinite sequences of just two symbols, again with the shift function, may be found by applying to them Delhommé's general transference theorem mentioned in *Delays*—for details, see his forthcoming paper [**2**]— and as pointed out to me by Señor Victor Jiménez López of Murcia, from the dynamics of the Cantor set to those of the real line is but a step.

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