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RUDIMENTARY RECURSION, GENTLE FUNCTIONS AND PROVIDENT SETS

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Abstract This paper, a contribution to "micro set theory", is the study promised by the first author in [M4], as improved and extended by work of the second. We use the *rudimentarily recursive* (set theoretic) functions and the slightly larger collection of *gentle* functions to initiate the study of *provident* sets, which are transitive models of PROVI, a subsystem of KP whose minimal model is Jensen's J_ω. PROVI supports familiar definitions, such as rank, transitive closure and ordinal addition—though not ordinal multiplication—and (shown in [M8]) Shoenfield's unramified forcing. Providence is preserved under directed unions. An arbitrary set has a *provident closure*, and (shown in [M8]) the extension of a provident M by a set-generic $\mathcal{G}$ is the provident closure of $M \cup \{\mathcal{G}\}$. The improvidence of many models of Z is shown. The final section uses similar but simpler recursions to show, in the weak system MW, that the truth predicate for $\dot{\Delta}_0$ formulæ is Δ_1.

0: Introduction

The research reported in this paper has evolved in response to the following question:

What is the minimal context in which set forcing works well?

It has long been known that the full power of ZF is not needed; but the results of [M4] show that forcing can go pathologically wrong if done over models of set theories which, even if strong in other ways, offer no support for set-theoretic recursion.

So let us ask a more specific question:

How much set-theoretic recursion is needed to do set forcing ?

Again, an upper bound has long been known, as Kripke-Platek set theory, KP, is certainly strong enough to allow recursive definitions of the right sort, such as defining the interpretation of names; the validity of such definitions follows easily from the Σ_1 recursion theorem which proves that, in KP, if G is a total Σ_1 function then so is the function F given by the recursion

$$F(x) = G(F \restriction x).$$

But what transpires is that even Σ_1 recursion is much stronger than needed for set forcing, and that a coherent and sufficiently strong recursion theory emerges if as our starting point we restrict attention to the above recursions when the defining function G is not merely Σ_1 but actually *rudimentary* in the sense of Jensen [J2]. In such cases we shall speak of F as given by a *rudimentary recursion*, or, more briefly, that F *is rud rec.*

In the present paper we present a theory that in the sequel [M8] supplies the answer to our initial question; and we give many counterexamples delimiting the scope of our current theory. But it is plain that, forcing aside, there are many aspects and applications yet to be explored.

Now for some examples: for the present we assume a knowledge of rudimentary functions, but shall develop their theory *ab initio* in §2.

Some rudimentary recursions

0·0 Example The definition of *rank*:

$$\varrho(x) = \bigcup\{\varrho(y) + 1 \mid_y y \in x\}$$

0·1 Example The definition of *transitive closure*:

$$\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid_y y \in x\}$$

0·2 Example Let $\mathcal{S}(x)$ be the set of finite subsets of x. *Restricted to ordinals*, this has a rudimentarily recursive definition:

$$\mathcal{S}(0) = \{\varnothing\}; \quad \mathcal{S}(\zeta+1) = \mathcal{S}(\zeta) \cup \{x \cup \{\zeta\} \mid_x x \in \mathcal{S}(\zeta)\}; \quad \mathcal{S}(\lambda) = \bigcup_{\nu<\lambda} \mathcal{S}(\nu).$$

All those are *recursions of type I*, meaning that no parameter occurs; and we may speak of such functions as *pure rud rec.*

0·3 Example Ordinal addition is given by the recursion

$$A(\alpha, 0) = \alpha;\ A(\alpha, \beta+1) = A(\alpha, \beta) + 1;\ A(\alpha, \lambda) = \bigcup_{\nu<\lambda} A(\alpha, \nu)$$

which is a rudimentary recursion on the second variable, the first remaining free; so the definition is of the form $F(\beta) = G(p, F\restriction\beta)$ with G rudimentary, where of course we set the parameter p equal to α. Such definitions we call *recursions of type II*, and we speak of such an F as *p-rud rec.*

0·4 Here is a good moment to remind the reader of some of our set-theoretic conventions. An *indecomposable* ordinal is an infinite ordinal closed under addition. The finite ordinals closed under addition are 0 and 1. We do not count 0 as a limit ordinal. We write the product of two ordinals α and β as $\alpha\beta$ or, for greater clarity, $\alpha \cdot \beta$. We should mention that, the power set axiom not usually being assumed in these weak systems, we do not assume when we write $\mathcal{P}(X)$ for the class $\{x \mid_x x \subseteq X\}$ of all subsets of X, that it is necessarily a set. "$A \subseteq \mathcal{P}(X)$" is simply a convenient way of saying that every member of A is a subset of X.

0·5 Example The relation $x \in^\star y$, meaning x is in the transitive closure of y, is given by a rud recursion on the second variable y, the first variable x remaining free:

$$x \in^\star y \iff x \in y \vee \exists z_{\in y}\, x \in^\star z$$

0·6 Example If M is an (intransitive) elementary submodel of a transitive set or class, then the Mostowski collapsing isomorphism ϖ_M is given by the recursion

$$\varpi_M(x) = \{\varpi_M(y) \mid_y y \in x \cap M\}$$

so that, in some sense, ϖ_M is rudimentarily recursive in the predicate M.

Rudimentary recursions in the theory of constructibility

0·7 EXAMPLE Let $\mathbb{T}$ be the unary rudimentary function introduced in [M3] and to be re-examined in §3. Then this rudimentary recursion on *ON*, the class of von Neumann ordinals,

$$T_0 = \varnothing; \qquad T_{\nu+1} = \mathbb{T}(T_\nu); \qquad T_\lambda = \bigcup_{\nu<\lambda} T_\nu$$

generates the constructible universe L and the Jensen hierarchy $(J_\nu)_\nu$, in that $L = \bigcup_{\nu\in ON} T_\nu$, and $J_\nu = T_{\omega\nu}$.

0·8 REMARK If challenged by a purist to define L by a recursion on V rather than on *ON*, we would define $T(x) = \bigcup_{y\in x} \mathbb{T}(T(y))$, and verify that $T(x)$ always equals $T_{\varrho(x)}$.

0·9 HISTORICAL NOTE Gödel evolved the notion of constructibility in the 1930s, and his first hierarchy was that now notated $\langle L_\nu \mid \nu \in ON\rangle$. He was implicitly doing Σ_1 recursion, a notion that became explicit in the 1960s.

His 1940 monograph aimed to present his relative consistency proof for AC to non-logicians, and therefore sought an exposition avoiding $\models$ and Def, and relying on what are now (unfairly to Bernays) called the Gödel functions. Here he comes much closer to a rudimentary recursion.

0·10 In the 1960s, Gandy and independently Jensen identified a more extensive and satisfactory collection of functions, called *basic* by Gandy and *rudimentary* by Jensen, which became the basis of Jensen's *fine structure* theory of L. It might be said that Jensen was implicitly doing rudimentary recursion, a notion that the present paper seeks to make explicit.

0·11 Indeed the definition given by Jensen of his auxiliary hierarchy is a rudimentary recursion, using the single rudimentary function S that he gave in [J2, p. 243], which lacks the property that its value for transitive argument is transitive. At each limit stage, he obtains the rud-closed set J_ν, and it is clear by induction that Jensen's J_ν equals our $T_{\omega\nu}$ for every ν.

0·12 COMMENT The referee asks us to comment on the relationship of our presentation to that of Schindler and Zeman in their article on fine structure for the *Handbook of Set Theory*.

At successor stages, they use not Jensen's S but another rudimentary function $\mathbf{S}$ which has the advantage over S that its value for transitive argument is always transitive; but their function raises rank by more than 1, which is a disadvantage not shared by $\mathbb{T}$.

The reader should note that Schindler and Zeman use only limit ordinals to index the Jensen hierarchy, so their $J_{\omega\nu}$ is exactly our $T_{\omega\nu}$. It should also be noted that Jensen defined rud(u) to be the rud closure of $u \cup \{u\}$ whereas Schindler and Zeman define it to be the rud closure of u. We follow Jensen.

Though the two papers both start from the theory of rudimentary functions, their main concern is the study of acceptable structures, that is, certain levels of Jensen-like hierarchies, which naturally come with a lot of "built-in" rudimentary recursion; but acceptability involves the notion of an initial ordinal, which is well beyond our present concern.

Relativisations of constructibility

0·13 Two ways of relativising constructibility have long been known; notation for them has varied, but we shall follow Jech's treatise in writing $L[A]$ for the result of constructing from a set or class A *as a predicate* and $L(a)$ for the result of constructing from a set a *as a set*. $L[A]$ will be the smallest inner model M with $x \cap A \in M$ for each $x \in M$; and $L(a)$ will be the smallest inner model M with $a \in M$.

0·14 Construction from a predicate A presents little difficulty: simply replace $\mathbb{T}$ by the function $\mathbb{T}_A$ defined either by setting $\mathbb{T}_A(u) = \mathbb{T}(u) \cup \{u \cap A\}$ or $\mathbb{T}_A(u) = \mathbb{T}(u) \cup \{x \cap A \mid x \in \mathbb{T}(u)\}$: the first is simpler but the second gives a faster construction. If A is a Δ_0 class, $\mathbb{T}_A$ (in either version) is still rudimentary, as what we shall call Δ_0 separators are rudimentary, indeed basic; and then the following will be a pure rudimentary recursion:

$$T_0[A] = \varnothing; \;\; T_{\nu+1}[A] = \mathbb{T}_A(T_\nu[A]); \;\; T_\lambda[A] = \bigcup_{\nu<\lambda} T_\nu[A];$$

and we may set $L[A] = \bigcup_{\nu \in ON} T_\nu[A]$.

0·15 Remark For A the class ON of ordinal numbers, $T_\nu[A] \cap ON = \nu$ for every ordinal ν.

0·16 For a a set let c be $\mathrm{tcl}(\{a\})$, the transitive closure of its singleton. Then the following rudimentary recursion in the parameter c is close to the traditional definition of construction from a as a set:

$$T_0(c) = c; \;\; T_{\nu+1}(c) = \mathbb{T}(T_\nu(c)); \;\; T_\lambda(c) = \bigcup_{\nu<\lambda} T_\nu(c).$$

Then $L(a) = L(c) = \bigcup_{\nu \in ON} T_\nu(c)$.

0·17 But if F is rud rec and we wish to compute $F(a)$, that start is too abrupt, even though $F(a) \in L(a)$, for we must first compute $F(b)$ for $b \in \mathrm{tcl}(a)$. We are therefore led to consider a different hierarchy, notated $(P^c_\nu)_\nu$, with $L(c) = \bigcup_{\nu \in ON} P^c_\nu$, which proves to be the central definition of this paper, as it is the rud recursion to which all other rud recursions reduce, as we shall show in §7. We outline the definition.

For a transitive set c, let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since c is transitive, $c_{\zeta+1}$ will be a set of subsets of c_ζ; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$, which we may use to give a direct recursive definition inspired by but not calling the rank function ϱ. If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Our definition of P^c_ν will have these properties:

$$P^c_\omega = T_\omega; \;\; P^c_{\zeta+1} = \{c_\zeta\} \cup c_{\zeta+1} \cup \mathbb{T}(P^c_\zeta); \;\; P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu; \;\text{ and } L(c) = \bigcup_{\nu \in ON} P^c_\nu.$$

The reader will notice that we have above used the definition of c_ν to define P^c_ν: so we appear to be using one rud rec function to define another; that creates a risk that our second function might not be not rud rec, so in our "official" definition we run the two definitions simultaneously.

Some illustrative counter-examples and more adventurous recursions

0·18 Remark In Model 13 of [M3], all axioms of Z hold but the rank function is not total, and therefore cannot be rudimentary.

0·19 Example In §12 of [M3], a transitive model of ZC is given in which TCo, the principle that every set is a member of a transitive set, fails. Thus tcl though rud rec, cannot be rud. Note that the rank of $\mathrm{tcl}(x)$ always equals that of x.

0·20 Remark The function $x \mapsto \mathcal{S}(x)$ of Example 0·2 is not given by a pure rud rec function, as we shall see below by estimating the rate of growth of its cardinality for $x \in \mathbf{HF}$. But we could define it by a recursion with parameter ω by remarking that for k a positive integer,

$$[a]^{k+1} = \Big\{ x \cup \{y\} \;\Big|_{x,y}\; x \in [a]^k \;\&\; y \in \bigcup [a]^k \Big\} \smallsetminus [a]^k.$$

0·21 Remark The function $\beta \mapsto \beta + \omega$, simple though it be, is not given by a pure rudimentary recursion, as we shall show in §6; still less are the other functions of ordinal arithmetic; nor is Jensen's map $\nu \mapsto J_\nu$. The reason is that, as was known to Gandy and to Jensen, to any rud function G there is a finite bound k, which we may call the *rudimentary constant* of G, such that for all arguments $\vec{x}$, $\varrho(G(\vec{x})) \leqslant \varrho(\vec{x}) + k$. From that it will follow that for a pure rud rec function F, for each argument x, $\varrho(F(x)) < \varrho(x) + \omega$.

0·22 Remark The functions $x \mapsto \mathcal{S}(x)$ and $\beta \mapsto \beta + \omega$ are examples of *recursions of type III*, a term we shall define in §5.

0·23 Example The function $\zeta \mapsto 2 \cdot \zeta$ is given by a rudimentary recursion. Nevertheless it is not rudimentary, for the rud closure of $\{\omega\}$ has ω as a member but, by Gandy [G], not $\mathit{EVEN} =_{\mathrm{df}} \{2 \cdot n \mid_n n \in \omega\}$.

0·24 Example The characteristic function of *EVEN* is given by a rud recursion on ω:

$$\chi(0) = 1; \qquad \chi(n+1) = 1 \smallsetminus \chi(n).$$

Note that $\chi \restriction \omega \notin \text{rud cl}\,\big(J_1 \cup \{\omega\}\big)$.

0·25 Remark Corollary 14·5 of *Weak Systems* shows that J_2 is not the rud closure of $J_1 \cup \{\omega\}$, J_1 not being a member of that latter set; but J_2 is the rud rec closure of $J_1 \cup \{\omega\}$; indeed of $\omega + 1$.

0·26 Remark The function g given by the recursion

$$g(0) = 1; \qquad g(\nu+1) = f(g(\nu)); \qquad g(\lambda) = \sup g``\lambda$$

where $f(\xi) = 2 \cdot \xi$ is given by a rud-rec recursion, but not by a rud recursion, as its rate of growth for finite arguments is too great. We shall explore iterated recursions of that sort in §6.

0·27 Remark We shall see in §6 that Gödel's original definition of L is ***not*** given by a rudimentary recursion, though every initial segment of it is.

0·28 Remark J_2 has recently been the object of study by Nik Weaver in his paper *Analysis in* J_2 [W].

Rudimentary recursions in the theory of forcing

0·29 Example Suppose we are making a forcing extension using a notion of forcing $\mathbb{P}$ that is a set of the ground model, assumed transitive. In the theory of forcing, a member y of the ground model is represented by the term $\hat{y}$ of the language of forcing, given by the recursion

$$\hat{y} =_{\mathrm{df}} \{(\mathbf{1}^{\mathbb{P}}, \hat{x}) \mid_x x \in y\}$$

That is a rudimentary recursion in a parameter, being of the form

$$F(a) = G(\mathbf{1}^{\mathbb{P}}, F \restriction a)$$

where G is the rudimentary function $(\mathbf{1}^{\mathbb{P}}, a) \mapsto \{\mathbf{1}^{\mathbb{P}}\} \times \mathrm{Im}(a)$: though it would be a simple matter to specify that $\mathbf{1}^{\mathbb{P}}$ is always to be some hereditarily finite set, for example 1, when G could be rewritten as a pure rud function.

0·30 Example If $\mathcal{G}$ is a generic filter on a notion of forcing $\mathbb{P}$ in a transitive model M, and we follow Shoenfield in treating all members of M as $\mathbb{P}$-names, the function $\mathrm{val}_{\mathcal{G}}(\cdot)$ defined for $a \in M$ is given by a rudimentary recursion with $\mathcal{G}$ as a parameter.

$$\mathrm{val}_{\mathcal{G}}(b) =_{\mathrm{df}} \{\mathrm{val}_{\mathcal{G}}(a) \mid_a \exists p_{\in \mathcal{G}}\, (p, a) \in b\}$$

The generic extension $M[\mathcal{G}]$ is then defined as $\{\mathrm{val}_{\mathcal{G}}(a) \mid_a a \in M\}$.

0·31 Remark Note that the definition of the forcing relation $\Vdash$ has not been invoked in making these definitions, but its properties would be needed to show that $M[\mathcal{G}]$ has properties of interest.

0·32 Remark The function $\mathrm{val}_{\mathcal{G}}(\cdot)$ combines two functions, which we might call *transforming* and *collapsing*. For example, if $\mathcal{G}$ is $(M, \mathbb{P})$-generic, one might first define for $x \in M$

$$\tilde{\pi}(x) = \{(\mathbf{1}^{\mathbb{P}}, \tilde{\pi}(a)) \mid_{p,a} (p, a) \in x \;\&\; p \in \mathcal{G}\},$$

thus transforming $\mathbb{P}$-names to $\mathbb{P}_1$-names, ($\mathbb{P}_1$ being the partial order whose sole member is $\mathbb{1}$); and then one would collapse the class of pure $\mathbb{P}_1$-names, to obtain the desired generic extension, by setting for $x \in \Im(\tilde{\pi})$,

$$\varpi(x) = \{\varpi(y) \mid_y (\mathbf{1}^{\mathbb{P}}, y) \in x\},$$

which of course is the inverse of the function $x \mapsto \hat{x}$ when the latter is taken to be defined on $M[\mathcal{G}]$.

Both recursions are rudimentary in appropriate parameters or classes.

We trust that these examples have given the reader a sense of the scope and limits of rudimentary recursion. We turn to the other unexplained terms in the title of the paper.

Gentle functions

A *gentle* function is one of the form $H \circ F$ where H is rudimentary and F is rud rec. The importance of the notion lies in the second author's results, presented in Section 4, that while the collection of rud rec functions is ***not*** closed under composition, the slightly larger collection of gentle functions ***is***.

0·33 EXAMPLE In their paper [SMcC], Scott and McCarty propose the following recursive definition of *ordered pair*:

$$\langle x, y\rangle_2^{\mathrm{SM}} = \{\langle 0, t\rangle_2^{\mathrm{SM}} \mid_t t \in x\} \cup \{\langle 1, u\rangle_2^{\mathrm{SM}} \mid_u u \in y\}$$

They show that if at least one of x and y is of infinite rank, the rank of the pair $\langle x, y\rangle_2^{\mathrm{SM}}$ equals the maximum of $\varrho(x)$ and $\varrho(y)$. We follow their alternative approach to that definition, but with slight changes to their notation and exposition.

Consider the four recursions, of which the first is taken from and the others inspired by their paper:

$$\tau(y) = \{\varnothing\} \cup \big\{\tau(u) \bigm|_u u \in y\big\}; \qquad \phi(y) = \{\phi(u) \mid_u u \in y \mathbin{\&} \varnothing \in u\};$$
$$\sigma(x) = \big\{\sigma(t) \cup \{\varnothing\} \bigm|_t t \in x\big\}; \qquad \psi(y) = \{\psi(u \smallsetminus \{\varnothing\}) \mid_u u \in y\}.$$

DEFINITION
$$\mathrm{left}^{\mathrm{SM}}(a) =_{\mathrm{df}} \psi``\big(a \cap \{d \mid_d \varnothing \notin d\}\big);$$
$$\mathrm{right}^{\mathrm{SM}}(a) =_{\mathrm{df}} \phi``\big(a \cap \{c \mid_c \varnothing \in c\}\big).$$

DEFINITION (Scott, McCarty) $\langle x, y\rangle_2^{\mathrm{SM}} =_{\mathrm{df}} \sigma``x \cup \tau``y$

REMARK τ, σ and ϕ are pure rud rec; results in §4 will show that ψ, $\mathrm{left}^{\mathrm{SM}}$ and $\mathrm{right}^{\mathrm{SM}}$ are gentle; $\langle\cdot,\cdot\rangle_2^{\mathrm{SM}}$ is a composite of gentle functions.

0·34 LEMMA *$\varnothing$ is a member of every $\tau(y)$ and of no $\sigma(x)$; and for all z, $\sigma(z) = \tau(z) \smallsetminus \{\varnothing\}$; $\phi(\tau(z)) = z$; and $\psi(\sigma(z)) = z$.*

0·35 LEMMA *Let $a = \langle x, y\rangle_2^{\mathrm{SM}}$: then $\mathrm{left}^{\mathrm{SM}}(a) = x$ and $\mathrm{right}^{\mathrm{SM}}(a) = y$.*

REMARK (Scott, McCarty) The two versions of the definition of $\langle\cdot,\cdot\rangle_2^{\mathrm{SM}}$ are equivalent as $\tau(v) = \langle 1, v\rangle_2^{\mathrm{SM}}$ and $\sigma(v) = \langle 0, v\rangle_2^{\mathrm{SM}}$.

Provident sets

A non-empty transitive set A is called *p-provident* if it is closed under all functions rudimentary recursive in the parameter p, (B)-*provident* if it is p-provident for all $p \in B$, and *provident* if it is (A)-provident.

For the more restrictive notions, it must be specified that A is closed under unordered pairs.

0·36 REMARK Natural examples of provident sets abound: for example Jensen's J_1 and J_ω. In §6 we shall give a very general notion of hierarchy such that the νth stage in any such hierarchy is provident whenever ν is an indecomposable ordinal: in particular, that will hold for the L and J hierarchies.

0·37 REMARK The main results of [M8] are that provident sets support the Shoenfield–Kunen approach to set forcing and that a set-generic extension of a provident set is provident. Those results taken with the counter-examples of [M4] are our grounds for asserting that the minimal context for set forcing is that afforded by provident sets.

The plan of the paper

Sections 1, 2 and 3 give back-ground material; much of the material here is taken from two previous papers, *The strength of Mac Lane set theory* [M2] and *Weak Systems of Gandy, Jensen and Devlin* [M3].

Section 1 prepares the reader for the study of rudimentary recursion by reviewing with some care the syntax and fundamental definitions of set theory.

Section 2 uses the first author's theory of companions to give a short proof of the fundamental theorem 2·16 concerning the collection of rudimentary functions. We give an example of a unary function with Δ_0 graph and of finite rank-bounded growth that is not rudimentary, thus answering negatively a question of Sy Friedman.

Section 3 re-examines the function $\mathbb{T}$ introduced in [M3]; this function is enormously helpful in the sequel. The subsections entitled "The intransitive case" and "Gandy reproved" are peripheral, but included for completeness: we use the function $\mathbb{T}$ to improve some arguments of Gandy.

Sections 4, 5 and 6 contain the hard work of the paper

Section 4 introduces *rudimentary recursion* without parameters, and the second author's analysis of the composition of rud rec functions. We enlarge our enquiry to include recursions from an additional predicate and show that a function that is gentle in a gentle predicate is gentle, which theorem is the key to simplifying the main proof of the sequel [M8].

Section 5 advances the discussion to include recursions from parameters and finds a single rudimentary recursion, with parameter, to instances of which all others reduce.

Section 6 introduces *provident sets* which are non-empty, transitive and closed under all rudimentarily recursive functions, allowing parameters from within the set in question. We obtain various characterizations of provident sets, and build many examples as the union of a sequence of transitive sets, the sequence being of a kind we call a *progress*. It turns out that to be provident it is enough for a transitive set to be closed under rather few rudimentary recursions; the main one being the one generating what we call *the canonical progress* $(P^c_\nu)_\nu$, which we have already mentioned, and which is discussed more fully on page 39.

0·38 We show in Theorem 6·12 that to every rudimentary function R there corresponds an integer c_R, which we call the *rudimentary constant of* R, such that for every progress $P_0, \ldots, P_{c_R}$, and all arguments $\vec{x} \in P_0$, the value $R(\vec{x})$ will lie in P_{c_R}. Armed with that result, forms of which were certainly known to Gandy and Jensen and other early workers in fine structure, we compute in Proposition 6·32 an equally uniform bound on the rate of evolution of a rudimentary recursion; that leads rapidly to the central result, Theorem 6·34, which implies for example that for every transitive set c and indecomposable ordinal θ, P^c_θ is provident.

0·39 We examine two related notions: $\varnothing$-providence involves closure only under pure, rather than parametrised, rudimentary recursive functions. Limit providence involves closure under functions produced by iterated recursion, such as that in Remark 0·20. We show in Proposition 6·39 that provident sets are closed under recursions of type III. We then make a first study of functions obtained by iterated recursions; this subsection is looking to future investigations of a hierarchy that is slowly emerging.

0·40 Our study of progresses enables us to show rapidly that the Gödel and Jensen segments $L_{\omega\nu}$ and J_ν are provident if and only if $\omega\nu$ is indecomposable. We show that the recursion underlying Gödel's original definition of the constructible hierarchy is not rudimentary. We show that each infinite level of the Gödel hierarchy is closed under the Scott–McCarty pairing and unpairing functions; and apply this observation to represent arbitrarily long segments of that hierarchy as rudimentary recursions in some parameter.

The going now becomes easier: **Sections 7, 8 and 9** apply the ideas developed in the preceding sections and in [M3].

Section 7 explains a simple construction that gives, for any set x, the minimal provident set $\mathrm{Prov}(x)$ including x. We call $\mathrm{Prov}(x)$ the *provident closure* of x. Provident closures allow the following transparent formulation of the relationship between providence and forcing, which will be proved in [M8]:

$$M[\mathcal{G}] = \mathrm{Prov}(M \cup \{\mathcal{G}\}) \text{ when } M \text{ is provident.}$$

Here M and $\mathcal{G}$ are as in Example 0·30.

Remark In [M4, §3], an even simpler definition was given of Prov(M) for non-empty M that are transitive and model AxPair and TCo. By Theorem 7·0 and [M4, Proposition 3·2], for such M the two definitions are equivalent.

We show that it is enough to require closure under a particular finite basis of rudimentary recursive functions, which leads to a finite axiomatisation of the notion of providence. The phenomena of finite axiomatisability and regular presence in natural hierarchies also hold for the collections of sets mentioned in 0·39.

Section 8 gives models of Zermelo set theory that fail in various ways to support rudimentary recursion; in one, the failure is of Scott's celebrated trick for defining cardinal number; in another, the addition of a Cohen generic real goes awry. Other models show the inability of Zermelo set theory to pass in either direction from the set of Zermelo naturals to that of von Neumann.

Other examples of the weakness for recursive definitions of the unimproved set theories of Zermelo and Mac Lane are given in [M1] and [M2]. In [M4] it is shown how passage to the provident closure of transitive models of those theories preserves the theories but adds the capacity for rudimentary recursion and therefore for doing set forcing. In [M2] it was shown that passage to what in [M4] is called the lune of such models again preserves the theory (Zermelo or Mac Lane as the case may be) but adds the capacity for Σ_1 recursion.

Section 9 shows, as promised in [M3], that the weak system MW supports a truth definition for $\dot{\Delta}_0$ formulæ.

In **the endmatter**, we record the origins of the paper and its sequel, and close with acknowledgments and references.

1: A rapid development of weak set theory

1·0 We regard set theory as formalised in a syntax with a class-forming operator and both restricted and unrestricted quantifiers. We have two two-place relation symbols $\in$ and $=$, propositional connectives $\neg$, $\&$, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$, unrestricted quantifiers $\forall$, $\exists$, restricted quantifiers $\forall_r$, $\exists_r$, a class forming operator $\rtimes$ and a supply of variables.

1·1 Our collection of *well-formed formulæ* is defined thus: atomic wffs are

$$x \in y, \qquad x = y$$

and if Φ and Ψ are well-formed, so are $\&\Phi\Psi$, $\vee\,\Phi\Psi$, $\neg\Phi$, $\forall x\Phi$, $\exists x\Phi$, $y \in \rtimes x\Phi$, $\forall_r xy\Phi$, and $\exists_r xy\Phi$, where in the last two, x and y are **distinct** variables, so that restricted quantifiers $Q_r xy$ bind x but not y, in harmony with the axioms, given below, that express their intended meaning. The expressions $\forall_r xx\Phi$ and $\exists_r xx\Phi$ are ill-formed.

We write $\forall x_{\in y}\,\Phi$ for $\forall_r xy\Phi$, $\exists x_{\in y}\,\Phi$ for $\exists_r xy$ and $\{y \mid \Phi\}$ for $\rtimes\, y\,\Phi$. Officially we use Polish notation and write $\&\Phi\Psi$; unofficially we use brackets, writing $(\Phi \;\&\; \Psi)$. Similarly we shall often adopt conventional ways of indicating negation, such as $\notin$ and $\neq$.

1·2 A string $\rtimes\, x\,\Phi$, where Φ is a wff, is a *class.* Here are five examples:

$$\begin{aligned} V &=_{\mathrm{df}} \{x \mid x = x\} \\ \varnothing &=_{\mathrm{df}} \{x \mid x \neq x\} \\ \{x,\, y\} &=_{\mathrm{df}} \{z \mid z = x \vee z = y\} \\ x \setminus y &=_{\mathrm{df}} \{z \mid z \in x \;\&\; z \notin y\} \\ \bigcup x &=_{\mathrm{df}} \{z \mid \exists y_{\in x}\, z \in y\} \end{aligned}$$

Since $\varnothing$ is the smallest von Neumann ordinal, we shall also set $0 =_{\mathrm{df}} \emptyset =_{\mathrm{df}} \varnothing$, and will tend to use the notation 0 when we are thinking of this set in its ordinal capacity, $\varnothing$ when thinking of it as the empty set, and $\emptyset$ when thinking of it as the sequence of length 0. In the review of set-theoretic notation which we now give, we are liable to omit definitions of familiar extensions such as writing $\{x\}$ for $\{x,\, x\}$.

1·3 We denote by $[\Phi\,{}^{y}_{x}]$ the result of substituting the variable x for the free occurrences of the variable y in the formula Φ, bound occurrences of x in Φ being first changed to an as yet unused variable. Less formally, we permit ourselves informally to indicate the result of substituting one variable for another by such usages as $\mathfrak{A}(x)$ and $\mathfrak{A}(y)$.

We progressively extend our notation to permit more liberal use of classes. Thus if Φ is a wff, t a variable or a class, and B a class, then

$$\begin{aligned} \exists z_{\in B}\;\Phi &\Longleftrightarrow_{\mathrm{df}} \exists z\,(z \in B \;\&\; \Phi) \\ \forall z_{\in B}\;\Phi &\Longleftrightarrow_{\mathrm{df}} \forall z\,(z \in B \implies \Phi) \\ t = B &\Longleftrightarrow_{\mathrm{df}} \forall x\,(x \in t \Longleftrightarrow x \in B) \\ B = t &\Longleftrightarrow_{\mathrm{df}} \forall x\,(x \in B \Longleftrightarrow x \in t) \\ B \in t &\Longleftrightarrow_{\mathrm{df}} \exists y_{\in t}\, y = B \end{aligned}$$

The first two would normally be used only when z is a variable not occurring in B, otherwise nonsense might result. In the last three x and y are presumed to be new variables occurring in neither B nor t.

1·4 DEFINITION Let x be a variable, B a class and Φ a wff. Then $[\Phi\ {}^{x}_{B}]$ is the result of

i) changing all bound occurrences of variables in Φ to occurrences of variables not occurring in B or free in Φ;
ii) replacing all free occurrences of x in the new formula by B;
iii) expanding occurrences of the strings "$B \in t$", "$B = t$", "$t = B$", "$\forall y_{\in B}$" and "$\exists y_{\in B}$" according to the definitions above.

Similarly one may define $[A\ {}^{x}_{B}]$ for A a class. Expressions such as $[\Phi\ {}^{A}_{B}]$ are not defined.

Axioms of logic

All our systems of set theory will have among their axioms those of classical propositional and predicate logic, these two schemes of axioms relating restricted quantifiers to unrestricted ones,

$$\forall x_{\in y}\, \Phi \iff \forall x(x \in y \implies \Phi)$$
$$\exists x_{\in y}\, \Phi \iff \exists x(x \in y\ \&\ \Phi)$$

and the Church conversion scheme

$$x \in \{y \mid \Phi\} \iff [\Phi\ {}^{y}_{x}]$$

by which all occurrences of the class-forming operator are in principle eliminable.

The system S_0

S_0	Extensionality:	$(\forall w_{\in x}\, w \in y \,\&\, \forall w_{\in y}\, w \in x) \implies x = y$	S_0
	Empty Set:	$\varnothing \in V$	
	Pair:	$\{x, y\} \in V$	
	Difference:	$x \setminus y \in V$	
	Union:	$\bigcup x \in V$	

1·5 DEFINITION We define a Δ_0 formula or a Δ_0 class to be one containing no unrestricted quantifiers; a Π_1 formula is one of the form $\forall x\mathfrak{A}$ where $\mathfrak{A}$ is Δ_0; a Σ_1 formula is one of the form $\exists x\mathfrak{A}$ where $\mathfrak{A}$ is Δ_0; a Σ_2 formula is one of the form $\exists y\mathfrak{B}$ where $\mathfrak{B}$ is Π_1; and so on.

1·6 DEFINITION $\mathsf{Foundation}$, the axiom of (set) foundation, is $x \neq \varnothing \implies \exists y_{\in x}\, x \cap y = \varnothing$.

S_0' S_0 + Foundation S_0'

1·7 DEFINITION If S is any system of set theory containing S_0 we say that a class A or a wff Φ is Δ_0^{S} iff there is a Δ_0 class B or a Δ_0 wff Ψ such that $\vdash_{\mathsf{S}} A = B$ or $\vdash_{\mathsf{S}} \Phi \iff \Psi$ respectively.

1·8 DEFINITION A class A is S-*suitable* if $\vdash_{\mathsf{S}} A \in V$ and for each Δ_0 wff Ψ and variable w **not occurring freely in** A, $\forall w_{\in A}\, \Psi$ is Δ_0^{S}.

1·9 REMARK If S is a subsystem of T, then all S-suitable classes are T-suitable.

This notion is important in building a calculus of Δ_0 wffs, which we now do.

1·10 PROPOSITION *If Φ and Ψ are Δ_0^{S}, so are $\exists w_{\in z}\, \Phi$, $\forall w_{\in z}\, \Phi$, where w and z are distinct variables, $(\Phi \,\&\, \Psi)$, $\neg\Phi$ and $x \in \{y \mid \Phi\}$.*

1·11 PROPOSITION *Let A be S-suitable.*
(i) if Φ is Δ_0^{S}, so is $\exists w_{\in A}\, \Phi$, provided w is not free in A;
(ii) $w \in A$, $w = A$, $A \in w$ are Δ_0^{S}, even if w occurs in A;
(iii) if Φ is Δ_0, $[\Phi \;{}^{x}_{A}]$ is Δ_0^{S};
(iv) if Φ is Δ_0^{S}, so is $[\Phi \;{}^{x}_{A}]$.

It is necessary to prove (iii) before (iv), since a subformula of a Δ_0^{S} formula need not be Δ_0^{S}.

1·12 PROPOSITION *If A and B are S-suitable, so is $[B \;{}^{x}_{A}]$.*

1·13 PROPOSITION *The classes $\varnothing$, $\{x, y\}$, $\{x\}$, $x \setminus y$ and $\bigcup x$ are S_0-suitable.*

Note that if we define

$$x \cup y =_{\mathrm{df}} \bigcup\{x, y\}$$
$$x \cap y =_{\mathrm{df}} x \setminus (x \setminus y),$$

we have $\vdash_{\mathsf{S}_0} x \cup y \in V \,\&\, x \cap y \in V$; indeed both $x \cup y$ and $x \cap y$ are S_0-suitable. We would wish to define

$$\bigcap x =_{\mathrm{df}} \{z \mid \forall y_{\in x}\, z \in y\};$$

but as $\vdash_{\mathsf{S}_0} \bigcap \varnothing = V$, and (by Russell) $\nvdash_{\mathsf{S}_0} V \in V$, $\bigcap x$ cannot be S_0-suitable. We therefore make an additional definition:

1·14 DEFINITION $\bigcap' x =_{\mathrm{df}} \bigcup x \cap \bigcap x$,

which will prove to be suitable in our next system ReS_0. For now, we can prove that $\bigcap x$ is nearly suitable:

1·15 PROPOSITION *If $\vdash_{\mathsf{S}} A \neq \varnothing$ and A is S-suitable, then so is $\bigcap A$.*

Descriptions are defined so that should the defining clause not have exactly one witness, the description is taken to mean the empty set:

1·16 DEFINITION $\iota x \Phi =_{\mathrm{df}} \bigcup\{x \mid \{x\} = \{x \mid \Phi\}\}$

The next proposition echoes the recursion-theoretic concept of a bounded search.

1·17 Proposition *Let A be* S*-suitable,* Φ Δ_0^{S} *and x a variable not free in A. If* $\vdash_{\mathcal{S}} \Phi \implies x \in A$, *then* $\iota x\, \Phi$ *is* S*-suitable.*

Ordered pairs

Following Kuratowski, we introduce a pairing function, in the definition of which we exploit our new freedom to compose suitable classes:

1·18 Definition $(x, y)_2 =_{\mathrm{df}} \{\{x\}, \{x, y\}\}$

1·19 Proposition $(x, y)_2$ *is* S_0*-suitable.*

1·20 Lemma *"w is a singleton", "w is an un-ordered pair", and "w is an ordered pair" are all* $\Delta_0^{\mathsf{S}_0}$

Now we define the un-pairing functions:

1·21 Definition $(x)_\ell =_{\mathrm{df}} \iota y(x$ is an ordered pair and $\bigcup\bigcap x = y)$

1·22 Proposition $(x)_\ell$ *is* S_0*-suitable.*

1·23 Definition $(x)_r =_{\mathrm{df}} \iota y(x$ an ordered pair and
either $\bigcup x = \bigcap x \,\&\, y = \bigcup\bigcap x$ **or** $\bigcup x \neq \bigcap x \,\&\, y = \bigcup(\bigcup x \setminus \bigcap x))$

1·24 Proposition $(x)_r$ *is* S_0*-suitable.*

Foundation, ordinals and the axiom of infinity

With $\mathsf{Foundation}$ added, the formulation of "ordinal" becomes Δ_0 and much of the elementary theory of ordinals can then be developed in S_0'. In this paper we shall usually be assuming the scheme of Foundation for Π_1 classes, which of course implies $\mathsf{Foundation}$.

Although in one model that we shall mention, we must use a different formulation, we shall usually take the axiom of infinity in the form $\omega \in V$, ω being defined as the class of all von Neumann ordinals such that they and all their predecessors are either 0 or successor ordinals.

PZ S_0 plus the Δ_0 separation scheme: $x \cap A \in V$ for A a Δ_0 class. PZ

ReS ReS_0 plus the scheme of Π_1 foundation: $A \neq \varnothing \implies \exists x_{\in A}\, x \cap A = \varnothing$ for A a Π_1 class. ReS

ReSI $\mathsf{ReS} + \omega \in V$. ReSI

Digression: models with failures of Δ_0 separation.

We digress to construct, in some conveniently strong system, a transitive model of the system S_0 in which an instance of Δ_0 separation fails. Recall that a set u is *transitive* if $\bigcup u \subseteq u$.

Let θ be a limit ordinal, for example ω^2. A θ-interval is a set $\{\alpha \in ON | \beta \leq \alpha < \gamma\}$ with $\beta, \gamma < \theta$. Let K_0^θ be the set of all finite unions of

θ-intervals. Note that K_0^θ is already a transitive model of all the axioms of S_0 except pairing, and that it is closed under finite unions. To get something which in addition models the pairing axiom, we define a sequence of sets K_n^θ, where K_{n+1}^θ is the set of all sets of the form $k \cup l$ with $k \in K_n^\theta$ and l a finite subset of K_n^θ, and we define $K^\theta = \bigcup_{n\in\omega} K_n^\theta$.

1·25 Proposition *i) Each K_n^θ is a transitive set.*

ii) K^θ is transitive and $K^\theta \cap ON = \theta$

iii) The axioms of extensionality, infinity, pairing, union and difference are all true in K^θ.

1·26 Proposition *If θ is a limit ordinal at least ω^2 then the set of limit ordinals less than θ is not a member of K^θ, and accordingly Δ_0 separation fails there.*

Proof : It is sufficient to note that every element of K^θ is a union of an element of K_0^θ with a finite set. ⊣ (1·26)

The definition of cartesian product

We introduce, successively, ordered $\mathfrak{k}$-tuples:

$$\begin{aligned} (y_1, y_2, y_3)_3 &=_{\mathrm{df}} (y_1, (y_2, y_3)_2)_2 \\ (y_1, y_2, y_3, y_4)_4 &=_{\mathrm{df}} (y_1, (y_2, y_3, y_4)_3)_2 \\ (y_1, y_2, y_3, y_4, y_5)_5 &=_{\mathrm{df}} (y_1, (y_2, y_3, y_4, y_5)_4)_2 \end{aligned}$$

and so on, and we may verify that all those are S_0-suitable.

1·27 Remark Thus all Kuratowski $\mathfrak{k}$-tuples are generated from the single binary function $\{x, y\}$.

1·28 Definition $x \times y \ =_{\mathrm{df}} \ \{z \mid \exists a_{\in x} \exists b_{\in y}\, z = (a, b)_2\}$

It would be more convenient to formulate such a definition in this way:

$$x \times y \ =_{\mathrm{df}} \ \{(a, b)_2 \mid a \in x \ \&\ b \in y\}$$

That is, though, ambiguous: where the context demands, we may remove the ambiguity by listing the variables to be quantified beside the | sign. Thus

$$\{(a, b)_2 \mid_b a \in x \ \&\ b \in y\}$$

would mean $\{a\} \times y$ if a is in x, and the empty set otherwise. Hence we make the following

1·29 Definition Let A be a class; then

$$\{A \mid_{x_1 \ldots x_n} \Phi\} \ =_{\mathrm{df}} \ \{y \mid \exists x_1 \ldots \exists x_n\ y = A \ \&\ \Phi\}.$$

1·30 Remark Define inductively $\bigcup^{\mathfrak{k}+1} x \ =_{\mathrm{df}} \ \bigcup(\bigcup^{\mathfrak{k}} x)$. Then each $\bigcup^{\mathfrak{l}} x$ is S_0-suitable. S_0 easily proves that if $x = (y, z)_2$, then $y \in \bigcup^2 x$ and $z \in \bigcup^2 x$; hence, using these S_0-suitable restrictions, one verifies easily that if $\mathfrak{A}$ is Δ_0 then the class $\big\{(y_1, y_2, \ldots, y_{\mathfrak{k}})_{\mathfrak{k}} \ \big|_{y_1, y_2, \ldots, y_{\mathfrak{k}}} \ \mathfrak{A}\big\}$ of $\mathfrak{k}$-tuples is equal, provably in

S_0, to a Δ_0 class. But in general, if A is S-suitable and Φ is Δ_0, $\{A \mid_{x,y} \Phi(x,y)\}$ might not be a Δ_0^{S} class.

1·31 Remark In both Models 1 and 2 of [M3, §4], ReSI holds but $\omega \times \omega$ is not a set.

Relations and functions

We may now develop the usual theory of relations, $\mathfrak{k}$-ary functions and so on: we treat functions as a subclass of their image $\times$ their domain. In discussing relations we shall often write Rxy to mean $(x,y)_2 \in R$, though this notation is perhaps too perilous to adopt in a general definition.

1·32 Definition Let R be a variable or class: write

$$Rel(R) \iff_{\text{df}} R = \{(x,y)_2 \mid Rxy\}.$$

We distinguish two relations by special, if inelegant, names:

1·33 Definition

$$\begin{aligned} id &=_{\text{df}} \{(x,x)_2 \mid x \in V\} \\ eps &=_{\text{df}} \{(x,y)_2 \mid x \in y\} \end{aligned}$$

Let F be a set or class.

1·34 Definition $Fn(F) \iff_{\text{df}} Rel(F)\ \&\ \forall x \forall y \forall z (Fxz\ \&\ Fyz \implies x = y)$

Note that we are following a convention in which $(x,y)_2 \in F$ corresponds to the statment $x = F(y)$, rather than $y = F(x)$.

1·35 Definition

$$\begin{aligned} F(t_1, \ldots t_n) &=_{\text{df}} \iota x\,(xt_1 \ldots t_n)_{n+1} \in F \\ \langle t \mid_{x_1, \ldots x_n} \Phi \rangle &=_{\text{df}} \{(t, x_1, \ldots x_n)_{n+1} \mid \Phi\} \end{aligned}$$

1·36 Proposition *$f(x)$ is ReS_0-suitable.*

1·37 Definition For R and t sets or classes, set

$$\begin{aligned} R``t &=_{\text{df}} \{y \mid \exists x\,(Ryx\ \&\ x \in t)\} \\ R{\restriction}t &=_{\text{df}} \{(x,y)_2 \mid Rxy\ \&\ y \in t\} \\ R^{-1} &=_{\text{df}} \{(x,y)_2 \mid Ryx\} \\ \mathrm{Dom}\,(R) &=_{\text{df}} R^{-1}``V \\ \Im(R) &=_{\text{df}} R``V \\ \mathrm{Field}\,(R) &=_{\text{df}} \mathrm{Dom}\,(R) \cup \Im(R) \end{aligned}$$

1·38 Remark Note that by our system of definitions, $f(x)$ is always defined, with default value $\varnothing$; hence $\exists y\, y = f(x)$ is **not** equivalent to $x \in Dom(f)$. We shall occasionally write $f(x)\downarrow$ for the latter.

Our definition of *well-founded relation* includes the concept of being "set-like":

1·39 Definition $Wf(R) \iff_{\text{df}} Rel(R)\ \&\ \forall x \exists y (x \in y\ \&\ R``y \subseteq y)\ \&$
$\forall x (x = 0 \vee \exists y_{\in x}\, x \cap R``\{y\} = 0)$

Relativisation of a formula to a class

1·40 DEFINITION Let M be a class. For each formula Φ of the language of set theory, we define $(\Phi)^M$, the *relativisation of* Φ *to* M by recursion on the length of Φ.

1·41 DEFINITION $(x \in y)^M$ is $x \in y$; $(x = y)^M$ is $x = y$; $(\neg\Phi)^M$ is $\neg(\Phi)^M$; $(\Phi \,\&\, \Psi)^M$ is $((\Phi)^M \,\&\, (\Psi)^M)$; $(\Phi \implies \Psi)^M$ is $((\Phi)^M \implies (\Psi)^M)$; and similarly for the other propositional connectives;

$$(\forall x\, \Phi)^M \text{ is } \forall x_{\in M}\, (\Phi)^M; \qquad (\forall x_{\in y}\, \Phi)^M \text{ is } \forall x_{\in y\cap M}\, (\Phi)^M;$$
$$(\exists x\, \Phi)^M \text{ is } \exists x_{\in M}\, (\Phi)^M; \qquad (\exists x_{\in y}\, \Phi)^M \text{ is } \exists x_{\in y\cap M}\, (\Phi)^M;$$
$$(x \in \{y \mid \Phi\})^M \text{ is } x \in \{y \mid y \in M \,\&\, (\Phi)^M\}.$$

We also define the relativisation of a class by:

1·41 DEFINITION $(\{y \mid \Phi\})^M$ is $\{y \mid y \in M \,\&\, (\Phi)^M\}$.

The systems DB, BS and MW

The next system, which we call DB for "Devlin Basic", adds the existence of cartesian product to PZ, but as it thereby becomes finitely axiomatisable, we give it officially as that finite axiomatisation.

DB_0 The system of which the set-theoretic axioms are Extensionality and the following nine set-existence axioms: DB_0

$\varnothing \in V$	$\bigcup x \in V$	$a \cap \{(x,y)_2 \mid_{x,y} x \in y\} \in V$
$\{x, y\} \in V$	$\mathrm{Dom}\,(x) \in V$	$\{(y,x,z)_3 \mid_{x,y,z} (x,y,z)_3 \in b\} \in V$
$x \setminus y \in V$	$x \times y \in V$	$\{(y,z,x)_3 \mid_{x,y,z} (x,y,z)_3 \in c\} \in V$

1·42 REMARK All those nine are theorems of PZ + cartesian product.

1·43 THEOREM (Bernays) *All instances of* Δ_0 *separation are provable in the system* DB_0

1·44 DEFINITION We shall call a function of the form $x \mapsto x \cap A$, where A is a class, a *separator*, or a Δ_0-*separator* if A is a Δ_0 class.

DB DB_0 plus Π_1 foundation. DB

$\mathsf{DB}_0\mathsf{I}$ DB_0 plus $\omega \in V$. $\mathsf{DB}_0\mathsf{I}$

DBI DB + $\omega \in V$. DBI

1·45 PROPOSITION ($\mathsf{DB}_0\mathsf{I}$) $[\omega]^1$ *and* $[\omega]^2$ *exist.*

Proof : $\omega \in V$ is an axiom of $\mathsf{DB}_0\mathsf{I}$. By the definition of ordered pair, $[\omega]^1 \cup [\omega]^2 \subseteq \bigcup(\omega \times \omega)$, and the result follows by Δ_0 separation. ⊣ (1·45)

If we add the axiom of infinity plus the scheme of foundation for all classes to DB we obtain the system BS as formulated on page 36 of Devlin's book *Constructibility*:

BS ReS_0 + Cartesian product + full foundation + $\omega \in V$. BS

The system BS is used extensively by Devlin in his study [De] of constructibility: for each limit ordinal ζ the set L_ζ in Gödel's constructible hierarchy models BS. But counterexamples of Solovay show that it is not quite strong enough for its intended tasks, one of which was to give a definition of the truth predicate $\models_u \varphi$ where u is a set and φ is a sentence of an appropriate object language. To decide whether an existential statement $\bigvee\mathfrak{x}\vartheta(\mathfrak{x})$ is true in a model $\mathfrak{M}$ (here the symbol $\bigvee$ is the existential quantifier of the object language), one considers the set $S_\vartheta =_{\mathrm{df}} \{\vartheta[\underline{a}] \mid a \in \mathfrak{M}\}$ of substitution instances of ϑ, where $\underline{a}$ is the constant of the relevant language interpreted by the element a.

1·46 Definition For each $\mathfrak{k} > 0$, we write $[\omega]^{\mathfrak{k}}$ for the class of subsets of ω of size $\mathfrak{k}$.

Now Model 6 of [M3, §5], where the defects of BS are discussed in detail, shows that although BS can prove the existence of $[\omega]^1$ and $[\omega]^2$ it cannot prove the existence of $[\omega]^3$, or indeed any $[\omega]^{\mathfrak{k}}$ for $\mathfrak{k} > 2$. Thus BS is unable to form the set S_ϑ and hence cannot define $\models$. The following strengthening suffices:

MW $\mathsf{DBI} + \forall a \forall k_{\in\omega}\, [a]^k \in V$ MW

That the truth relation $\models_u \varphi$ is, provably in MW, Δ_1-definable was shown in [M3, §10]. §9 of this paper will give a new proof of that, and also of the corresponding result for $\overset{0}{\models}$, which has this interesting consequence:

1·47 Theorem MW *is finitely axiomatisable, modulo one subtlety.*

Proof : We already know that DB_0 is; to that we have added an axiom of infinity, the axiom just given, and the scheme of Π_1 foundation. The subtlety is this: we use the truth definition for $\dot{\Delta}_0$ wffs: what are they ? Here we are quantifying in the language of discourse, not in the metalanguage, so we are getting slightly more than the scheme, but only in non-standard models will we be able to tell the difference. We invite the reader to complete the proof by using $\overset{0}{\models}$ to formulate $\dot{\Pi}_1$ foundation. ⊣ (1·47)

1·48 Remark In the transitive Model 7 of [M3, §5] MW is true but for some element a, $\{\bigcup x \mid_x x \in a\}$ is absent.

The system GJ

1·49 We now reach a system of the greatest importance in the study of constructibility, which was discovered independently by Gandy [G] and by Jensen

[J2]. The transitive models of this system are precisely the transitive sets closed under a certain collection $\mathcal{R}$ of functions, which we have yet to define. The members of this collection were called *basic* by Gandy and *rudimentary* by Jensen; the second adjective has been generally adopted in the literature, and is customarily shortened to *rud*. We follow that usage, and shall define a subcollection $\mathcal{B}$ of $\mathcal{R}$, calling the members of $\mathcal{B}$ *basic* functions. The transitive sets closed under the members of $\mathcal{B}$ are the transitive models of DB.

GJ_0 $\quad \mathsf{DB}_0 + \{x``\{w\} \mid w \in y\} \in V$ $\quad \mathsf{GJ}_0$

GJ $\quad \mathsf{GJ}_0$ + the scheme of Π_1 foundation. $\quad$ GJ

GJI $\quad \mathsf{GJ} + \omega \in V$. $\quad$ GJI

1·50 Proposition *The class* $\{x``\{w\} \mid w \in y\}$ *is* GJ_0*-suitable.*

In the next section we shall prove the important *eyebrow principle* that if F is a rudimentary function so is $F``$. For its proof we shall introduce *companions* and establish the Gandy–Jensen Lemma.

1·51 Remark An application of that principle is that $a \in V \implies \{\bigcup x \mid_x x \in a\} \in V$ is provable in GJ_0.

Again using the eyebrow principle, we may prove the following scheme of theorems:

1·52 Proposition (GJ_0) *For each set* a, $[a]^{\mathfrak{k}}$ *exists.*

Proof : $[a]^0 = \{\varnothing\} \in V$. $[a]^1 = A_0``a \in V$. $[a]^{\mathfrak{k}+1} = \bigl\{s \cup \{x\} \bigm| (s,x)_2 \in ([a]^{\mathfrak{k}} \times a) \cap \{(s,x)_2 \mid x \notin s\}\bigr\}$, which is in $V$, being of the form $h``b$ for some set b and rudimentary function h. ⊣ (1·52)

That scheme becomes a single theorem once the right instances of Π_1 foundation are available:

1·53 Theorem (GJ) $\forall a \forall k_{\in\omega}\ [a]^k \in V$.

Proof : Once we know Theorem 2·93 of [M3], which runs:

Theorem (GJ) $\forall a \forall m_{\in\omega}\ {}^m a \in V$.

where ${}^m a$ is the set of functions from m to a, and which is proved by using Π_1 foundation to find for given a the least counterexample m, we may again invoke the eyebrow principle to obtain the desired result, since

$$[a]^k = \{\Im(f) \mid_f f \in {}^k a \ \&\ f \text{ is injective}\}. \qquad \dashv (1{\cdot}53)$$

1·54 Corollary MW *is a subsystem of* GJI.

1·55 Remark §6 of [M3] recalls the result of Gandy [G] that GJI does not prove the existence of $\mathcal{S}(\omega)$. Thus by Theorem 1·53 the function $(a,k)_2 \mapsto [a]^k$ is not rudimentary.

2: Review of the elementary theory of rudimentary functions

We introduce the rudimentary functions $R_0, \dots R_8$ and certain auxiliary functions $A_0 \dots A_{14}$ generated by them under composition: this is not the shortest possible list, but one that conveniently extends the list, given in the axioms of DB_0, that generates the Δ_0 separators.

$$
\begin{aligned}
&R_0(x,y) = \{x,y\}\\
&\quad A_0(x) = \{x\}\,[= R_0(x,x)]\\
&\quad A_1(x,y) = (x,y)_2\,[= R_0(A_0(x), R_0(x,y))]\\
&\quad A_2(x,y,z) = \{x,(y,z)_2\}\\
&\quad A_3(x,y,z) = (x,y,z)_3\,[= A_1(x, A_1(y,z))]\\
&R_1(x,y) = x \setminus y\\
&\quad A_4(x,y) = x \cap y\,[= x \setminus (x \setminus y)]\\
&\quad A_5(x) = \varnothing\,[= x \setminus x]\\
&\quad A_6(x) = x\,[= x \setminus \varnothing]\\
&R_2(x) = \bigcup x\\
&R_3(x) = \mathrm{Dom}\,(x)\\
&R_4(x,y) = x \times y\\
&R_5(x) = x \cap \{(a,b)_2 \mid_{a,b} a \in b\}\\
&\quad A_7(x) = eps \restriction x\,[= R_5(\bigcup x \times x)]\\
&R_6(x) = \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}\\
&R_7(x) = \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}\\
&\quad A_8(x) = \{(a,c,b)_3 \mid_{a,b,c} (a,b,c)_3 \in x\}\,[= R_6(R_7(R_7(x)))]\\
&\quad A_9(x) = x^{-1}\,[= \mathrm{Dom}\,(\{(a,c,b)_3 \mid_{a,b,c} (a,b,c)_3 \in \{\varnothing\} \times x\})\\
&\qquad\qquad\qquad\qquad = R_3(A_8(R_4(A_0(A_5(x)), x)))]\\
&\quad A_{10}(x) = \Im(x)\,[= \mathrm{Dom}\,(x^{-1})]\\
&\quad A_{11}(x,y) = eps \cap (x \times y)\,[= R_5(x \times y)]\\
&\quad A_{12}(x,y) = \{w \mid_w x \in w \in y\}\,[= \mathrm{Dom}\,(A_{11}(\{x\} \times y))]\\
&\quad A_{13}(x,y) = id \cap (x \times y)\\
&\quad A_{14}(x,y) = x``\{y\}\,[= \mathrm{Dom}\,((x \cap ([\bigcup\bigcup x] \times \{y\}))^{-1})]\\
&R_8(x,y) = \{x``\{w\} \mid_w w \in y\}
\end{aligned}
$$

2·0 Proposition *Each of $R_0 \dots R_7$ and $A_0, \dots A_{14}$ is DB_0-suitable;*

Separators, basic functions and Δ_0 branching

2·1 Definition Let $\mathcal{R}$, the collection of *rudimentary functions*, be the closure of $R_0 \dots R_8$ under composition. Let $\mathcal{B}$, the collection of *basic functions*, be the closure of $R_0 \dots R_7$ under composition.

2·2 Proposition *(i) For each Δ_0 class A the map $x \mapsto x \cap A$ is in $\mathcal{B}$.*

(ii) It is a theorem of MW *that for each $\dot{\Delta}_0$ wff φ the map $a \mapsto a \cap \{x \mid \models^0 \varphi[x]\}$ is in $\mathcal{B}$.*

The Proposition is well expressed by the slogan "Δ_0 separators are basic".

The difference between the two results lies in the quantification, which in part (i) is in the metalanguage and in part (ii) in the language of discourse. So really we have cheated in not specifying in which language $\mathcal{B}$ is being defined. A similar ambiguity is inherent in our definition of $\mathcal{R}$.

Proposition 2.1 implies that branching over Δ_0 choices can be coded by rudimentary functions.

2·3 Proposition *For each Δ_0 class A the map* $x, y, z \mapsto \begin{cases} x & \text{if } z \in A \\ y & \text{otherwise.} \end{cases}$ *is rudimentary.*

Proof : The map can be expressed as

$$x, y, z \mapsto \mathrm{Dom}\,(x \times (\{z\} \cap A) \cup \mathrm{Dom}\,(y \times (\{z\} \cap (V \setminus A)) \qquad \dashv (2{\cdot}3)$$

Resolution of a question of Sy Friedman

We may now answer a question of Sy Friedman, whether a unary function F with a Δ_0 graph and such that for some $k \in \omega$ and all x, $\varrho(F(x)) \leqslant \varrho(x)+k$ is necessarily rudimentary.

Write **HF** for the class of hereditarily finite sets, defined as the union of all finite transitive sets.

2·4 Lemma *Let $k \in \omega$; then for any x, $x \subseteq V_{k+1} \iff \bigcup x \subseteq V_k$; hence $x \subseteq V_{k+1} \iff \bigcup^{k+1} x \subseteq V_0 = \varnothing$.*

2·5 Lemma *The predicate $a = \mathbf{HF}$ is Δ_0.*

Proof : To say $a = \mathbf{HF}$, say that $\varnothing \in a$, that a is transitive and closed under $\bigcup$ and (unordered) pairing, that no member of a is a limit ordinal, and if $b \in a$ then there is an $f \in a$ with domain a successor ordinal $\ell + 1$ such that $f(0) = b$, for every $k < \ell$, $f(k+1) = \bigcup f(k)$ and $f(\ell) = \varnothing$. $\dashv$

2·6 Proposition **HF** *is not a member of the rud closure of* $\{\omega\}$.

Proof : otherwise $\mathcal{S}(\omega)$ would be, contradicting the result of Gandy mentioned in Remark 1·55. $\dashv$ (2·6)

Now let F be $\{(x,y)_2 \mid_{x,y} (y = \omega \ \&\ x = \mathbf{HF}) \vee (y \neq \omega \ \&\ x = \varnothing)\}$. Then F is a function, its graph is Δ_0 and for any y, $\varrho(F(y)) \leqslant \varrho(y)$. But F is not rudimentary, for $F(\omega) = \mathbf{HF}$.

Companions for rudimentary functions

The collection of functions in $\mathcal{R}$ is closed under formation of images: by which is meant that if F is in $\mathcal{R}$ so is $x \mapsto F``x$. To prove that, we introduce the notion of a *companion*—we will actually have two such notions—and establish the Gandy–Jensen Lemma.

Let S be some system of set theory extending DB_0, and let G and F be Δ_0 classes such that S proves that both G and F are total functions.

2·7 Definition G is a *1-companion* of F in S if G is S-suitable and

$$\vdash_{\mathsf{S}} \vec{x} \in \vec{u} \implies F(\vec{x}) \downarrow\in G(\vec{u})$$

2·8 Definition H is a *2-companion* of F in S if H is S-suitable and

$$\vdash_{\mathsf{S}} \vec{x} \in \vec{u} \implies F(\vec{x}) \downarrow\subseteq H(\vec{u})$$

where $\vec{x} \in \vec{u}$ abbreviates $x_1 \in u_1$ & $\dots x_n \in u_n$ for an appropriate n.

2·9 PROPOSITION *If G^1 is a 1-companion of G in S and H^1 is a 1-companion of H in S, then $G^1 \circ H^1$ is a 1-companion of $G \circ H$ in S.*

The function F“, if available in S, is the best 1-companion of F in S, and in favourable cases separators may be used to reduce a given 1-companion F^1 of F to that one, since

$$\vdash_{\mathsf{S}} F\text{“}a = F^1(a) \cap \{y \mid \exists x_{\in a}\; y = F(x)\}$$

so that if F is given by an S-suitable term,

$$\vdash_{\mathsf{S}} y = F(x) \iff \forall w_{\in y}\, w \in F(x) \;\&\; \forall w_{\in F(x)}\, w \in y$$

2·10 PROPOSITION *Each one of the nine functions $R_0, \dots, R_7$ and A_{14} has a 2-companion in DB.*

Proof :
R_0: $a \in x \,\&\, b \in y \implies \{a, b\} \subseteq x \cup y = \bigcup\{x, y\}$.
R_1: $a \in x \,\&\, b \in y \implies a \setminus b \subseteq a \subseteq \bigcup x$.
R_2: $a \in x \implies \bigcup a \subseteq \bigcup\bigcup x$.
R_3: $a \in x \implies \mathrm{Dom}\,(a) \subseteq \bigcup\bigcup x$.
R_4: $a \in x \,\&\, b \in y \implies a \times b \subseteq \bigcup x \times \bigcup y$.
R_5: $t \in x \implies t \cap \{(a, b)_2 \mid a \in b\} \subseteq t \subseteq \bigcup x$.
R_6: If $t \in x$, $\{(b, a, c)_3 \mid (a, b, c)_3 \in t\}$ is, by reasoning similar to that given below for R_7, a subset of $\Im(\mathrm{Dom}\,(\bigcup x)) \times (\Im(\bigcup x) \times \mathrm{Dom}\,(\mathrm{Dom}\,(\bigcup x)))$.
R_7: Let $t \in x$. It is enough to show that $\{(b, c, a)_3 \mid (a, b, c)_3 \in t\}$ is a subset of $\Im(\mathrm{Dom}\,(\bigcup x)) \times (\mathrm{Dom}\,(\mathrm{Dom}\,(\bigcup x)) \times \Im(\bigcup x))$. To see that, note that $\{(b, c, a)_3 \mid (a, b, c)_3 \in t\} \subseteq \Im(\mathrm{Dom}\,(t)) \times (\mathrm{Dom}\,(\mathrm{Dom}\,(t)) \times \Im(t))$, and apply these principles: $t \in x \implies t \subseteq \bigcup x$; $t \subseteq s \implies \mathrm{Dom}\,(t) \subseteq \mathrm{Dom}\,(s)$; $t \subseteq s \implies \Im(t) \subseteq \Im(s)$; and $t \subseteq s \,\&\, v \subseteq u \implies t \times v \subseteq s \times u$.
A_{14}: $a \in x$ & $b \in y \implies a\text{“}\{b\} \subseteq \Im(\bigcup x)$. ⊣ (2·10)

2·11 REMARK The above 2-companions are generated by the four functions $\Im$, Dom , $\bigcup$ and $\times$. We can get that down to two, $\bigcup$ and $\times$, by using the above principles. For u transitive, a single generator, namely the function $u \mapsto u^\star =_{\mathrm{df}} u \cup [u]^{\leqslant 2} \cup (u \times u)$, is enough.

2·12 PROPOSITION *If F has a 1-companion F^1 then $\bigcup F^1$ is a 2-companion of F.*

2·13 PROPOSITION *If G has a 2-companion G^2 and H has a 1-companion H^1, then $G^2 \circ H^1$ is a 2-companion of $G \circ H$.*

The Gandy–Jensen Lemma

The Gandy–Jensen Lemma is the core of the proof that $\mathcal{R}$ is closed under formation of images. Versions of it are to be found in the papers of Gandy [G] and Jensen [J2]. We discuss it only for 1-ary functions. The extension to n-ary functions poses no problems.

2·14 THE GANDY–JENSEN LEMMA *Let S be a system extending DB_0. Suppose that H is a 2-companion of F in S, and that ‘$a \in F(b)$’ is Δ_0^{S}. Then F is* *generated by composition from H and members of $\mathcal{B}$, and so is* S*-suitable; if in addition* S *extends* GJ*, then* $\vdash_{\mathsf{S}} F``x \in V$ *and* $F``$ *(as a function) is generated by H and members of $\mathcal{R}$ and (as a term) is* S*-suitable and is a 1-companion of F in* S.

Proof : We have

$$\vdash_{\mathsf{S}} x \in u \implies F(x) \subseteq H(u).$$

Working in S, form

$$h(u) =_{\mathrm{df}} \big(H(u) \times u\big) \cap \{(a,b)_2 \mid_{a,b} b \in u \;\&\; a \in F(b)\}.$$

Since "$a \in F(b)$" is Δ_0^{S} and for each Δ_0 A, the function $x \mapsto x \cap A$ is in $\mathcal{B}$ and is DB-suitable, we have that h is S-suitable, and is generated by H and functions in $\mathcal{B}$.

Now note that for $b \in u$, $F(b) = h(u)``\{b\} = A_{14}(h(u), b)$, so $F$ is built from $H$ and functions in $\mathcal{B}$; if $R_8$ is available in the system $\mathsf{S}$, we may argue further that $F``u = R_8(h(u), u)$ so $F``$ is built from $H$ and rudimentary functions, and is thus $\mathsf{S}$-suitable; hence $\vdash_{\mathsf{S}} F``u \in V$, and the function $F``$ now forms a 1-companion of F in S. ⊣ (2·14)

2·15 Proposition *R_8 has a 2-companion in* GJ.

Proof : By the Gandy–Jensen lemma, $A_{14}``$ is GJ-suitable, and so for a in x and b in y,

$$R_8(a,b) = \{A_{14}(a,w) \mid w \in b\} = A_{14}``(\{a\} \times b) \subseteq A_{14}``(x \times \bigcup y). \quad \dashv (2\cdot15)$$

2·16 Corollary *R_8 has a 1-companion in* GJ.

Proof : by the Gandy–Jensen Lemma. ⊣ (2·16)

2·17 Theorem *$\mathcal{R}$ is closed under formation of images and of unions of images.*

Proof : We have seen that each of $R_0, \ldots R_8$ has a 1-companion in GJ; the collection of functions possessing a 1-companion is closed under composition, and hence each function in $\mathcal{R}$ has a 1-companion in GJ; but if G is a 1-companion of F then $u \mapsto \bigcup(G(u))$ is a 2-companion of F. Hence each function F in $\mathcal{R}$ has a 2-companion in GJ; each such function is GJ-suitable, Proposition 1·11 proving the survival of suitability under composition, and so by the Gandy–Jensen lemma, $F``$ is in $\mathcal{R}$; composition with $\bigcup$ yields the last clause. ⊣ (2·17)

2·18 Remark Gandy shows in [G] that these three are equivalent: (i) F is rudimentary; (ii) "$a \in F(b)$" is Δ_0 and F has a 1-companion in GJ; (iii) "$a \in F(b)$" is Δ_0 and F has a 2-companion in GJ.

2·19 Remark Gandy in [G] and Jensen in [J2] supply other characterisations of $\mathcal{R}$ and other axiomatisations of GJ.

3: A single generating function for rud(u)

In developing further properties of the collection of rudimentary functions we shall use the function $\mathbb{T}$ introduced in Definition 2·73 of *Weak Systems*.

The function $\mathbb{T}$

3·0 Definition
$$\begin{aligned}\mathbb{T}(u) =_{\text{df}}\ & u \cup \{u\} \\ & \cup [u]^1 \cup [u]^2 \\ & \cup \{x \smallsetminus y \mid_{x,y} x, y \in u\} \\ & \cup \left\{\bigcup x \,\Big|_x\, x \in u\right\} \\ & \cup \left\{\mathrm{Dom}\,(x) \,\Big|_x\, x \in u\right\} \\ & \cup \{u \cap (x \times y) \mid_{x,y} x, y \in u\} \\ & \cup \left\{x \cap \{(a,b)_2 \mid_{a,b} a \in b\} \,\Big|_x\, x \in u\right\} \\ & \cup \left\{u \cap \{(b,a,c)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \,\Big|_x\, x \in u\right\} \\ & \cup \left\{u \cap \{(b,c,a)_3 \mid_{a,b,c} (a,b,c)_3 \in x\} \,\Big|_x\, x \in u\right\} \\ & \cup \left\{x``\{w\} \,\Big|_{x,w}\, x \in u, w \in u\right\} \\ & \cup \left\{u \cap \left\{x``\{w\} \,\Big|_w\, w \in y\right\} \,\Bigg|_{x,y}\, x, y \in u\right\}.\end{aligned}$$

3·1 Remark The successive lines of the definition of $\mathbb{T}$, after the first, may be written more prosaically as $R_0``(u \times u)$, $R_1``(u \times u)$, $R_2``u$, $R_3``u$, $\{u \cap R_4(x,y) \mid_{x,y} x, y \in u\}$, $R_5``u$, $\{u \cap R_6(x) \mid_x x \in u\}$, $\{u \cap R_7(x) \mid_x x \in u\}$, $A_{14}``(u \times u)$ and $\{u \cap R_8(x,y) \mid_{x,y} x, y \in u\}$. It will be notationally convenient to treat all these functions as having three variables, so let us define $S_i(u; x, y) := R_i(x,y)$ for $i = 0, 1$; $S_i(u; x, y) := R_i(x)$ for $i = 2, 3, 5$; $S_i(u; x, y) := u \cap R_i(x,y)$ for $i = 4, 8$; $S_i(u; x, y) := u \cap R_i(x)$ for $i = 6, 7$; and $S_9(u; x, y) := A_{14}(x, y)$. Then each of those lines now takes the form $S_i``(\{u\} \times (u \times u))$ for some i.

We have proved the first clause of the following, and the others are easy.

3·2 Proposition *$\mathbb{T}$ is rudimentary, $u \subseteq \mathbb{T}(u)$ and $u \in \mathbb{T}(u)$. Further, if u is transitive, then $\mathbb{T}(u)$ is a set of subsets of u, and hence $\mathbb{T}(u)$ is transitive.*

3·3 Remark It will not in general be true that $u \subseteq v \implies \mathbb{T}(u) \subseteq \mathbb{T}(v)$, the problem being that $u \in \mathbb{T}(u)$, but if v is countably infinite, so is $\mathbb{T}(v)$ which therefore cannot contain all the subsets of v. Fortunately, $u \subseteq \mathbb{T}(u) \subseteq \mathbb{T}^2(u) \ldots$

3·4 Lemma *If x and y are in u, then $R_0(x,y)$, $R_1(x,y)$, $R_2(x)$, $R_3(x)$, and $R_5(x)$ are all in $\mathbb{T}(u)$.*

In the next five results, it is supposed that u is transitive.

3·5 Lemma *For x, y in u, $R_4(x,y) = x \times y \subseteq u \times u \subseteq \mathbb{T}^2(u)$.*

3·6 Corollary *For x, y in u, $R_4(x,y) \in \mathbb{T}^3(u)$.*

3·7 Lemma *For a, b, c in u, $(a,c)_2 \in \mathbb{T}^2(u)$ and $(b,a,c)_3 \in \mathbb{T}^4(u)$.*

3·8 Corollary *For $x \in u$, $R_6(x)$ and $R_7(x)$ are in $\mathbb{T}^5(u)$.*

3·9 LEMMA *For x, $y \in u$, $R_8(x,y) \in \mathbb{T}^2(u)$.*

Proof : For x, w in u, $x``w \in \mathbb{T}(u)$, so $R_8(x,y) = \mathbb{T}(u) \cap \{x``w \mid_w w \in y\}$; $x, y \in \mathbb{T}(u)$, so $R_8(x,y) \in \mathbb{T}^2(u)$. ⊣ (3·9)

Those remarks, which were proved in *Weak Systems*, though regrettably without the requirement that u be transitive being clearly stated, and of which more general forms will be proved below, immediately yield:

3·10 PROPOSITION *If $F(\vec{x})$ is a rudimentary function of several variables, there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in \mathbb{T}^\ell(u)$.*

Proof : The stated property holds of the nine generating functions and is preserved under composition. ⊣ (3·10)

3·11 REMARK Strictly, we should give this as two different results, like Proposition 2·1, in one of which we quantify in the metalanguage (and so get a fact about each externally definable rudimentary function) and in the other of which we quantify internally, and so get a single fact about the internal set of all (codes for) rudimentary functions.

3·12 COROLLARY (Gandy; Jensen) *If F is rudimentary, then there is a finite ℓ such that the rank of the value is at most the maximum of the ranks of the arguments, plus ℓ.*

Proof : the function $\mathbb{T}$ increases rank by exactly 1. ⊣ (3·12)

3·13 COROLLARY *For any transitive u, $\bigcup_{n\in\omega}\mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$ and models* TCo.

Functions rudimentary in a predicate

Let B be a unary predicate. The collection of functions *rudimentary in* B is that obtained by adding to the generators of $\mathcal{R}$ the function $x \mapsto x \cap B$.

To extend Proposition 3·10 to the collection of functions rudimentary in B, we introduce a function $\mathbb{T}_B$, rudimentary in B, given by

$$\mathbb{T}_B(u) = \mathbb{T}(u) \cup \{x \cap B \mid x \in \mathbb{T}(u)\}\,.$$

3·14 PROPOSITION *If $F(\vec{x})$ is a rudimentary function in B of several variables, then there is an $\ell \in \omega$ such that for all transitive u, if each argument in $\vec{x}$ is in u, then $F(\vec{x}) \in (\mathbb{T}_B)^\ell(u)$.*

EXAMPLE $\mathbb{T}_{ON}$, used in 6·86, is itself rudimentary, by Proposition 2·1(i).

The intransitive case

The function $\mathbb{T}$ works very happily for transitive argument, but for intransitive argument it starts to create non-trivial problems. The aim, in the two cases, is not quite the same. The purpose of $\mathbb{T}$ is to proceed by rud steps from any transitive set u to $\mathrm{rud}(u)$, which will be of strictly greater rank; with an intransitive argument of limit rank, our first concern would be to fatten it to a transitive rud closed set, without raising rank. Here are two ways of doing so, using the new functions trud and krud.

3·15 DEFINITION $\mathrm{trud}(u) \ =_{\mathrm{df}} \ \bigcup\{F(\vec{x}) \mid F \text{ rud} \ \& \ \vec{x} \in u\}$.

That is a legitimate definition because we are quantifying over programs for rud functions; the axiom of infinity is at work here. Here $\vec{x}$ denotes a finite sequence of arguments of F, and we follow Devlin's convention that $\vec{x} \in u$ means that each argument is in u; if we wanted to say that the sequence is in u we would write $\langle\vec{x}\rangle \in u$.

3·16 PROPOSITION *For any set u,* $\mathrm{trud}(u)$ *is transitive, rud closed and includes u; and if A is transitive, rud closed and includes u, then* $\mathrm{trud}(u) \subseteq A$. *The rank of* $\mathrm{trud}(u)$ *will be the least limit ordinal greater than or equal to the rank of u.*

Proof : If $a \in b \in F(\vec{x})$, then $a \in \bigcup F(\vec{x}) \subseteq \mathrm{trud}(u)$, $\bigcup \circ F$ being rud; and so $\mathrm{trud}(u)$ is transitive.

If $G(\cdot,\cdot)$ is rud, $b_1 \in F_1(\vec{x})$, $b_2 \in F_2(\vec{y})$, then $G(b_1, b_2) \in G``H(\vec{x}, \vec{y})$ for some rud $H$; $G``\circ H$ is rud, and so $G(b_1, b_2) \in \mathrm{trud}(u)$. Similarly for functions of a different number of variables.

If $a \in u$ then $a \in \{a\} \subseteq \mathrm{trud}(u)$.

If $\vec{x} \in u$ then $\vec{x} \in A$ as A includes u; then $F(\vec{x}) \in A$, A being rud closed; so $F(\vec{x}) \subseteq A$, as A is transitive. Thus $\mathrm{trud}(u) \subseteq A$. $\dashv$ (3·16)

The definition of trud can be given recursively.

3·17 DEFINITION $\mathbb{K}(u) = u \cup \bigcup u \cup \{R_i(x, y, z) \mid 0 \leqslant i \leqslant 8 \ \& \ x, y, z \in u \cup \bigcup u\}$.

That definition is intended for use even when u is intransitive. Note that $\mathbb{K}$ is rudimentary, and that it has the agreeable property that $u \subseteq v \implies \mathbb{K}(u) \subseteq \mathbb{K}(v)$.

3·18 DEFINITION $\mathbb{K}_0(u) = u$; $\mathbb{K}_{n+1}(u) = \mathbb{K}(\mathbb{K}_n(u))$; $\mathrm{krud}(u) = \bigcup_{n\in\omega} \mathbb{K}_n(u)$.

3·19 PROPOSITION *For any u,* $\mathrm{krud}(u) = \mathrm{trud}(u)$.

Proof : plainly $\mathrm{krud}(u)$ includes u, is transitive and is rud closed; so $\mathrm{trud}(u) \subseteq \mathrm{krud}(u)$.

If $u \subseteq A$ where A is transitive, rud closed and includes u then one verifies by an easy induction that each $\mathbb{K}_n(u) \subseteq A$. Hence $\mathrm{krud}(u) \subseteq \mathrm{trud}(u)$. $\dashv$ (3·19)

3·20 REMARK $\mathbb{K}$ has the property that for any rud function R there is a d such that $\mathbb{K}^d$ is a 1-companion of R.

Gandy reproved

The proofs of a couple of the very interesting results of Gandy's paper *Set theoretic functions* are unfortunately flawed, which may have resulted from Gandy encountering similar difficulties to those created by "the intransitive case". We shall give a brief review of the problems, and shall explain how to obtain proofs of those results which are right but not supported by Gandy's arguments as they stand. See especially Propositions 3·25 and 3·27 below.

The first problem is in his Lemma 1.5.3. on page 111. We will begin our discussion from his definition 1.5.2: he uses a bold-face $\mathbf{x}$ to denote the (meta) finite sequence $x_1, \ldots x_m$: cf the bottom of page 105. This usage is a little ambiguous; the letter m here may be a variable of the meta-language.

Let us for simplicity take the case $m = 1$, and write x for x_1. Then the first part of his Definition 1.5.2 runs

$$\mathrm{Cc}_0\{x\} = \{x\};$$
$$\mathrm{Cc}_{q+1}\{x\} = Cc_q\{x\} \cup \{\mathrm{Cc}_q\{x\}\} \cup \{\mathbf{F}_i uv : 1 \leqslant i \leqslant 9 \;\&\; u, v \in \mathrm{Cc}_q\{x\}\}.$$

For the purposes of this discussion, we shall take the letter q here to be a variable of the language of discourse.

3·21 PROPOSITION *For any $q \in \omega$ and any x, $\mathrm{Cc}_q\{x\}$ is a finite set.*

Proof : by induction on q. Indeed, for a given x, let n_q be the number of elements in $\mathrm{Cc}_q\{x\}$. Then $n_0 = 1$; $n_{q+1} \leqslant n_q + 1 + 9 \cdot {n_q}^2$. ⊣ (3·21)

3·22 Thus the second statement of part (ii) of Lemma 1.5.3 is false: if x is actually an infinite set, it cannot be a subset of any $\mathrm{Cc}_q\{x\}$.

Similarly, $\mathrm{Cc}\{x\}$ is defined as $\bigcup_{q\in\omega} \mathrm{Cc}_q\{x\}$; which will be a countable infinite set; so if x is uncountable, it cannot be a subset of $\mathrm{Cc}\{x\}$, even if it is transitive.

Lemma 1.5.4 is also incorrect—the difficulty is with step (C) of the proof. The 'only if' direction of Theorem 1.5.5, which relies on Lemma 1.5.4, is also wrong. Theorem 1.5.6 is false: $\mathrm{Bc}\{x\}$ is always transitive but $\mathrm{Cc}\{x\}$ need not be.

We now turn to the ways in which some of the correct results may be recovered.

3·23 LEMMA *If u is a finite transitive set with $\overline{\overline{u}} = \ell$, then $\overline{\overline{\mathbb{T}(u)}} \leqslant \frac{1}{2}(2+13\ell+9\ell^2)$.*

Proof : by inspection. ⊣ (3·23)

3·24 DEFINITION (Gandy) $\eta(x) =_{\mathrm{df}}$ the cardinal of the transitive closure of x.

3·25 PROPOSITION (Gandy) *If F is rud, then there is a k such that $\eta(F(\vec{x}))$ is less than $(\eta(\{\vec{x}\}) + 1)^k$.*

Here $\{\vec{x}\}$ for many variables means the set of them.

Proof : we know that there is an ℓ such that for u transitive and the arguments of F in u, $F(\vec{x}) \in \mathbb{T}^\ell(u)$. For u transitive, $\mathbb{T}(u)$ is transitive, and iterating the previous estimate, we find that there is a polynomial $Q(X)$ of degree 2^ℓ, (for example $13^{2^\ell-1}X^{2^\ell}$) such that $x \in u$ implies that $\eta(F(\vec{x}))$ is at most $Q(\overline{\overline{u}} + 1)$. ⊣ (3·25)

3·26 REMARK We may now justify our earlier remark that there is no pure rud recursion for $\mathcal{S}(x)$ for x an arbitrary set. If we look at $\mathcal{S}(x)$ for $x \in \mathbf{HF}$, we see that $\mathcal{S}(V_n) = V_{n+1}$; if $\mathcal{S}(x)$ were pure rud rec, given by G, we would have

$$G(\mathcal{S} \restriction V_n) = V_{n+1}.$$

But if $\overline{\overline{V_n}} = N$, $\overline{\overline{V_{n+1}}} = 2^N$, whereas

$$\mathrm{tcl}(\mathcal{S} \restriction V_n) \subseteq \{(\mathcal{S}(x), x) \mid x \in V_n\} \cup \{\{\mathcal{S}(x)\} \mid x \in V_n\}$$
$$\cup \{\{\mathcal{S}(x),\, x\} \mid x \in V_n\} \cup \{\mathcal{S}(x) \mid x \in V_n\} \cup V_n$$

which has cardinality at most $5N$; but for each k, $(5N)^k$ will, for large N, be much less than 2^N. ⊣ (3·26)

Gandy remarks on page 114 that there is a primitive recursive function which returns the value ω given any argument of infinite rank. Indeed the example he gives is rud rec: define

$$F(x) = \omega \cap \bigcup \{ F(y) \cup \{F(y)\} \mid_y y \in x \},$$

which is rud rec as intersection with ω is given by a Δ_0 separator; and show first that if $x \in \mathbf{HF}$, then $F(x) = \varrho(x)$. His Theorem 2.1.3 then states:

3·27 Proposition (Gandy) *There is a set c of infinite rank such that for no rud function G is $G(c) = \omega$.*

Indeed there will be many such sets in any transitive model of Z containing sets of infinite rank but not ω, as such models are automatically rud closed and absolute for rud functions. Proposition 8·12 and [M7, §2] give examples.

4: The collections of pure rud rec and gentle functions

4·0 Definition (Mathias): By type I or pure rudimentary recursions we mean those given by a recursion equation of the form

$$F(x) = G(F \restriction x)$$

where G is a pure rud function with no hidden parameters. We call functions which may be defined in this way *rudimentary recursive*, or *rud rec*. For example, as was shown in the introduction, the rank function ϱ is rud rec. We will now explore the closure properties of rud rec functions.

4·1 Proposition *Every (unary) rud function is rud rec.*

Proof: If $F(\cdot)$ is unary and rud, let $G(f) =_{\mathrm{df}} F(\mathrm{Dom}\,(f))$; then G is rud and $\forall x\ F(x) = G(F \restriction x)$. Other rud functions can be transformed to unary functions by using the pairing and un-pairing functions, which are rudimentary. ⊣ (4·1)

4·2 Proposition *If F_1 and F_2 are rud rec, so is $x \mapsto (F_1(x), F_2(x))_2$.*

Proof: Let $K(x) = \big(F_1(x), F_2(x)\big)_2$. Then $K(x) = \big(G_1(F_1 \restriction x), G_2(F_2 \restriction x)\big)_2$, and $K \restriction x = \{\big((F_1(a), F_2(a))_2, a\big)_2 \mid_a a \in x\}$. There are rud G_3 and G_4 such that $G_3(K \restriction x) = F_1 \restriction x$ and $G_4(K \restriction x) = F_2 \restriction x$. So

$$K(x) = \big(G_1(G_3(K \restriction x)), G_2(G_4(K \restriction x))\big)_2 = G_5(K \restriction x)$$

where $G_5(z) =_{\mathrm{df}} \big(G_1(G_3(z)), G_2(G_4(z))\big)_2$. G_5 is rudimentary. ⊣ (4·2)

4·3 Proposition *Let G_1 and G_2 be rudimentary, and suppose that F_1 and F_2 are defined by the simultaneous recursion*

$$F_1(x) = G_1(F_1 \restriction x, F_2 \restriction x); \qquad F_2(x) = G_2(F_1 \restriction x, F_2 \restriction x).$$

Then the function $x \mapsto (F_1(x), F_2(x))_2$ is rud rec.

Proof : Let $K(x) = \big(F_1(x), F_2(x)\big)_2$. Then $K(x) = (G_1(F_1\restriction x, F_2\restriction x), G_2(F_1\restriction x, F_2\restriction x))_2$, and $K\restriction x = \{\big((F_1(a), F_2(a))_2, a\big)_2 \mid_a a \in x\}$. There are rud G_3 and G_4 such that $G_3(K\restriction x) = F_1\restriction x$ and $G_4(K\restriction x) = F_2\restriction x$. So

$$K(x) = \big(G_1(G_3(K\restriction x), G_4(K\restriction x)), G_2(G_3(K\restriction x), G_4(K\restriction x))\big)_2 = G_6(K\restriction x)$$

where $G_6(z) =_{\mathrm{df}} \big(G_1(G_3(z), G_4(z)), G_2(G_3(z), G_4(z))\big)_2$. G_6 is rudimentary. ⊣ (4·3)

4·4 Corollary *Let F be a rud rec function and H a rud function. Then $H\circ F$ is a projection of a rud rec function.*

Proof : Suppose that F is given by $F(x) = G(F\restriction x)$. Then F and $H\circ F$ are definable by the simultaneous recursion given by that equation and $H\circ F(x) = H(G(F\restriction x))$, and hence Proposition 4·3 applies. ⊣ (4·4)

Significantly, $H\circ F$ need not be rud rec:

4·5 Proposition (Bowler) *The function* $H : x \mapsto \begin{cases} \omega & \text{if } \varrho(x) = \omega \\ 0 & \text{otherwise.} \end{cases}$ *is a composite of a rud function with a rud rec fuction, but is not rud rec.*

Proof : H is the composite $\delta_\omega\circ\varrho$, where the function $\delta_\omega : x \mapsto \begin{cases} \omega & \text{if } x = \omega \\ 0 & \text{otherwise} \end{cases}$ is rudimentary by Proposition 2·2. For any unary rud G and ℓ as in Proposition 3·10, (in fact $\ell = c_G$ as in Definition 6·13) and any transitive x,

$$\bigcup{}^{\ell+1} G(x) \subseteq \bigcup{}^{\ell+1} G''(x\cup\{x\}) \subseteq \bigcup{}^{\ell+1} \mathbb{T}^\ell(x\cup\{x\}) = \bigcup(x\cup\{x\}) = x.$$

Suppose that H were rud rec, given by G_0 say. Let $\ell = c_G$ where G is the rud function $G : y \mapsto G_0(\{0\}\times y)$. Let Z be the transitive set, of rank ω, of Zermelo integers: $Z = \{s^n(\varnothing) \mid n\in\omega\}$, where $s : x\mapsto \{x\}$. Then

$$\begin{aligned}\omega = \bigcup{}^{\ell+1}\omega = \bigcup{}^{\ell+1} H(Z) = \bigcup{}^{\ell+1}(G_0(H\restriction Z)) &= \bigcup{}^{\ell+1}(G_0(\{0\}\times Z)) \\ &= \bigcup{}^{\ell+1}(G(Z)) \subseteq Z\text{—a falsehood !}\end{aligned}$$ ⊣ (4·5)

We therefore turn our attention to a collection of functions with better closure properties: those of the form $G\circ F$ with G rud and F rud rec. We call such functions *gentle*. Our first concern will be to show that, unlike the collection of rud rec functions, the collection of gentle functions is closed under composition.

4·6 Lemma *Let F be rud rec, given by $F(x) = G(F\restriction x)$ where G is rud. Then there is a rud function H_G obtainable uniformly from G such that for every u, not necessarily transitive, and every $v\subseteq \mathcal{P}u$, $F\restriction v = H_G(v, F\restriction u)$.*

Proof : For $x\in v$, $F\restriction x = (F\restriction u)\restriction x$. Let $\phi(f, x) = (G(f\restriction x), x)_2$. Then ϕ is rud, and

$$F\restriction v = \{\phi(F\restriction u, x) \mid_x x\in v\} = H_G(v, F\restriction u)$$

where H_G is rud. ⊣ (4·6)

4·7 Corollary *Let F be rud rec, given by $F(x) = G(F\restriction x)$ where G is rud. Then there is a rud function $H^{\mathbb{T}}_G$ obtainable uniformly from G such that for every transitive u, $F\restriction\mathbb{T}(u) = H^{\mathbb{T}}_G(F\restriction u)$.*

Proof : We take $H_G^{\mathbb{T}}(f) = H_G(\mathbb{T}(\mathrm{Dom}\,(f)), f)$. ⊣ (4·7)

4·8 PROPOSITION (Bowler) *If F_1 and F_2 are rud rec, then $F_1 \circ F_2$ is gentle.*

Proof : Let F_1 be given by $F_1(x) = G_1(F_1 \restriction x)$ and F_2 by $F_2(x) = G_2(F_2 \restriction x)$.

We say that a set is *sufficient for* x if it is the restriction of F_1 to a transitive set u containing $(F_2(x), x)_2$. We proceed by showing that there is a function F, definable by mutual rudimentary recursion with F_2 as in Proposition 4·3, with the property that for any x $F(x)$ is sufficient for x. For that, we must find a rudimentary function E in two variables such that, for any function f with domain x and sending each y in x to a set sufficient for y, $E(F_2 \restriction x, f)$ is sufficient for x.

Suppose we have such an f, with $f(y) = F_1 \restriction u(y)$ for each $y \in x$ (where we write $u(y)$ for the domain of $f(y)$). Then $\bigcup \Im(f) = F_1 \restriction \bar{u}$, where $\bar{u} =_{\mathrm{df}} \bigcup_{y \in x} u(y)$ is a transitive set of which $F_2 \restriction x$ is a subset. Thus $H_{G_1}(\bar{u} \cup \{F_2 \restriction x\}, F_1 \restriction \bar{u}) = F_1 \restriction (\bar{u} \cup \{F_2 \restriction x\})$ is a restriction of F_1 to a transitive set containing $F_2 \restriction x$. The rudimentary function $K : f \mapsto (G_2(f), \mathrm{Dom}\,(f))_2$ has the property that for any x we have $K(F_2 \restriction x) = (F_2(x), x)_2$. So if we choose ℓ as in Proposition 3·10 for this K, then $(H_{G_1}^{\mathbb{T}})^{\ell}(F_1 \restriction (\bar{u} \cup \{F_2 \restriction x\})) = F_1 \restriction \mathbb{T}^{\ell}(\bar{u} \cup \{F_2 \restriction x\})$ is sufficient for x. So the rudimentary function

$$E : g, f \mapsto (H_{G_1}^{\mathbb{T}})^{\ell}(H_{G_1}(\mathrm{Dom}\,(\bigcup \Im(f)) \cup \{g\}, \bigcup \Im(f)))$$

has the property stated above: the function F defined by $F(x) = E(F_2 \restriction x, F \restriction x)$ sends each x to something sufficient for x.

By Proposition 4·3, $x \mapsto (F_2(x), F(x))_2$ is rud rec. Thus $F_1 \circ F_2$ is gentle, as it can be obtained by precomposing this rud rec function with the rudimentary function $q \mapsto \mathrm{right}(q)(\mathrm{left}(q))$. ⊣ (4·8)

4·9 THEOREM (Bowler) *Any composite of gentle functions is gentle.*

Proof : Suppose that H_1 and H_2 are gentle, with H_i given by $H_i = G_i \circ F_i$ with G_i rud and F_i rud rec. Then by Propositions 4·1 and 4·8 $F_2 \circ G_1$ is gentle—say it is given by $G \circ F$ with G rud and F rud rec. By Proposition 4·8 again, $F \circ F_1$ is gentle—say it is given by $G' \circ F'$ with G rud and F rud rec. Thus $H_2 \circ H_1 = G_2 \circ F_2 \circ G_1 \circ F_1 = G_2 \circ G \circ F \circ F_1 = (G_2 \circ G \circ G') \circ F'$ is gentle. ⊣ (4·9)

The collection of gentle functions is closed in other good ways: for example, by Proposition 4·2 if H_1 and H_2 are gentle then so is $x \mapsto (H_1(x), H_2(x))_2$.

4·10 PROPOSITION *If F is rud rec, so is $x \mapsto F \restriction x$.*

Proof : Let F be given by G, and let $H(x) = F \restriction x$. Then

$$\begin{aligned} H(x) &= F \restriction x \\ &= \{(F(a), a)_2 \mid_a a \in x\} \\ &= \{\big(G(F \restriction a), a\big)_2 \mid_a a \in x\} \\ &= \{\big(G(H(a)), a\big)_2 \mid_a a \in x\} \\ &= G_2(H \restriction x) \end{aligned}$$

where, setting G_1 to be the rud function $x \mapsto \big(G(\mathrm{left}(x)), \mathrm{right}(x)\big)_2$, we take $G_2(x) =_{\mathrm{df}} G_1``x$. ⊣ (4·10)

4·11 Corollary *If H is gentle, then $H``$, being equal to $Im \circ (H\restriction)$, is also gentle.*

Thus any gentle function has a gentle 1-companion. It is also clear that any gentle function has a gentle 2-companion, obtained by precomposing this 1-companion with $\bigcup$.

Gentle predicates

4·12 Proposition *Let B be a predicate. The following are equivalent:*
(i) *The characteristic function of B is gentle.*
(ii) *The separator $x \mapsto x \cap B$ is gentle.*

Proof : $(i) \Rightarrow (ii)$ is immediate from Proposition 4·10, and $(ii) \Rightarrow (i)$ follows from Theorem 4·9 and the fact that $x \in B$ iff $\{x\} \cap B \neq \varnothing$. ⊣ (4·12)

We call predicates with those properties *gentle*, and may view the Proposition as saying that gentle predicates give gentle separators. There is a variant of Corollary 4·7 for $\mathbb{T}^B$ with B gentle.

4·13 Lemma *If B is a gentle predicate, with the function $x \mapsto x \cap B$ given by $H \circ F$ with H rudimentary and F rud rec, then there is a rudimentary function $G^{\mathbb{T}^B}$ such that, for any transitive set u, $G^{\mathbb{T}^B}(F\restriction u) = F\restriction \mathbb{T}^B(u)$.*

Proof : Let F be given by $F(x) = G(F \restriction x)$. We take the function $f \mapsto H_G(\mathbb{T}(\mathrm{Dom}\,(f)) \cup \Im(H \circ f), f)$. ⊣ (4·13)

4·14 Lemma *If B is a gentle predicate, with the function $x \mapsto x \cap B$ given by $H \circ F$, H rud and F rud rec, and G is a unary function which is rudimentary in B, then there is a binary rudimentary function $\hat{G}$ such that $\hat{G}(x, y) = G(x)$ whenever y is a restriction of F to a transitive set containing x.*

Proof : Using Proposition 3·10, we can find some ℓ such that, for any transitive set u containing x and any subterm G' of some fixed term representing G as a function rudimentary in B, $G'(x) \in (\mathbb{T}^B)^\ell(u)$. Thus for y as in the statement, $f = (G^{\mathbb{T}^B})^\ell(y)$ is a restriction of F to a transitive set containing all of the $G'(x)$. Thus $G(x)$ can be obtained by using $H \circ f$ in place of $z \mapsto z \cap B$ in the term defining G. ⊣ (4·14)

Variants

There are some natural variations on the definition of rudimentary recursion, which we now show do not give more general collections of functions. For example, we could vary the relation used in the recursion.

4·15 Proposition *Let F be defined by $F(x) = G(x, F \restriction \mathrm{tcl}(x))$, where G is rudimentary. Define H by $H(x) = F \restriction \mathrm{tcl}(\{x\})$. Then H is rud rec and therefore F is gentle.*

Proof : $F\restriction \mathrm{tcl}(x) = \bigcup_{y\in x} H(y)$, so

$$\begin{aligned} H(x) &= \{(F(x), x)_2\} \cup \bigcup_{y\in x} H(y) \\ &= \{\big(G(x, \bigcup_{y\in x} H(y)), x\big)_2\} \cup \bigcup_{y\in x} H(y) \\ &= G_1(H\restriction x) \end{aligned}$$

where $G_1(h) = \{\big(G(\mathrm{Dom}\,(h), \bigcup \Im(h)), \mathrm{Dom}\,(h)\big)_2\} \cup \bigcup \Im(h)$, so that G_1 is rudimentary and H is rud rec. Then $F(x) = [H(x)](x)$, the evaluation of $H(x)$ at argument x, and is thus a trivial rud function of x and $H(x)$.

⊣ (4·15)

4·16 Corollary *Recursions of the form* $F(x) = G(x, F\restriction \bigcup\bigcup x)$, *where again* G *is rudimentary, thus yield gentle functions.*

4·17 Remark Recursions of that kind occur in the definition of forcing.

We could also restrict the domain of the recursion, for example to the ordinals.

4·18 Remark For G a rudimentary function, define $G'(f) = G(f) \cap \{z \mid \mathrm{Dom}\, f \in On\}$. Then G' is rudimentary, by Proposition 2·1; and if we recursively define $F(x) = G'(F\restriction x)$, then F is rudimentarily recursive and

$$F(x) = \begin{cases} G(F\restriction x) & \text{if } x \in On \\ \varnothing & \text{otherwise.} \end{cases}$$

We could also consider gentle functions of more than one variable—for example, any gentle function H can be considered as giving the function $x, y \mapsto H((x, y)_2)$ of two variables. Gentle functions in multiple variables are still closed under composition. We could also consider functions defined by mutual recursions—but as Proposition 4.3 shows, that does not take us outside the collection of gentle functions.

The final variant we shall consider is rudimentary recursion in a predicate. We call a function *rud rec in* B if it is of the form

$$F(x) = G(F\restriction x)$$

where G is rud in B. We say K is *gentle in* B iff it is of the form $H \circ F$ with H rud in B and F rud rec in B. It is clear that rudimentary recursion in arbitrary predicates is more general than pure rudimentary recursion. However, rudimentary recursion in gentle predicates is not.

4·19 Theorem (Bowler) *Let* F_2 *be a gentle function in a gentle predicate* B. *Then* F_2 *is gentle.*

Proof : Suppose that $x \mapsto x \cap B$ is given by $H_1 \circ F_1$, with F_1 given by $F_1(x) = G_1(F_1\restriction x)$, where G_1 and H_1 are rud. Since any gentle function in B is a composite of rud rec functions in B and any composite of gentle functions is gentle, we may suppose without loss of generality that F_2 is rud rec in B, given by $F_2(x) = G_2(F_2\restriction x)$, where G_2 is rud in B.

We say that a set is *sufficient for x* if it is the restriction of F_1 to a transitive set u containing $(F_2(x), x)_2$. We proceed, as in the proof of Proposition 4·8, by showing that there is a function F, definable by mutual rudimentary recursion with F_2 as in Proposition 4·3, with the property that for any x $F(x)$ is sufficient for x. As in that proof (but using Lemma 4·13 in place of Corollary 4·7), we can find a rudimentary function E in two variables such that, for any function f with domain x and sending each y in x to a set sufficient for y, $E(F_2 \restriction x, f)$ is sufficient for x. It also follows from this construction that $F_2 \restriction x$ is in the domain of $E(F_2 \restriction x, f)$. Thus $F_2(x) = \hat{G}_2(F_2 \restriction x, E(F_2 \restriction x, f))$, where $\hat{G}_2$ is as given by Lemma 4·14. Therefore there is an F which is definable together with F_2 by the simultaneous rudimentary recursion

$$F(x) = E(F_2 \restriction x, F \restriction x); \qquad F_2(x) = \hat{G}_2(F_2 \restriction x, E(F_2 \restriction x, F \restriction x)).$$

and so by Proposition 4·3 F_2 is gentle. ⊣ (4·19)

An illusory recursion

Just to warn the reader:

4·20 Proposition *There are rud functions G and H such that for any function F, $F(x) = G(F \restriction H(x))$.*

5: Rudimentary recursion from parameters

5·0 We have defined functions of type I, or pure rud rec functions to be those given by a recursion equation of the form

$$F(x) = G(F \restriction x)$$

where G is a pure rud function with no hidden parameters.

5·1 Definition (Mathias) For recursions involving parameters, the following definition seems the most satisfactory, which we call type II.

$$F(x) = G(p, F \restriction x)$$

Here G is a pure rud function of two variables and p is some set. We shall call such an F *p-rud rec* or *a function of type II*. Similarly, we call F *p-gentle* if it is a composite of a rudimentary function with a p-rud rec function.

5·2 It might be asked whether a simpler kind of recursion, which we might call type II', will suffice. Let us say that F is *rud rec from p*, where p is some set, if there are G_0 and G, pure rud functions of one variable, such that

$$F(x) = \begin{cases} G_0(p) & \text{if } x = \varnothing; \\ G(F \restriction x) & \text{if } x \neq \varnothing. \end{cases}$$

For such an F and for any rudimentary function H we shall say $H \circ F$ is *gentle from p*.

Thus in type II recursion the parameter p may be re-used throughout the recursion, whereas in type II', use of the parameter p occurs only at the beginning.

5·3 Example To form $L(d)$, the constructible closure of d, a transitive set, requires a rud recursion in the parameter d: define

$$D(x) = d \cup \bigcup_{y \in x} \mathbb{T}(D(y)).$$

Then $D(x) = D_{\varrho(x)}$ where $D_0 = d$; $D_{\nu+1} = \mathbb{T}(D_\nu)$; $D_\lambda = \bigcup_{\nu<\lambda} D_\nu$, which is the usual ordinal recursion for this purpose. $L(d) = \bigcup_x D(x) = \bigcup_\nu D_\nu$.

5·4 A delicate distinction has to be made here. The two collections of functions given by recursions of type II and of type II' from a given parameter are not the same: for example, for p of infinite rank, the function $F : x \mapsto p \times x$ is p-rud rec but not rud rec from p, since there is no rud G with $p \times \{\varnothing\} = F(\{\varnothing\}) = G(F \restriction \{\varnothing\}) = G(\{(\varnothing, \varnothing)_2\})$. The closure property given in Proposition 4·10 holds for the collection of p-rud rec functions, by essentially the same proof, but fails for the collection of functions rud rec from p, since if K is the constant function with value ω, K is rud rec from ω, but $x \mapsto K \restriction x = \{\omega\} \times x$ is not. It is for such reasons that we have preferred type II to type II'.

But when we pass to the associated gentle collections, we may breathe again, as that distinction no longer applies:

5·5 Proposition (Bowler) *A function F is p-gentle iff it is gentle from p.*

Proof : The 'if' direction is clear from the definitions and from Proposition 2·2. For the 'only if' direction, note that without loss of generality F is p-rud rec, given by $F(x) = G(p, F \restriction x)$. Let $K : x \mapsto (p, F(x))_2$. There is a rudimentary function G_1 such that for any x we have $G_1(K \restriction x) = F \restriction x$, and so

$$K(x) = \begin{cases} (p, G(p, \varnothing))_2 & \text{if } x = \varnothing; \\ \left(\bigcup \Im(\Im(K \restriction x)), G\left(\bigcup \Im(\Im(K \restriction x)), G_1(K \restriction x)\right)\right)_2 & \text{if } x \neq \varnothing. \end{cases}$$

Thus K is rud rec from p and so F is gentle from p. ⊣ (5·5)

5·6 Essentially the same arguments as in the last section show that the p-gentle functions have good closure properties. For example, if F is p-gentle then so is $x \mapsto F \restriction x$. However, it is not true that any composite of p-gentle functions is p-gentle: for example, the function $x \mapsto \omega + x$ is ω-gentle, but its composite with itself is not. This composite is, however, $\omega + \omega$-gentle and there is a similar phenomenon in general.

5·7 Proposition (Bowler) *Let F_1 be p_1-rud rec and F_2 be p_2-rud rec. Then $F_1 \circ F_2$ is $(p_1, F_1 \restriction \mathrm{tcl}\{p_2\})_2$-gentle.*

5·8 Proposition (Bowler) *Let B be a p_1-gentle predicate, with $x \mapsto x \cap B$ represented as $H_1 \circ F_1$, and let F_2 be p_2-gentle in B. Then F_2 is $(p_1, F_1 \restriction \mathrm{tcl}\{p_2\})_2$-gentle.*

The proofs are like those in the last section. Apart from these two cases, the results of the last section transfer directly to p-gentle functions, and we may refer to them in future as if they were stated in those terms. Specifically, in Propositions 4·1, 4·2, 4·3, 4·12 and 4·15, Corollary 4·16 and Remark 4·18, we may replace rud, rud rec and gentle respectively by p-rud (that is, of the form $x \mapsto G(p, x)$ with G rud), p-rud rec and p-gentle.

The parametrized forms of Lemma 4·6 and its corollary are now given: note the uniformity. H depends only on G and not on the specific parameter p.

5·9 The Propagation Lemma *Let G be a binary rudimentary function. Then there is a ternary rudimentary function H_G, obtainable uniformly from G, such that for any set p, if F be the p-rud rec function given by the recursion $F(x) = G(p, F \restriction x)$, and if P^+ and P be transitive sets with $P^+ \subseteq \mathcal{P}(P)$, then*

$$F \restriction P^+ = H_G(p, F \restriction P, P^+).$$

Proof : If $x \in P^+$, then $x \subseteq P$, so $F \restriction x = (F \restriction P) \restriction x$ so $F(x) = G(p, (F \restriction P) \restriction x)$. Hence

$$F \restriction P^+ = \{(G(p, (F \restriction P) \restriction x), x)_2 \mid_x x \in P^+\}.$$

We take $H_G(p, f, q) \equiv \{(G(p, f \restriction x), x)_2 \mid_x x \in q\}$. ⊣ (5·9)

5·10 Corollary *Let G be rud. Then there is a binary rud function $H^{\mathbb{T}}_G$ obtainable uniformly from G such that for every set p F rud rec, given by $F(x) = G(p, F \restriction x)$, and every transitive u, $F \restriction \mathbb{T}(u) = H^{\mathbb{T}}_G(p, F \restriction u)$.*

Proof : We take $H^{\mathbb{T}}_G(p, f) = H_G(p, \mathbb{T}(\mathrm{Dom}\,(f)), f)$. ⊣ (5·10)

5·11 Remark Type II recursions will underlie the discussion of rudimentary forcing in [M8], with the poset $\mathbb{P}$ of conditions as an ever-present parameter.

5·12 Remark The first Jensen fragment after J_1 that is closed under functions of type II is J_ω, as given J_k we could set $f(0) = J_k; f(n+1) = \mathbb{T}(f(n)); f(\lambda) = \bigcup f``\lambda$, and then $f(\omega) = J_{k+1}$.

Recursions of Type III

5·13 Finally, we ask what happens to type II if we turn the parameter back into a variable and consider recursion equations of the following form

$$F(v, x) = G(v, F \restriction (\{v\} \times x))$$

which we shall call type III.

5·14 Remark The recursion here is on the second variable, in harmony with the form of the definition of ordinal addition as given in Example 0·3.

5·15 Proposition *For each fixed v the map $x \mapsto F(v, x)$ is rud recursive of type II, in the parameter v.*

Proof : Let $E(x) = F(v, x)$. Then $E \restriction x = \{(F(v, b), b)_2 \mid_b b \in x\}$ whereas

$$\begin{aligned} F \restriction (\{v\} \times x) &= \{\big(F(v, b), (v, b)_2\big)_2 \mid_b b \in x\} \\ &= \{\big(E(b), (v, b)_2\big)_2 \mid_b b \in x\} \\ &= H(v, E \restriction x) \end{aligned}$$

for a certain rud function H; so $E(x) = G(v, H(v, E \restriction x)) = G_1(v, E \restriction x)$, for some rud function G_1. ⊣ (5·15)

5·16 Remark Since x is recoverable by a rud function from $F \restriction (\{v\} \times x)$, as the domain of its domain, no new functions result from equations of the form

$$F(v, x) = H(v, x, F \restriction (\{v\} \times x)).$$

6: Provident sets

6·0 Definition (Mathias) A set A is *p-provident*, where p is a set, if it is nonempty, transitive, closed under pairing and for all p-rud rec F (or equivalently all p-gentle F) and all x in A, $F(x) \in A$.

6·1 Remark If A is p-provident, $p \in A$.

6·2 Example We shall see that the Jensen fragment J_ν is $\varnothing$-provident for all $\nu \geqslant 1$.

6·3 Theorem *Any directed union of $\varnothing$-provident sets is $\varnothing$-provident. Explicitly, if $\mathcal{A}$ is a nonempty set of $\varnothing$-provident sets such that for any $A, B \in \mathcal{A}$ there is $C \in \mathcal{A}$ with $A \cup B \subseteq C$, then $\bigcup \mathcal{A}$ is $\varnothing$-provident.*

Proof: $\bigcup \mathcal{A}$ is nonempty since $\mathcal{A}$ and all $A \in \mathcal{A}$ are nonempty. It is transitive since each $A \in \mathcal{A}$ is. For any $x, y \in \bigcup \mathcal{A}$, we can find $A, B \in \mathcal{A}$ with $x \in A$ and $y \in B$, and we can find $C \in \mathcal{A}$ with $A \cup B \subseteq C$. Since C is provident, $\{x, y\} \in C \subseteq \bigcup \mathcal{A}$. Finally, $\bigcup \mathcal{A}$ is closed under $\varnothing$-rud rec functions since each such function is unary. ⊣ (6·3)

6·4 Definition (Mathias) A is *provident* if it is p-provident for every $p \in A$.

6·5 Remark The only provident set not containing an infinite set is **HF**.

6·6 Remark For provident sets, it is unnecessary to demand that they be closed under pairing, for if $x \in A$, the function $y \mapsto \{x,\, y\}$ is x-rud rec, being given by the recursion $F(y) = \{x, \mathrm{Dom}\, F \restriction y\}$. But the union of two sets each closed under $\varnothing$-rud rec functions might not be closed under pairing, though as rud rec functions are unary, that union would be closed under $\varnothing$-rud rec functions: for example, let a and b be mutually Cohen-generic subsets of ω and consider the model $J_2(a) \cup J_2(b)$.

6·7 Theorem *Any directed union of provident sets is provident.*

Ranks of provident sets

If A is an $\varnothing$-provident set, then for $\nu < \varrho(A)$ we have $\nu = \varrho(x)$ for some $x \in A$ and so $\nu \in A$. Thus $\varrho(A) = On \cap A$. Since the function $\nu \mapsto \nu + 1$ is rudimentary, we can deduce that $\varrho(A)$ is a limit ordinal. If A is provident, then since the function $\nu \mapsto \mu + \nu$ is μ-rud rec, $\varrho(A)$ is closed under addition.

6·8 Remark The discussion above shows that the rank of any provident set is an indecomposable ordinal, as defined in 0·4.

6·9 Lemma *An ordinal θ is indecomposable iff it is of the form ω^α for some $\alpha > 0$.*

Proof: If $\alpha = \beta + 1$ then for $\mu, \nu < \omega^\alpha$ we can choose $m, n < \omega$ with $\mu \leq \omega^\beta \cdot m$ and $\nu \leq \omega^\beta \cdot n$, so that $\mu + \nu \leq \omega^\beta(m+n) < \omega^\alpha$. If α is a limit, then for $\mu, \nu < \omega^\alpha$ we can choose $\kappa < \alpha$ with $\mu, \nu < \omega^\kappa$ and so $\mu + \nu < \omega^\kappa < \omega^\alpha$.

Conversely, suppose that θ is indecomposable. Let β be minimal such that $\omega^\beta > \theta$. Since exponentiation by ω is continuous, β must be a successor: say $\beta = \alpha + 1$. Now choose $n < \omega$ maximal so that $\omega^\alpha . n \leq \theta$. If $n \neq 1$, then the identity $\omega^\alpha \cdot (n-1) + \omega^\alpha = \omega^\alpha \cdot n$ contradicts indecomposability of θ, so we must have $n = 1$. Let $\theta = \omega^\alpha + \gamma$. Since $n = 1$, $\gamma < \omega^\alpha$ and so since θ is indecomposable we must have $\gamma = 0$. Thus $\theta = \omega^\alpha$, as required. ⊣ (6·9)

A familiar provident set is $J_{\omega^\nu}(c)$ provided ω^ν is greater than the rank of the transitive set c. But we shall replace the traditional definition of $L(a)$ recalled in 0·16 by the one outlined in 0·17.

Bounding rudimentary functions in a finite progress

6·10 Definition Let ξ be an ordinal or *ON*. A *ξ-progress* is a sequence $\langle P_\nu \mid \nu < \xi\rangle$ of transitive sets such that for each ν with $\nu{+}1 < \xi$, $\mathbb{T}(P_\nu) \subseteq P_{\nu+1}$ and for each limit ordinal $\lambda < \xi$, $\bigcup_{\nu<\lambda} P_\nu \subseteq P_\lambda$; the progress is *strict* if for each ν with $\nu + 1 < \xi$, $P_{\nu+1} \subseteq \mathcal{P}(P_\nu)$; *continuous* if for each limit $\lambda < \xi$, $P_\lambda = \bigcup_{\nu<\lambda} P_\nu$; and *solid* if it is strict and continuous and $P_0 = \varnothing$.

6·11 Proposition *If the progress is strict and continuous then for each* $\nu < \xi$, $\varrho(P_\nu) = \varrho(P_0) + \nu$.

Proof : by induction on ν. ⊣ (6·11)

6·12 Theorem *Let R be a rudimentary function of n variables. There is a $c_R \in \omega$ such that for every $(c_R + 1)$-progress $P_0, P_1, \dots, P_{c_R}$, $R``P_0^n \subseteq P_{c_R}$.*

6·13 Definition We call c_R the *rudimentary constant* of R. For $R : a \mapsto a \cap \{x \mid \overset{0}{\models} \varphi(x, b)\}$ with φ a $\dot{\Delta}_0$ formula, we also call c_R the *separational delay.*

6·14 Remark More precisely, there is a recursive function sending a program for R to a bound; but the function sending a program for R to the minimal bound is not recursive.

We prove the theorem in a series of lemmata.

6·15 Lemma *If x and y are in P_ν then* $\{x, y\} \in P_{\nu+1}$, $x \smallsetminus y \in P_{\nu+1}$, $\bigcup x \in P_{\nu+1}$ *and* $\mathrm{Dom}\,(x) \in P_{\nu+1}$.

Proof : Immediate from lines 2, 3, 4 and 5 of the definition of $\mathbb{T}$. ⊣ (6·15)

6·16 Lemma $x, y \in P_\zeta \implies x \times y \in P_{\zeta+3}$.

Proof : If x and y are in P_ν then both $\{x\}$ and $\{x, y\}$ are in $P_{\nu+1}$; so $\{\{x\}, \{x, y\}\}$ are in $P_{\nu+2}$; P_ν being transitive, we may infer that if $a \in x$ and $b \in x$, then $(a, b)_2$ is in $P_{\nu+2}$; thus $x \times y \subseteq P_{\nu+2}$, which, since $P_\nu \subseteq P_{\nu+2}$, implies that $x \times y \in P_{\nu+3}$. ⊣ (6·16)

6·17 Lemma $x, y \in P_\zeta \implies R_5(x, y) \in P_{\zeta+1}$.

6·18 Lemma $a, b, c \in P_\zeta \implies [(a, c)_2 \in P_{\zeta+2}\ \&\ (b, a, c)_3 \in P_{\zeta+4}]$.

6·19 Lemma $x \in P_\zeta \implies R_6(x) \in P_{\zeta+5}$.

6·20 Lemma $x \in P_\zeta \implies R_7(x) \in P_{\zeta+5}$.

6·21 Lemma $x, w \in P_\zeta \implies x``\{w\} \in P_{\zeta+1}$.

6·22 Lemma $x, y \in P_\zeta \implies R_8(x, y) \in P_{\zeta+2}$.

Proof of Theorem 6·12: The lemmata show that for $i = 0, \dots 8$, we may take c_{R_i} to be 1, 1, 1, 1, 3, 1, 5, 5, 2 respectively. The theorem now follows by remarking that if S and T_i are rudimentary and for all x, $Q(\vec{x}) = S(T_0(\vec{x}), \dots, T_k(\vec{x}))$, we may take $c_Q = c_S + \max_i c_{T_i}$. ⊣ (6·12)

6·23 Corollary *If* $\langle P_\nu \mid \nu < \xi\rangle$ *is a* ξ*-progress, then at each limit ordinal* $\lambda \leqslant \xi$, $\bigcup_{\nu<\lambda} P_\nu$ *is rud closed.*

The canonical progress towards a given transitive set

6·24 Let c be a transitive set. Let $c_\zeta = c \cap \{x \mid \varrho(x) < \zeta\}$. Since c is transitive, $c_{\zeta+1}$ will be a set of subsets of c_ζ; in fact $c_{\zeta+1} = c \cap \{x \mid x \subseteq c_\zeta\}$, which we shall use below as a direct recursive definition.

If $c_{\zeta+1} = c_\zeta$, then $c_\zeta = c$ and for all $\xi > \zeta$, $c_\xi = c_\zeta$; so that that first happens when $\zeta = \varrho(c)$.

Using c as a parameter we define a sequence of pairs $((c_\nu, P^c_\nu))_\nu$ by a rud recursion on ν. Each P^c_ν will be of rank ν; we shall use the function $\mathbb{T}$, but we shall also "feed" stages of c into the process.

The sequence $(P^c_\nu)_\nu$ forms a solid progress, which we shall call the canonical progress *towards, to*, or *through* c, the choice of preposition depending on the length of the sequence as compared to the rank of c.

6·25 Definition

$$\begin{array}{lllllllll} c_0 & = & \varnothing & c_{\nu+1} & = & c \cap \{x \mid x \subseteq c_\nu\} & c_\lambda & = & \bigcup_{\nu<\lambda} c_\nu \\ P^c_0 & = & \varnothing & P^c_{\nu+1} & = & \mathbb{T}(P^c_\nu) \cup \{c_\nu\} \cup c_{\nu+1} & P^c_\lambda & = & \bigcup_{\nu<\lambda} P^c_\nu \end{array}$$

6·26 Lemma *Each* P^c_ν *is transitive.* $P^c_\nu \subseteq P^c_{\nu+1}$. $P^c_\nu \in P^c_{\nu+1}$*; and so for* $\nu < \zeta$, $P^c_\nu \subseteq P^c_\zeta$ *and* $P^c_\nu \in P^c_\zeta$.

6·27 Remark $c_\nu = c \cap P^c_\nu$; $\varrho(P^c_\nu) = \nu$.

6·28 Remark P^c_ν may be defined by a single rud recursion on ordinals:

$$P^c_0 = \varnothing; \qquad P^c_{\nu+1} = \mathbb{T}(P^c_\nu) \cup \{c \cap P^c_\nu\} \cup (c \cap \{x \mid x \subseteq P^c_\nu\}); \qquad P^c_\lambda = \bigcup_{\nu<\lambda} P^c_\nu.$$

With that definition, one should then verify by induction that for each ν, $c \cap P^c_\nu = c \cap \{x \mid \varrho(x) < \nu\}$, and thence that the two definitions agree.

6·29 Remark Each P^c_λ is rud closed, for λ a limit ordinal, by Theorem 6·12.

6·30 Remark $P^c_\omega = V_\omega$: for each $P^c_n \subseteq V_n$ and so $P^c_\omega \subseteq V_\omega$; equality will follow from the fact that P^c_ω is a non-empty rud closed set, by the previous remark.

Bounding rudimentarily recursive functions in a progress

To see why and how quickly progresses tend to become closed under p-rud rec functions, we recall the notion of an F-attempt.

6·31 Definition Let F be the p-rudimentarily recursive function defined by $F(x) = G(p, F\restriction x)$. A set f is an F*-attempt* iff it satisfies

$$Fn(f) \;\&\; \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f) \;\&\; \forall x_{\in \mathrm{Dom}\,(f)}\; f(x) = G(p, f\restriction x).$$

Note that that is Δ_0 in f and p; we will denote the separational delay of that predicate by s_F.

We say that an F-attempt f *attains* x iff $x \in \mathrm{Dom}\,(f)$.

6·32 Proposition *Let F be a p-rud rec function. Then there is a natural number c_F such that for any set x and any (c_F+1)-progress $P_0, P_1, \ldots, P_{c_F}$ such that P_0 contains p and x and contains, for each $y \in x$, an F-attempt attaining y, the set P_{c_F} contains an F-attempt attaining x.*

Proof : Under those hypotheses, $f_0 =_{\mathrm{df}} \bigcup\{f \in P_0 \mid f \text{ is an } F\text{-attempt}\}$ is an F-attempt attaining every $y \in x$. Further, $f_0 \in P_{s_F+2}$, as $P_0 \in P_1$, so the set $P_0 \cap \{f \mid f \text{ is an } F\text{-attempt}\}$ is in P_{1+s_F}, and its union will be in P_{1+s_F+1} by the definition of $\mathbb{T}$.

Now $F \restriction x = f_0 \restriction x$, and so $f_0 \cup \{(G(p, f_0 \restriction x), x)_2\}$ is an F-attempt attaining x. It is therefore enough to take $c_F = s_F + c_R + 2$, where R is the rudimentary function $(x, p, f) \mapsto f \cup \{(G(p, f \restriction x), x)_2\}$. ⊣ (6·32)

6·33 Theorem *Let F be a p-rud rec function, x a set, and $\langle P_\nu \mid \nu < \xi \rangle$ a ξ-progress with $\xi > c_F \cdot (\varrho(x)+1)$, and p and x in P_0. Then $P_{c_F \cdot (\varrho(x)+1)}$ contains an F-attempt attaining x.*

Proof : By induction on $\varrho(x)$. For each $y_0 \in x$, we have $\varrho(y_0) + 1 \leqslant \varrho(x)$ and so the induction hypothesis will imply that $P_{c_F.\varrho(x)}$ contains an F-attempt attaining y_0. By Proposition 6·32, $P_{c_F.\varrho(x)+c_F}$ contains an F-attempt attaining x, which is the desired result, as $c_F.\varrho(x) + c_F = c_F.(\varrho(x)+1)$. ⊣ (6·33)

6·34 Theorem *Let $\langle P_\nu \mid \nu \leqslant \theta \rangle$ be a solid $(\theta+1)$-progress. Then P_θ is provident iff θ is an indecomposable ordinal.*

Proof : The ‘only if’ direction is immediate from Remark 6·8. For the ‘if’ direction, let $x, p \in P_\theta$; choose $\nu < \theta$ with $x, p \in P_\nu$. Let F be p-rud rec. Then $\varrho(x) < \nu$ and so $F(x) \in P_{\nu + c_F \cdot \nu} \subseteq P_\theta$. ⊣ (6·34)

6·35 Proposition *Let c be a transitive set and θ an indecomposable ordinal. Then P^c_θ is provident, and*

$$P^c_\theta = P^{c_\theta}_\theta = \bigcup_{\lambda<\theta} P^{c_\lambda}_\theta.$$

Proof : That P^c_θ is provident is an immediate corollary of Theorem 6·34.

If $x \in P^c_\theta$, then for some $\lambda < \theta$, $x \in P^c_\lambda = P^{c_\lambda}_\lambda \subseteq P^{c_\lambda}_\theta$.

Conversely, if $\lambda < \theta$, c_λ is in P^c_θ, which we now know to be provident, and the map $\nu \mapsto P^{c_\lambda}_\nu$ is given by a c_λ-rudimentary recursion, and so each $P^{c_\lambda}_\nu$, for $\nu < \theta$, is in P^c_θ; thus $P^{c_\lambda}_\theta \subseteq P^c_\theta$. ⊣ (6·35)

In fact, the inductive argument of Theorem 6·33 gives the following slightly sharper version:

6·36 Theorem *Let F be a p-rud rec function and $\langle P_\mu \mid \mu < \xi \rangle$ a solid ξ-progress, and let $p \in P_\kappa$ and $x \in P_\nu$, where $\kappa + c_F \cdot \nu < \xi$. Then there is an F-attempt attaining x in $P_{\kappa + c_F \cdot \nu}$.*

6·37 Corollary *Let $\langle P_\nu \mid \nu \le \theta \rangle$ be a solid $(\theta+1)$-progress. Then P_θ is $\varnothing$-provident iff θ is a limit ordinal.*

Indeed something a little more general is true.

6·38 Theorem *Let $\langle P_i \mid i < \omega \rangle$ a strict ω-progress with P_0 p-provident. Then $\bigcup_{i<\omega} P_i$ is also p-provident.*

Proof : Let F be p-rud rec. As in Proposition 4·15, $x \mapsto F \restriction \mathrm{tcl}\{x\}$ is p-gentle and so for any $x \in P_0$ there is an F-attempt attaining x in P_0. Then by induction on i, with $i < \omega$, using Proposition 6·32, we obtain that for any $x \in P_i$ there is an F-attempt attaining x in $P_{c_F \cdot i}$, and in particular that $F(x) \in P_{c_F \cdot i}$. ⊣ (6·38)

A criterion for providence in terms of Type III

6·39 PROPOSITION *A transitive set A is provident iff it contains the graph of the restriction of F to $X \times X$ for any $X \in A$ and any F which is recursive of type III.*

Proof : It is clear that if A contains all these graphs then it is provident.

Conversely, suppose A is provident of rank θ and $X \in A$. If $A = \mathbf{HF}$, the result is clear, so we assume $\theta > \omega$. Let F be defined by $F(v, x) = G(v, F \restriction (\{v\} \times x))$. Then for each $v \in X$ we have by Theorem 6·36 that there is an $F(v, -)$-attempt attaining X in $P^X_{\varrho(X).2+\omega}$. The graph in question is then given by

$$[P^X_{\varrho(x).2+\omega} \times (X \times X)] \cap \bigcup \Big\{ (y, (v, x)_2)_2 \;\Big|\; \exists f \in P^X_{\varrho(x).2+\omega} \; f \text{ is an } F(v, -)\text{-attempt attaining } x \;\&\; f(x) = y \Big\}. \quad ⊣ (6·39)$$

Iterated recursion and limit provident sets

We can obtain similar bounds on the growth of functions obtained by recursing rud rec functions, or by recursing functions obtained in that way, and so on. More precisely:

6·40 DEFINITION A unary class function $F : V \to V$ is p-rud $[\mathrm{rec}]^0$ iff it is rud. F is p-rud $[\mathrm{rec}]^{n+1}$ iff there is a p-rud $[\mathrm{rec}]^n$ function G such that for all x we have $F(x) = G((p, F \restriction x)_2)$. F is p-rud $[\mathrm{rec}]^{<\omega}$ iff it is p-rud $[\mathrm{rec}]^n$ for some $n < \omega$.

Thus F is p-rud $[\mathrm{rec}]^1$ iff it is p-rud rec.

6·41 REMARK That is more powerful than rudimentary recursion, but it is still fairly weak. For example, as we shall see in Corollary 6·52, for no p is $\nu \mapsto \nu + \omega$ p-rud $[\mathrm{rec}]^{<\omega}$, in contrast to the fact that $\nu \mapsto \alpha + \nu$ is α-rud rec for each α. Similarly these recursions are too weak to define ordinal multiplication.

6·42 REMARK Provident sets need not be closed under p-rud $[\mathrm{rec}]^n$ functions for $n > 1$. For example, the ordinal function $x \mapsto \omega + x$ is ω-rud rec, and so the ordinal function $F : x \mapsto \omega^2 \cap (\omega \cdot x)$ obtained from it by recursion is ω-rud $[\mathrm{rec}]^2$. But $P^{\varnothing}_{\omega^2}$ is not closed under F, since $F(\omega) = \omega^2$.

However, we shall find that to check whether a provident set is closed under such recursions it is enough to know the rank of that provident set. To prove that, we shall consider bounds on the growth of such functions; and for that we must first consider a notion of limitation for ordinal functions.

6·43 DEFINITION For λ an ordinal, we say that an ordinal function $l : On \to On$ is *λ-restrained* iff for all ordinals ν we have $l(\nu) < \lambda + \nu + \omega$.

6·44 LEMMA *If l is λ-restrained, then it is λ'-restrained for any $\lambda' \geq \lambda$.*

6·45 LEMMA *If l_1 and l_2 are λ-restrained, then so is $\nu \mapsto l_1(\nu) \cup l_2(\nu)$.*

6·46 LEMMA *If l_1 is λ_1-restrained and l_2 is λ_2-restrained then $l_1 \circ l_2$ is $\lambda_1+\lambda_2$-restrained.*

Proof : For any ν, we can pick $m \in \omega$ with $l_2(\nu) \leq \lambda_2 + \nu + m$ and so $l_1(l_2(\nu)) < \lambda_1 + l_2(\nu) + \omega \leq \lambda_1 + \lambda_2 + \nu + m + \omega = \lambda_1 + \lambda_2 + \nu + \omega$. ⊣ (6·46)

6·47 LEMMA *If l is λ-restrained and increasing then the function l' defined by $l'(\nu) = l\left(\bigcup_{\mu<\nu} l'(\mu)\right)$ is $\lambda \cdot \omega$-restrained and increasing.*

Proof : l' is clearly increasing. $l'(0) = l(0) < \lambda + \omega \leq \omega \cdot \lambda + 0 + \omega$. For ν a successor, say $\nu = \mu+1$, we can pick $m < \omega$ such that $l'(\mu) \leq \lambda\cdot\omega+\mu+m$, so that $l'(\nu) = l(l'(\mu)) \leq l(\lambda\cdot\omega+\mu+m) < \lambda+\lambda\cdot\omega+\mu+m+\omega = \lambda\cdot\omega+\nu+\omega$. Finally, for ν a limit, for every $\mu < \nu$ we have $l'(\mu) < \lambda\cdot\omega+\mu+\omega \leq \lambda\cdot\omega+\nu$ and so $\bigcup_{\mu<\nu} l'(\mu) \leq \lambda\cdot\omega+\nu$, and so $l'(\nu) \leq l(\lambda+\nu) < \lambda+\lambda\cdot\omega+\nu+\omega = \lambda\cdot\omega+\nu+\omega$. ⊣ (6·47)

6·48 DEFINITION For ordinals κ and λ and a set p, a unary class function F is (p,κ,λ)-restrained if there are an increasing λ-restrained ordinal function l and a rudimentary function H such that for any solid progress P with $p \in P_\kappa$ and $x \in P_\nu$ and any $\alpha \geq l(\nu)$ we have $H(p, P_\alpha, x) = F(x)$.

That notion is designed to make the following true:

6·49 LEMMA *Any p-rud rec function is (p,κ,κ)-restrained for every κ.*

Proof : Let $H : p, P, x \mapsto [\bigcup\{f \in P \mid f \text{ is an } F\text{-attempt}\}](x)$. Let $l : \nu \mapsto \kappa + c_F \cdot \nu$. By Proposition 6·11 and Theorem 6·36, if P, ν and x are as in Definition 6·48, and $\alpha \geq l(\nu)$, then P_α contains an F-attempt attaining x, and so $H(p, P_\alpha, x) = F(x)$. ⊣ (6·49)

We now mimic the argument used to obtain Theorem 6·36, to show that functions obtained by recursion from restrained functions are still restrained.

6·50 THEOREM (Bowler) *Suppose that F is defined by $F(x) = G((p, F \restriction x)_2)$, where G is (p,κ,λ)-restrained. Then F is $(p,\kappa,(\lambda+\kappa)\cdot\omega)$-restrained.*

Proof : Let H and l witness the fact that G is (p,κ,λ)-restrained, as in Definition 6·48. We say that f is an *F-attempt using P* if

$$Fn(f) \;\&\; \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f) \;\&\; \forall y{\in}\mathrm{Dom}\,(f)\; f(y) = H(p, P, (p, f \restriction y)_2).$$

We shall refer to this Δ_0 formula again, so we denote it by $A(p,P,f)$. Let $K : p, P, x \mapsto [\bigcup\{f \in P \mid A(p,P,f)\}](x)$. We say that x is *attained by P* iff there is some $f \in P$ such that $A(p,P,f)$ and $x \in \mathrm{Dom}\,(f)$, and for every $y \in \mathrm{tcl}\{x\}$ we have $H(p, P, (p, f \restriction y)_2) = G((p, f \restriction y)_2)$. Thus if x is attained by P then $K(p,P,x) = F(x)$.

Next we define a sequence of variously restrained ordinal functions which will help us restrain the growth of F. Let $l_1 : \nu \mapsto \kappa + \nu + c_{R_1} + 1$, where R_1 is the rudimentary function $P, p \mapsto \bigcup\{f \in P \mid A(p,P,f)\}$. Let $l_2 : \nu \mapsto l_1(\nu) + c_{R_2}$, where R_2 is the rudimentary function $p, f, x \mapsto (p, f \restriction x)_2$. Let $l_3 = l \circ l_2$. Let $l_4 : \nu \mapsto l_3(\nu) + c_{R_5}$, where R_4 is the rudimentary function

$p, P, f, x \mapsto H(p, P, (p, f \restriction x)_2)$. Let $l_5 : \nu \mapsto (l_4(\nu) \cup c_{\mathrm{tcl}} \cdot \nu) + c_{R_5}$, where R_5 is the rudimentary function $f, t, v, x \mapsto f \restriction t \cup \{(v, x)_2\}$. Finally, let l_6 be defined by $l_6(\nu) = l_5(\bigcup_{\mu<\nu} l_6(\mu))$. Each of the l_i is increasing. Evidently l_1 and l_2 are κ-restrained. Thus by Lemma 6·46, l_3 is $\lambda + \kappa$-restrained and therefore so are l_4 and l_5. Therefore by Lemma 6·47, l_6 is $(\lambda + \kappa) \cdot \omega$-restrained.

Now suppose that we have a solid progress P with $p \in P_\kappa$. We shall show by induction on ν that for $x \in P_\nu$ and $\alpha \geq l_6(\nu)$ x is attained by P_α. For given ν, write $l_7(\nu) = \bigcup_{\mu<\nu} l_6(\mu)$. Let

$$f_0 = \bigcup \{f \in P_{l_7(\nu)} \mid A(p, P_{l_7(\nu)}, f)\},$$

which is in $P_{l_1(l_7(\nu))}$. By our induction hypothesis, for any $y \in x$ we have $f_0(y) = K(p, P_{l_7(\nu)}, y) = F(y)$. Thus $(p, F \restriction x)_2 = (p, f_0 \restriction x)_2 \in P_{l_2(l_7(\nu))}$. Therefore for $\alpha \geq l_3\,(l_7(\nu))$, $F(x) = G((p, F \restriction x)_2) = H(p, P_\alpha, (p, f_0 \restriction x)_2)$, and in particular $F(x) \in P_{l_4(l_7(\nu))}$. Thus $f_1 = f_0 \restriction \mathrm{tcl}(x) \cup \{(F(x), x)_2\} \in P_{l_5(l_7(\nu))} = P_{l_6(\nu)}$ and $A(p, P, f_1)$, so x is attained by any $\alpha \geq l_6(\nu)$.

Thus K and l_6 witness that F is $(p, \kappa, (\lambda + \kappa) \cdot \omega)$-restrained. ⊣ (6·50)

6·51 Theorem *Any p-rud [rec]^n function is $(p, \kappa, \kappa \cdot (\omega^{n-1} + n - 1))$-restrained for every κ.*

Proof : By induction on n. The base case follows from Lemma 6·49, and for the induction step let F be p-rud $[\mathrm{rec}]^{n+1}$, given by $F(x) = G((p, F \restriction x)_2)$ for some p-rud $[\mathrm{rec}]^n$ function G. Then by the induction hypothesis G is $(p, \kappa, \kappa \cdot (\omega^{n-1} + n - 1))$-restrained, and so by Theorem 6·50 F is $(p, \kappa, (\kappa \cdot (\omega^{n-1} + n - 1) + \kappa) \cdot \omega))$-restrained, which is the desired result as $(\kappa \cdot (\omega^{n-1} + n - 1) + \kappa) \cdot \omega = \kappa \cdot (\omega^{n-1} + n) \cdot \omega \leq \kappa \cdot (\omega^n + n)$. ⊣ (6·51)

6·52 Corollary *For no p is $F : \nu \mapsto \nu + \omega$ is p-rud [rec]$^{<\omega}$.*

Proof : Suppose for a contradiction that F is p-rud $[\mathrm{rec}]^n$ for some p and n. Let $c = \mathrm{tcl}\{p\}$, $\kappa = \varrho(p) + 1$ and $\lambda = \kappa \cdot (\omega^{n-1} + n - 1)$. Then F is (p, κ, λ)-restrained: let l and H witness that. As $p \in P^c_\kappa$, we must have $\lambda \cdot \omega + \omega = F(\lambda \cdot \omega) \in P^c_{l(\lambda\cdot\omega)+c_H}$, which is the desired contradiction as $\varrho(P^c_{l(\lambda\cdot\omega)+c_H}) = l(\lambda \cdot \omega) + c_H < \lambda + \lambda \cdot \omega + \omega = \lambda \cdot \omega + \omega$.

6·53 Theorem (Bowler) *Let A be a provident set other than* **HF**. *The following are equivalent:*

i. $\varrho(A)$ is of the form ω^α for some ordinal α which is a limit.

ii. A is closed under p-rud [rec]^2 functions for $p \in A$.

iii. A is closed under p-rud [rec]$^{<\omega}$ functions for $p \in A$.

Proof : It is clear that $(iii) \Rightarrow (ii)$. To see that $(ii) \Rightarrow (i)$, we know by Remark 6·8 and Lemma 6·9 that there is some $\alpha > 0$ with $\varrho(A) = \omega^\alpha$. Suppose for a contradiction that α is a successor ordinal, say $\alpha = \beta + 1$. Then $\nu \mapsto \omega^\beta + \nu$ is ω^β-rud rec and so $F : \nu \mapsto \omega^\alpha \cap \omega^\beta \cdot \nu$ is ω^β-rud $[\mathrm{rec}]^2$, which contradicts (ii), as $\omega \in A$ and $\omega^\beta \in A$ but $F(\omega) = \omega^\alpha \notin A$.

For $(i) \Rightarrow (iii)$, let $p, x \in A$ and let F be p-rud $[\mathrm{rec}]^n$. Let $\kappa = \varrho(\{p, x\})$. By Theorem 6·51, F is $(p, \kappa, \kappa \cdot (\omega^{n-1} + n - 1))$-restrained: let H and l witness this. We have $x, p \in P^{\{p,x\}}_\kappa$, and so $F(x) \in P^{\{p,x\}}_{l(\kappa)+c_H+1}$. Since $\kappa < \omega^\alpha$, we can pick some $\beta < \alpha$ with $\kappa \leq \omega^\beta$, and so $l(\kappa) + c_H + 1 \leq \kappa \cdot (\omega^{n-1} + n -$

$1) + \kappa + \omega + c_H + 1 \leq \omega^\beta(\omega^{n-1}.\omega) = \omega^{\beta+n} < \omega^\alpha$ and so, since A is provident, $P^{\{p,x\}}_{l(\kappa)+c_H+1} \in A$ and so $F(x) \in A$ as required. $\dashv$ (6·53)

6·54 Definition We call such sets *limit provident sets.*

6·55 Theorem *Any directed union of limit provident sets is limit provident.*

Proof : The rank of such a directed union is a union of ordinals of the form ω^α with α limit, and so is of the same form. $\dashv$ (6·55)

6·56 Theorem *Let $\langle P_\nu \mid \nu \leqslant \theta\rangle$ be a solid $(\theta+1)$-progress. Then P_θ is limit provident iff θ is of the form ω^α with α a limit ordinal.*

Proof : Immediate from the definition, Theorem 6·34 and Proposition 6·11. $\dashv$ (6·56)

Provident levels of the Gödel, Jensen and related hierarchies

Let us start with parameter-free versions of results already proved.

6·57 Lemma *Let F be pure rud recursive, given by G. Then "f is an F-attempt" is a Δ_0 predicate of f.*

Proof : Here the formula required is

$$Fn(f)\ \&\ \bigcup \mathrm{Dom}\,(f) \subseteq \mathrm{Dom}\,(f)\ \&\ \forall x_{\in \mathrm{Dom}(f)}\ f(x) = G(f \restriction x). \qquad \dashv (6{\cdot}57)$$

6·58 Proposition *If u is transitive and $\varnothing$-provident then so is rud(u).*

Proof : We take $P_n = \mathbb{T}^n(u)$, and $P_\omega = \bigcup_n P_n$. $\langle P_\nu \mid \nu \leqslant \omega\rangle$ is then a strict continuous ω-progress, so we may apply Theorem 6·38 with $p = \varnothing$. $\dashv$ (6·58)

6·59 Corollary *Each non-empty J_ν is $\varnothing$-provident,*

Proof : $J_1 = \mathbf{HF}$; $J_{\nu+1} = \mathrm{rud}(J_\nu)$; the induction at limit stages is trivial. $\dashv$ (6·59)

6·60 Remark More generally, although for a given p in L we must go to the first indecomposable ordinal above the moment of construction of p to find a J_ν which is p-provident, every subsequent J_ξ will also be p-provident.

The following is a corollary of Theorem 6·34.

6·61 Theorem *J_ν is provident iff $\omega\nu$ is indecomposable.*

6·62 Example J_ω is provident. The next one will be J_{ω^2}.

To summarise: J_ν is $\varnothing$-provident iff $\nu > 0$, provident iff ν is positive and closed under addition, and limit provident iff ν is of the form ω^α with α a limit ordinal.

The $T_\nu[A]$'s

They were defined in 0·14 and form a solid progress, so similar remarks will apply.

The $T_\nu(c)$'s and $J_\nu(c)$'s

Let c be a transitive set; in 0·16 we defined

$$T_0(c) = c;\ T_{\nu+1}(c) = \mathbb{T}(T_\nu(c));\ T_\lambda(c) = \bigcup_{\nu<\lambda} T_\nu(c);\ L(c) = \bigcup_\nu T_\nu(c).$$

That sequence is not solid, but since $\varrho(T_\nu(c)) = \varrho(c)+\nu$, we have by Theorem 6·34 that $T_\nu(c)$ is provident iff ν is an indecomposable ordinal greater than $\varrho(c)$, and is limit provident iff ν is greater than $\varrho(c)$ and is of the form ω^α with α a limit. An induction argument using Theorem 6·38 then shows that $T_\nu(c)$ is $\varnothing$-provident for ν a limit ordinal at least as big as the least indecomposable ordinal greater than $\varrho(c)$. The condition that ν exceed the rank of c is inescapable, as $\varnothing$-provident sets contain the ranks of their members. For that reason we have in 6·24 preferred the solid progress P^c_ν.

With the $T_\nu(c)$'s in hand, we turn to the Jensen hierarchy: by an induction argument, for any ordinal ν we have $J_\nu(c) = T_{\omega\cdot\nu}(c)$. Thus $J_\nu(c)$ is provident iff ν is nonzero and closed under addition with $\omega \cdot \nu > \varrho(c)$. It is limit provident iff ν is bigger than $\varrho(c)$ and of the form ω^α with α a limit ordinal. Finally, $J_\nu(c)$ is $\varnothing$-provident if $\nu > 0$ and $\omega \cdot \nu$ is at least as big as the least indecomposable ordinal greater than $\varrho(c)$; as before, that condition is necessary for $\varnothing$-providence.

Provident levels of the L hierarchy

6·63 Gödel in his original paper used the function Def, where for transitive u, Def(u) is the set of subsets of u definable over u in the language of set theory allowing constants for members of u. Thus his recursion reads

$$L_0 = \varnothing; \quad L_{\nu+1} = \mathrm{Def}(L_\nu); \quad L_\lambda = \bigcup_{\nu<\lambda} L_\nu. \qquad \text{Then we set } L = \bigcup_{\nu\in ON} L_\nu.$$

6·64 Proposition *The sequence $\langle L_\nu \mid \nu \in ON\rangle$ is a solid progress.*

Proof : Two of the requirements are clear from the definition; we must show that for any ν, $\mathbb{T}(L_\nu) \subseteq L_{\nu+1}$. But it is immediate from the definition of $\mathbb{T}$ that each member of $\mathbb{T}(L_\nu)$ is a definable subset of L_ν. ⊣ (6·64)

6·65 Corollary *L_ν is $\varnothing$-provident iff ν is a limit, provident iff ν is indecomposable, and limit provident iff ν is of the form ω^α with α a limit ordinal.*

Proof : immediate from Theorems 6·37, 6·34 and 6·56.

6·66 Corollary *For each limit λ, $L_\lambda \supseteq T_\lambda$.*

6·67 Let us remark next that Gödel's recursion is not rudimentary. Note that for finite n, $L_n = V_n$ and thus by Remark 3·26 the rate of growth is too large to be that of a pure rudimentary recursion. But that particular argument collapses if we admit parameters, for $P_n^{V_\omega} = V_n$ for every n. Suppose, towards a contradiction, that there is a rud function G and parameter $p \in L$ with $L_{\nu+1} = G(p, L_\nu)$ for every ν. Choose θ indecomposable with $\theta = \omega\theta$ and

$p \in J_\theta = L_\theta = T_\theta$. Then for every limit ordinal $\lambda \geqslant \theta$, L_λ and T_λ are both p-provident, so we should have $L_\lambda = T_\lambda$. But that is false for $\lambda = \theta + \omega$:

6·68 PROPOSITION *If $L_\theta = T_\theta$ then $T_{\theta+\omega} \subsetneqq L_{\theta+\omega}$.*

Proof in outline: for transitive u, every element in $\mathbb{T}(u)$ is of the form $G(\vec{y})$ with G one of the rudimentary functions used in defining $\mathbb{T}$ and the arguments $\vec{y}$ in u. Iterating that observation shows us that there is a definable subset $\mathcal{C}_n$ of T_θ which codes $T_{\theta+n}$ in a sufficiently simple way to permit a diagonal argument to show that $\mathcal{C}_n \notin T_{\theta+n}$; but $\mathcal{C}_n \in L_{\theta+1}$. Hence $L_{\theta+1}$ is not a member of T_λ, but it is a member of L_λ. ⊣ (6·68)

The proof of [M3, Theorem 9·7] includes details, which readily generalise, of this argument for the case $\theta = \omega$.

6·69 The $L_\nu[A]$ behave well: since they too form a solid progress, the same remarks apply to them as to the L_ν.

REMARK Fairly brutal methods will show that for each κ the initial segment $\langle L_\nu \mid_\nu \nu < \kappa \rangle$ can be represented as a progress rudimentary in a well-chosen set parameter.

6·70 Construction from a set a: the sequence $(L_\nu(a))_{\nu \in ON}$ as traditionally defined is a strict and continuous progress, but it is not solid. Thus, since $\varrho(L_\nu(a)) = \varrho(a) + 1 + \nu$, we have by Theorem 6·34 that $L_\nu(a)$ is provident iff ν is an indecomposable ordinal greater than $\varrho(a)$, and is limit provident iff ν is greater than $\varrho(a)$ and of the form ω^α with α a limit. An induction argument using Theorem 6·38 then shows that $L_\nu(a)$ is $\varnothing$-provident for ν a limit ordinal at least as big as the least indecomposable ordinal greater than $\varrho(a)$; the need for that condition will be illustrated in Example 8·10.

Two other progresses

6·71 We mention briefly that it is possible to combine construction from a set and from a predicate. For example, we might wish to define a progress $\left(P_\nu^{c;B}\right)_\nu$ where c is a transitive set and B a class. The simplest method would be to replace $\mathbb{T}$ by the simpler form of $\mathbb{T}_B$, and to do nothing else; thus we should have this definition:

6·72 DEFINITION

$$c_0 = \varnothing \quad c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\} \qquad c_\lambda = \bigcup_{\nu<\lambda} c_\nu$$
$$P_0^{c;B} = \varnothing \quad P_{\nu+1}^{c;B} = \mathbb{T}(P_\nu^{c;B}) \cup \{P_\nu^{c;B} \cap B\} \cup \{c_\nu\} \cup c_{\nu+1} \quad P_\lambda^{c;B} = \bigcup_{\nu<\lambda} P_\nu^{c;B}$$

6·73 In [M8] we shall have a use for a progress $P^{c;D}$ where the relation D is itself being defined as the progress advances.

Suppose that A is provident and that $D \subseteq A$ is a relation, defined by a p-rud recursion, using the rud function G_D; and that H_D is the rud function given by the Propagation Lemma. Let c be a transitive set of which p is a member. We may define by a simultaneous p-rudimentary recursion sequences $(c_\nu)_\nu$, $(P_\nu)_\nu$, $(D_\nu)_\nu$ thus:

6·74 Definition

$$\begin{array}{llll} c_0=\varnothing & c_{\nu+1}= & c\cap\{x\mid x\subseteq c_\nu\} & c_\lambda=\bigcup_{\nu<\lambda}c_\nu\\ P_0=\varnothing & P_{\nu+1}= & \mathbb{T}(P_\nu)\cup\{c_\nu\}\cup c_{\nu+1}\cup\{P_\nu\cap D_\nu\} & P_\lambda=\bigcup_{\nu<\lambda_\nu}\\ D_0=\varnothing & D_{\nu+1}= & H_D(p,D_\nu,P_{\nu+1}) & D_\lambda=\bigcup_{\nu<\lambda}D_\nu \end{array}$$

In fact in [M8] we shall use Theorem 4·19 and its parametrized form above to simplify the further discussion.

Closure of Gödel levels under Scott–McCarty pairing

In Example 0·33 we considered these recursions:

$$\begin{array}{ll} \tau(y)=\{\varnothing\}\cup\big\{\tau(u)\,\big|_u\,u\in y\big\}; & \phi(y)=\{\phi(u)\mid_u u\in y\ \&\ \varnothing\in u\};\\ \sigma(x)=\big\{\sigma(t)\cup\{\varnothing\}\,\big|_t\,t\in x\big\}; & \psi(y)=\{\psi(u\setminus\{\varnothing\})\mid_u u\in y\}. \end{array}$$

and stated this lemma, which may be proved by an induction on $\varrho(z)$:

Lemma *$\varnothing$ is a member of every $\tau(y)$ and of no $\sigma(x)$; and for all z, $\sigma(z)=\tau(z)\setminus\{\varnothing\}$; $\phi(\tau(z))=z$; and $\psi(\sigma(z))=z$.*

6·75 Lemma *i) For $k\in\omega$, $\tau(k)=k+1$ and $\sigma(k)=k+1\setminus\{\varnothing\}$;*
ii) for $\zeta\geqslant\omega$, $\tau(\zeta)=\zeta$ and $\sigma(\zeta)=\zeta\setminus\{\varnothing\}$;
iii) restricted to ordinals, σ and τ are rudimentary.

6·76 Definition We introduce four pure rudimentary functions:

$$\begin{array}{ll} A(x)\ =_{\rm df}\ x\cup\{\varnothing\}; & B(y)\ =_{\rm df}\ y\cap\{x\mid\varnothing\in x\};\\ C(x)\ =_{\rm df}\ x\setminus\{\varnothing\}; & D(y)\ =_{\rm df}\ y\cap\{x\mid\varnothing\notin x\}. \end{array}$$

In terms of those functions, the Lemma states, in part, that $\sigma(x)=C(\tau(x))$; and the definitions of σ, τ, ϕ and ψ simplify to:

$$\sigma(x)=A``\sigma``x;\quad \tau(y)=A(\tau``y);\quad \phi(y)=\phi``B(y);\quad \psi(y)=\psi``C``y.$$

6·77 Definition If u is transitive and f is a unary function, we shall say that u is *definably closed under* f if $x\in u\implies f(x)\in u$ and for x,y in u, the relation $y=f(x)$ is definable over u (by a formula $\dot\Phi^f(\mathfrak{y},\mathfrak{x})$ of a set-theoretic object language, possibly with constants for members of u occurring as parameters); and for binary f the corresponding definition would require the relation $z=f(x,y)$ to be definable over u.

6·78 Proposition *Let χ be one of the functions σ, τ, ϕ, or ψ. Then each infinite L_ν is definably closed under χ and under $\chi``$.*

6·79 Lemma *Suppose that u is transitive and definably closed under a unary function f; specifically by the formula $\dot\Phi^f$. Then $\mathrm{Def}(u)$ is definably closed under $f``$.*

Proof : If $x=\big\{a\in u\ \big|\models_u\vartheta[a,p]\big\}$, $f``x=\Big\{b\in u\ \Big|\models_u\bigvee\mathfrak{a}\big[\vartheta(\mathfrak{a},p]\wedge\dot\Phi^f[b,\mathfrak{a})\big]\Big\}$.
Further, for y and x in $\mathrm{Def}(u)$,

$$y=f``x\iff\models_{\mathrm{Def}(u)}\bigwedge\mathfrak{a}_{\epsilon y}\bigvee\mathfrak{c}_{\epsilon x}\big(\dot\Phi^f(\mathfrak{a},\mathfrak{c})\big)^u\wedge\bigwedge\mathfrak{c}_{\epsilon x}\bigvee\mathfrak{a}_{\epsilon y}\big(\dot\Phi^f(\mathfrak{a},\mathfrak{c})\big)^u.\ \dashv(6\cdot79)$$

Remark The convention followed in the notation of the first line of that proof is this: when ϑ is a formula of an object language, we use Fraktur lower case letters for formal variables and indicate their occurrence by writing $\vartheta(\mathfrak{a}, \mathfrak{y})$; when those formal variables are interpreted, say by (names for) elements a and y of the model in question, we write $\vartheta[a, y]$; and usages such as $\vartheta(\mathfrak{a}, p]$ indicate that the second but not the first of the variables is being interpreted. The cumbersome use of an explicit substitution function Subst is thus avoided.

6·80 Lemma *Each (infinite) L_ν is closed under A, B, D, C, A“ and C“; all those six are rudimentary.*

Proof : If $F(x) \subseteq x$ and is rud, then every L_ν will be closed under it: at limits by rud closure; at successor stages by adding a condition to the definition of x as a subset of the stage before. That argument does B, C and D; and closure under A happens as each L_ν is closed under union of two members, and, if non-empty, contains $\varnothing$. Lemma 6·75 now does A“ and C“. ⊣ (6·80)

Proof of Proposition 6·78: χ is pure rud rec; so the relation $x = \chi(y)$ is definable over every L_λ (λ a limit ordinal) as also is χ“, as it is (pure) rud rec or gentle and we know from Corollary 6·65 that L_λ is $\varnothing$-provident; further the definition is independent of λ. Fix λ and write M_k for $L_{\lambda+k}$.

Suppose that we have reached a k where M_k is definably closed under χ and χ“. By Lemma 6·79, M_{k+1} is definably closed under χ“. We then use the recursion equation for χ, as simplified in 6·76, to deduce that M_{k+1} is definably closed under χ. Depending on which of the four functions χ is, we may have to invoke Lemma 6·80 for A“, A, B, or C“. ⊣ (6·78)

We recall and reformulate definitions given in §0:

6·81 Definition (Scott, McCarty) $\langle x, y\rangle_2^{\mathrm{SM}} =_{\mathrm{df}} \sigma“x \cup \tau“y$

6·82 Definition $\mathrm{left}^{\mathrm{SM}}(a) =_{\mathrm{df}} \psi“\big(D(a)\big)$; $\mathrm{right}^{\mathrm{SM}}(a) =_{\mathrm{df}} \phi“\big(B(a)\big)$.

6·83 Theorem *For each $\nu \geqslant \omega$, L_ν is definably closed under Scott-McCarty pairing and unpairing functions.*

Proof : Immediate from the above. ⊣ (6·83)

6·84 Remark In fact, for a an SM pair, $\mathrm{right}^{\mathrm{SM}}(a) = \phi(a)$; and if $d = D(a)$ then $\psi(d) = \psi“(d)$, so that $\mathrm{left}^{\mathrm{SM}}(a) = \psi(D(a))$: but those simplifications are misleading, as in the proof of closure in Proposition 6·78, χ“ precedes χ !

6·85 Remark The above proof generalises easily to show that for a predicate A and each infinite ν the levels $L_\nu[A]$ are closed under Scott–McCarty pairing and unpairing: but if c is a transitive set of infinite rank which is not SM-closed, the levels $L_\nu(c)$ are liable not to be SM-closed for small ν. For ν at least the first indecomposable ordinal exceeding the rank of c, all will be well, as the limit levels thereafter will be c-provident, and the successor levels will be covered by the arguments of Lemma 6·79.

6·86 Remark $T_{\omega+1} = T_\omega \cup \{T_\omega\}$ and is thus not closed under Scott-McCarty pairing. If we used $\mathbb{T}_{ON}$ instead of $\mathbb{T}$ then, whichever of the two definitions suggested in 0·14 we adopt, we would get at level $\omega + 1$ the set $T_\omega \cup \{T_\omega\} \cup \{\omega\}$ and do no better.

7: Provident closures and the Finite Basis Theorem

Provident closures

7·0 THEOREM *Suppose that M is a non-empty set. Let θ be the least indecomposable ordinal not less than $\varrho(M)$. Set*

$$\mathrm{Prov}(M) =_{\mathrm{df}} \bigcup\{P_\theta^{\mathrm{tcl}(s)} \mid s \in \mathcal{S}(M)\}.$$

Then $\mathrm{Prov}(M)$ *is provident and includes M, and if P is any other such,* $\mathrm{Prov}(M) \subseteq P$.

Here the notation $\mathcal{S}(M)$ is as introduced in Example 0·2.

Proof : $\mathrm{Prov}(M)$ is provident by Theorems 6.7 and 6.35. Suppose that P is provident and $M \subseteq P$. Then $\mathcal{S}(M) \subseteq P$; $\theta \leqslant On \cap P$; for each $s \in \mathcal{S}(M)$, $\mathrm{tcl}(s) \in P$, and for $\nu < \theta$, $P_\nu^{\mathrm{tcl}(s)} \in P$, and so $\mathrm{Prov}(M) \subseteq P$. ⊣ (7·0)

7·1 DEFINITION We call $\mathrm{Prov}(M)$ the *provident closure* of M.

7·2 THEOREM *Suppose that M is a non-empty set. Let θ be the least ordinal not less than $\varrho(M)$ and of the form ω^α with α a limit ordinal. Set*

$$\mathrm{LProv}(M) =_{\mathrm{df}} \bigcup\{P_\theta^{\mathrm{tcl}(s)} \mid s \in \mathcal{S}(M)\}.$$

Then $\mathrm{LProv}(M)$ *is limit provident and includes M, and if P is any other such,* $\mathrm{LProv}(M) \subseteq P$.

7·3 DEFINITION We call $\mathrm{LProv}(M)$ the *limit provident closure of M*.

The theories PROV, PROVI and LPROV

Theorem 7·0 implies that there is a finitely axiomatisable set theory (which we call PROV) of which the transitive models are the provident sets.

Let PROV be the following axioms

(7·3·0) extensionality;

(7·3·1) the ten axioms of GJ_0, as given in Section 1:

$\varnothing \in V$	$\bigcup x \in V$	$a \cap \{(x,y)_2 \mid x \in y\} \in V$
$\{x, y\} \in V$	$\mathrm{Dom}\,(x) \in V$	$\{(y,x,z)_3 \mid (x,y,z)_3 \in b\} \in V$
$x \setminus y \in V$	$x \times y \in V$	$\{(y,z,x)_3 \mid (x,y,z)_3 \in c\} \in V$
	$\{x``\{w\} \mid w \in y\} \in V$	

(7·3·2) `each set is in the domain of an attempt at the rank function`;

(which implies both TCo and set foundation)

(7·3·3) `any two ordinals are in the domain of an attempt at ordinal addition`;

(7·3·4) `for each transitive` $\mathfrak{c}$ `each ordinal is in the domain of an attempt at the sequence` $\langle \mathsf{P}^{\mathfrak{c}}_{\nu} \mid \nu \in ON \rangle$.

We write PROVI for PROV $+ \ \omega \in V$.

Theorem 7·0 will suffice to prove that the transitive models of PROV are the provident sets; the reasoning in this paper has been mainly semantic, but experience of the weak systems in [M3] suggest that if one wished to use PROV for syntactical reasoning, it would be desirable to enhance it by adding the axiom of infinity and the scheme of Π_1 foundation. The result will be finitely axiomatisable in the subtle sense of 1·47.

We write LPROV for the theory obtained from PROVI by adding the axiom

(7·3·5) `for each ordinal` α, ω `is in the domain of an attempt at the recursion` $F(\nu) = \alpha + \bigcup_{\mu<\nu} F(\nu)$.

Note that a formal statement of this axiom should include the postulation of a sufficiently long attempt at the function $\nu \mapsto \alpha + \nu$.

Then the transitive models of LPROV are the limit provident sets.

The theory $\varnothing$-PROV

Next, we will obtain a finite theory whose transitive models are the $\varnothing$-provident sets. For this, we will present a finite collection of rud rec functions which capture all rud rec functions, in the same sense that the canonical progresses (P^c_ν) capture all parametrized rud rec functions.

7·4 Definition Let $Y_1 : x \mapsto \{Y_1(y) \mid y \in x\} \cup \{x\}$, $Y_2 : x \mapsto \{\{Y_2(y)\} \mid y \in x\} \cup x$ and $Y_3 : x \mapsto \bigcup_{y\in x} \mathbb{T}(Y_3(y)) \cup x$.

Each of these is rud rec, and $Y_3(x)$ is transitive for any x.

7·5 Lemma *For any $n < \omega$ and any x we have $x \in Y_2^n(Y_1(x))$.*

Proof : By induction on n. The case $n = 0$ is immediate from the definition of Y_1, and for the induction step, for any x we have $x \in Y_2^n(Y_1(x)) \subseteq Y_2(Y_2^n(Y_1(x)))$. ⊣ (7·5)

7·6 Definition We define the relations $\in_n$ for each $n < \omega$ by $x \in_0 y$ iff $x = y$, and $x \in_{n+1} y$ iff there is z with $x \in_n z \in y$. Thus $y \in_n x$ iff there is a sequence $y = y_0 \in y_1 \in y_2 \in ... \in y_n = x$.

7·7 Lemma *For any $n < \omega$ and any $y \in x$ we have $Y_2^n(y) \in_{2^n} Y_2^n(x)$.*

Proof : By induction on n. The case $n = 0$ holds by definition. For the induction step, we have $Y_2(y) \in \{Y_2(y)\} \in Y_2(x)$ and so by the induction hypothesis $Y_2^n(Y_2(y)) \in_{2^n} Y_2^n(\{Y_2(y)\}) \in_{2^n} Y_2^n(Y_2(x))$, so that $Y_2^{n+1}(y) \in_{2^{n+1}} Y_2^{n+1}(x)$. ⊣ (7·7)

7·8 Theorem (Bowler) *Let F be rud rec, and let $n < \omega$ with $2^n \geq c_F$. Then for any x, and any $y \in x$, $Y_3(Y_2^n(Y_1(x)))$ contains an F-attempt attaining y.*

Proof : By induction on $\varrho(x)$. Let $y \in x$. Then $Y_1(y) \in Y_1(x)$ and so by Lemma 7·7 $Y_2^n(Y_1(y)) \in_{2^n} Y_2^n(Y_1(x))$ and so we can find a sequence $Y_2^n(Y_1(y)) = y_0 \in y_1 \in \ldots \in y_{2^n} = Y_2^n(Y_1(x))$. Define a $2^n + 1$-progress P by $P_i = Y_3(y_i)$. By the induction hypothesis P_0 contains, for each $z \in y$, an F-attempt attaining z, and by Lemma 7·5 and the definition of Y_3 we have

$y \in P_0$. So by Proposition 6·32 we know that P_{c_F} contains an F-attempt attaining y, which F-attempt must then be contained in each P_j with $j \geq c_F$, and in particular in $P_{2^n} = Y_3(Y_2^n(Y_1(x)))$. ⊣ (7·8)

Thus if we let ∅-PROV be the following axioms then the transitive models of ∅-PROV are the ∅-provident sets:

(7·8·0) extensionality

(7·8·1) the ten axioms of GJ_0, as given in Section 1.

(7·8·6) `each set is in the domain of an attempt at each of` Y_1, Y_2 `and` Y_3.

In fact, we need only add a very simple kind of parametrized recursion to obtain a theory equivalent to PROV. The recursive definition of ordinal addition makes sense even if the first input is not an ordinal: for any set x define $x + \beta$ by recursion on the ordinal β, as $x + 0 = x$ and $x + \beta = \bigcup_{\gamma<\beta}((x+\gamma) \cup \{x+\gamma\})$ for $\beta > 0$. (This definition is intermediate between the ordinary $\alpha+\beta$ of ordinal addition and the definition of $A+B$ for arbitrary sets A and B given in [SMcC]). We get a theory whose transitive models are provident sets by adding the following axiom to ∅-PROV:

(7·8·3) `for any set` x`, each ordinal is in an attempt at the function` $\beta \mapsto x + \beta$.

This works because for any set x, the sequence $(Y_3(x+\beta))_\beta$ is a progress, so we can get any value of any parametrized rud rec function using Theorem 6·33.

8: Models of stunted growth

We have mentioned "Model $\mathbf{M}_{13,\lambda}$" studied in *Weak Systems* [M3], which is supertransitive and a proper class but which contains only the ordinals $< \lambda$; so in that model rank is stunted.

8·0 Aside Consider that model in the special case $\lambda = \omega$; ω is not a member of $\mathbf{M}_{13,\omega}$, which is otherwise a model of Z, save for the axiom of infinity in its customary form. But that axiom is not used in defining the finite basis of rudimentary functions; so $\mathbf{M}_{13,\omega}$ is rud closed; and therefore ω is not of the form $F(x)$ for any rud function F and $x \in \mathbf{M}_{13,\omega}$.

That is the promised sketch of the argument for Gandy's Theorem 2.1.3. It also demonstrates the claim in Remark 0·18 that the rank function is not rudimentary. Note that in Model $\mathbf{M}_{13,\lambda}$, TCo holds; by supertransitivity, the actual transitive closure of each member of the model is a member of the model.

8·1 Historical Note Priority for the underlying idea of the definition of the model $\mathbf{M}_{13,\omega}$, in a different context, must go to Jonathan Stavi. In Example 3 on page 610 of his paper [Stav], he considers a countable admissible set M and the set T of those $x \in M$ with ω not a subset of $\mathrm{tcl}(x)$, and shows that T is not closed under rank and is not a union of admissible sets.

The first author records his gratitude to Zachary McKenzie for drawing his attention to Stavi's paper.

We may generalise the idea behind model $\mathbf{M}_{13}$ thus:

8·2 Definition Suppose that $F : On \longrightarrow V$ is a function such that for $\mu < \nu$ we have $F(\mu) \in F(\nu)$. For limit λ, set

$$A_{F,\lambda} =_{\mathrm{df}} \left\{u \mid \bigcup u \subseteq u \ \&\ \sup\{\nu \in On \mid F(\nu) \in u\} < \lambda\right\}; \quad M_{F,\lambda} = \bigcup A_{F,\lambda}.$$

8·3 Proposition *If there is $\nu < \lambda$ with $F(\nu) \notin \omega$ then $M_{F,\lambda}$ is a supertransitive model of* Z *for which $F(\xi) \in M_{F,\lambda} \iff \xi < \lambda$. For any F and λ, the model $M_{F,\lambda}$ will be a proper class.*

Proof : as in Section 7 of [M3]. The union of two members of $A_{F,\lambda}$ is in $A_{F,\lambda}$, and if $u \in A_{F,\lambda}$, so is $\mathcal{P}(u)$; so that $M_{F,\lambda}$ will be a supertransitive model of Z. If $F \restriction \lambda$ only takes ordinals as values, the argument in [M3, p. 182] shows that $M_{F,\lambda}$ will contain sets of all ranks. Otherwise, there is some $\eta < \lambda$ such that $F(\xi)$ is not an ordinal for $\xi \geqslant \eta$, and in that case $M_{F,\lambda}$ will contain all ordinals. ⊣ (8·3)

8·4 Definition For limit λ, set $\mathbf{A}_{17,\lambda} =_{\mathrm{df}} \left\{u \mid \bigcup u \subseteq u \ \&\ \sup\{\nu \mid P^{\varnothing}_{\nu} \in u\} < \lambda\right\}$; $\quad \mathbf{M}_{17,\lambda} =_{\mathrm{df}} \bigcup \mathbf{A}_{17,\lambda}$.

8·5 Proposition $\mathbf{M}_{17,\lambda}$ *is a supertransitive proper class, containing all ordinals but the T hierarchy only up to λ but no further. In this model the rud recursion defining rank is total but that defining the growth of the Jensen auxiliary hierarchy stops prematurely.*

8·6 Proposition *There is a supertransitive class model $\mathbf{M}_{18,\lambda}$ of* Z *which contains a Cohen generic real c, and all constructible sets, but such that neither $L_{\omega+\omega}(c)$ nor $P^c_{\omega+\omega}$ is in $\mathbf{M}_{18,\lambda}$.*

Proof : This time take $\lambda = \omega + \omega$ and $F(\zeta) = P^c_\zeta$ and $\mathbf{M}_{18,\lambda} = M_{F,\lambda}$. $c \in P^c_\zeta$ whenever $\zeta \geqslant \omega + 1$, so that each $L_\eta \in A_{F,\lambda}$ and $L \subseteq M_{F,\lambda}$. ⊣ (8·6)

In the above model the Jensen hierarchy exists for all ordinals, but the same hierarchy relativised to c is defined before but not at level $\omega + \omega$.

8·7 Remark We have seen that in the model $\mathbf{K}$, which should have been called $\mathbf{M}_{16}$, of section 12 of [M3], the definition of tcl is stunted, and therefore also the definition of rank, for if every set is a member of the domain of some attempt at ϱ, that domain will be a transitive set; so TCo holds, and hence tcl may be recovered using the full strength of the axioms of Z.

$\mathbf{M}_{13}$ is a model of ZC in which rank is stunted but tcl not; $\mathbf{M}_{17}$ is a model of ZC in which the Jensen hierarchy is stunted but tcl and rank not; $\mathbf{M}_{18}$ is a model of ZC in which the relative Gödel and Jensen hierarchies $L_\nu(c)$ and $J_\nu(c)$ are stunted but the hierarchies L_ν and J_ν and tcl and rank are not. So there is a certain ordering to some rudimentary recursions; but we have seen in Section 7 that there is, in a sense, a finite basis to the collection of rud recursions.

Failure of Scott's trick in a model of Zermelo

We record here another variant of the above construction.

8·8 Definition Let $A = \leqslant_R$ be a well-ordering, viewed as a binary relation $\leqslant_R$ on the set $\{x \mid (x, x)_2 \in A\}$. For such A, define $I(A)$ to be the class of well-orderings isomorphic to A, and, in imitation of Scott's celebrated trick for reducing equivalence classes to equivalence sets, let $ST(A)$ be the class $\{B \in I(A) \mid \forall C_{\in I(A)}\, \varrho(C) \geqslant \varrho(B)\}$, the class of wellorderings of minimal rank isomorphic to A.

The following shows that Z is too weak a set theory for Scott's trick to work.

8·9 Theorem *Let $\kappa = \beth_\kappa$ be a beth fixed point. Let A_κ be the epsilon relation restricted to κ; thus a well-ordering of length κ. There is a supertransitive, proper class, model $\mathbf{M}_{19}$ of Z containing all ordinals and the well-ordering A_κ, in which every set has a rank, but in which $(ST(A_\kappa))^{\mathbf{M}_{19}}$, though a definable class of the model, is not a set.*

Proof : Take $F(\nu) = V_\nu$ and $\mathbf{M}_{19} = M_{F,\kappa}$. $V_\nu \in \mathbf{M}_{19} \iff \nu < \kappa$. As κ is a beth fixed point, $V_\kappa = H_\kappa$, so that all well-orderings of length κ in the universe must be of rank at least κ. Thus $A_\kappa \in I(A_\kappa)$. Let B_ξ be the well-ordering $\{(b_\nu, b_\zeta)_2 \mid_{\nu,\zeta} \nu \leqslant \zeta < \kappa\}$ where for $\zeta \leqslant \xi$, $b_\zeta = V_\zeta$, and for $\xi < \zeta < \kappa$, $b_\zeta = \zeta$.

Then each $B_\xi \in \mathbf{M}_{19}$, being obtained from $V_{\xi+1}$ and A_κ by rudimentary operations. Further each B_ξ is of rank κ, and thus is in $ST(A_\kappa)$, even as defined in $\mathbf{M}_{19}$. Thus the class $ST(A_\kappa)$ cannot be a set in $\mathbf{M}_{19}$, as $\bigcup^4 ST(A_\kappa) = V_\kappa$. ⊣ (8·9)

Remarks on Zermelo and von Neumann natural numbers

Definition The Zermelo natural numbers are those in the set $D =_{\rm df} \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, ...\}$. The von Neumann natural numbers are of course the members of ω.

With the help of the set D we may illustrate a point arising in §6.

8·10 Example By induction on ν, $L_\nu(D) \cap ON \subseteq 2+\nu$, whereas $\varrho(L_\nu(D)) = \omega + 1 + \nu$. Thus $L_\nu(D)$ cannot be closed under the rank function for $\nu < \omega^2$, and in particular $L_\nu(D)$ is $\varnothing$-provident only at limit ordinals at least as big as ω^2. A similar argument shows that $J_\nu(D)$ is $\varnothing$-provident only for $\nu \geqslant \omega$.

8·11 Proposition *There is a supertransitive model of Z of which ω but not D is a member.*

Proof : Proposition 14·7 of *Weak systems* reads

Proposition *Suppose that $(x_n)_n$ and $(u_n)_n$ are two sequences of sets such that for each $n < \omega$:*

(14·7·0) $x_n \in u_n$;
(14·7·1) $u_n \subseteq u_{n+1}$;
(14·7·2) u_n *is transitive;*
(14·7·3) $x_n \in \mathrm{tcl}(x_{n+1})$;
(14·7·4) $x_{n+1} \notin u_n$.

Then $\bar u =_{\rm df} \bigcup_n u_n$ is transitive and if w is a transitive set with $x_0 \notin w$, the set $\bar x =_{\rm df} \{x_n \mid n \in \omega\}$ is not a member of the rud closure of $\bar u \cup w \cup \{w\}$. If in addition $\omega \subseteq w$, then there is a supertransitive model of Zermelo set theory of which $\bar u \cup w \cup \{w\}$ is a subset but $\bar x$ and $\bar u$ are not members.

Take $x_n =$ Zermelo's $n+2$, so $x_0 = \{\{\varnothing\}\}$ which is not a member of ω, and take $u_n = V_{n+3}$. Then conditions (14·7·0) – (14·7·4) are satisfied.

Take $w = \omega$ and let K be the supertransitive model of Z supplied by the last sentence of the Proposition. Then $\omega \in K$ and $\bar{x} \notin K$. As $\bar{x} = D \setminus \{0, 1\}$ and K is rud closed, $D \notin K$. ⊣ (8·11)

8·12 Proposition *There is a supertransitive model of* Z *of which D but not ω is a member.*

Proof : Such a model is model $\mathbf{M}_{13,\omega}$; for $D \cup \{D\}$ is transitive and contains only the ordinals 0 and 1, and is thus a subset of $\mathbf{M}_{13,\omega}$. ⊣ (8·12)

9: The truth predicate for $\dot{\Delta}_0$ sentences

In [M3, section 10, culminating in page 211] it is shown that the truth predicate $\models_u \varphi$ is, provably in MW, Δ_1-definable. The comment on Devlin's Lemma VI.1.14 in [M3, page 210, lines 6 and 7] inadvertently omitted the words "defer to" between "but" and "*Rudimentary Recursion*". We now exceed our intended promise by proving the following sharpening.

9·0 Theorem (Mathias) *Truth for $\dot{\Delta}_0$ sentences is uniformly Δ_1 for transitive models of* MW.

Our method derives from those of [M3]: the notion of "attempt at integer addition", used below, is that of the discussion of [M3, paragraph 2·56, page 165], and "sufficiently long" is to be understood in the sense explained in [M3, paragraph 10·3, page 200].

Proof : We must begin by introducing some notation for $\dot{\Delta}_0$ formulae, in order to maintain the distinction between formulae in the object language and those in the language of discourse. Thus we denote conjunction, disjunction, and negation in the object language by $\wedge$, $\vee$ and $\neg$ rather than & , V and $\neg$. We denote unversal and existential quantification by $\bigwedge$ and $\bigvee$, and we denote the equality and membership relations by = and ϵ. We shall typically denote variables in the object language by lowercase letters in the Fraktur font, such as $\mathfrak{x}$ or $\mathfrak{y}$, and formulae in the object language with variant forms of lowercase Greek letters, such as ϑ or φ. The notation for restricted quantifiers in the object language is also new: for example, instead of $\forall x_{\in y}$, we write $\bigwedge \mathfrak{x}_{\epsilon \mathfrak{y}}$. For any set a, the object language contains a name $\mathring{a}$ for a.

Let M be a transitive model of MW, and φ a $\dot{\Delta}_0$ sentence of $\mathcal{L}_M$. It suffices to find a Σ_1 definition of $\models_M \varphi$, for if a truth predicate for a class of sentences that is closed under negation is Σ_1 it will automatically be Π_1, since $\models \varphi \iff \neg \models \neg\varphi$.

We have a sentence φ; let k be its length; let N_φ be the finite set comprising those members of M of which the names occur in φ; let q_φ be the number of occurrences of quantifiers in φ.

Step 1: we rewrite φ by de-nesting restricted quantifiers: for example,

$$\text{replace } \bigwedge \mathfrak{x}_{\epsilon \mathring{a}} \bigvee \mathfrak{y}_{\epsilon \mathfrak{x}}\, \vartheta \text{ by } \bigwedge \mathfrak{x}_{\epsilon \mathring{a}} \bigvee \mathfrak{y}_{\epsilon \mathring{c}} [\mathfrak{y} \epsilon \mathfrak{x} \wedge \vartheta], \text{ where } c = \bigcup a.$$

We thus reach within q_φ steps a formula φ' in which all quantifiers are restricted by *free terms*, each of the form <name of> $\bigcup^m a$, where $a \in N_\varphi$ and $m < q_\varphi$. As the Axiom of Union is among those of MW, each such $\bigcup^m a$

will be in M. Let F_φ be the finite set comprising those members of M of which the names occur in φ'.

A similar process is described in some detail in section 8 of [M3], though there, but not here, the formalism admits limited quantifiers as well as restricted ones.

Step 2: using the usual procedures of predicate logic, we rewrite φ' in prenex form, thus reaching a sentence φ^* in which a string of quantifiers, all restricted by free terms, precedes a quantifier-free formula ϑ.

These two steps may be achieved by primitive recursive processes applied to the formulæ in question.

We must now show that M contains a set which contains all the constants that will occur in substitution instances of subformulæ of φ^*: but such a set will be $P_\varphi \ =_{\mathrm{df}}\ F_\varphi \cup \bigcup F_\varphi$

Let S_φ be the class of quantifier-free sentences, of length no longer than k, in which the only names occurring are those of members of P_φ, That, provably in MW, will be a set.

Step 3: we show that truth for members of S_φ is uniformly Σ_1 for transitive models of MW.

Specifically, we show that there is an *evaluation* $g_\varphi : S_\varphi \longrightarrow 2$, that is, a function which obeys the rules for evaluation of Boolean combinations of atomic statements. These rules are:

$$g(\mathring{x} = \mathring{y}) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$
$$g(\mathring{x} \mathrel{\epsilon} \mathring{y}) = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{if } x \notin y \end{cases}$$
$$g(\neg\vartheta) = 1 - g(\vartheta)$$
$$g(\vartheta_1 \wedge \vartheta_2) = \inf\{g(\vartheta_1), g(\vartheta_2)\}$$
$$g(\vartheta_1 \vee \vartheta_2) = \sup\{g(\vartheta_1), g(\vartheta_2)\}$$

and similarly for other propositional connectives if they have also been taken as primitive.

We saw in [M3] that a statement of the form $\vartheta = \vartheta_1 \wedge \vartheta_2$ is not Δ_0 but will be Δ_0 in any sufficiently long attempt at addition. As the sentences to be considered are all of length not exceeding that of φ, a single sufficiently long attempt, α, will exist, and we shall be able to express the above rules for g as a statement that is $\Delta_0(\alpha, g, S_\varphi)$. Thus the desired Σ_1 truth predicate for sentences ϑ in S_φ will be of the form

$$\exists\alpha\text{, a sufficiently long attempt at addition, and}$$
$$\exists g : S_\varphi \longrightarrow 2,\ g \text{ an evaluation, with } g(\vartheta) = 1.$$

Step 4: we show how to reduce the computation of truth of φ^* to that of numerous substitution instances in S_φ.

9·1 Remark This step would be possible even if we had not done Steps One and Two, but would be more complicated to express.

Suppose that φ^* has $n+1$ quantifiers, so that there are $a_0, \ldots, a_n$ in M such that $\varphi_\emptyset$ is

$$\mathcal{Q}_0 \mathfrak{x}_{0 \epsilon \mathring{a}_0} \mathcal{Q}_1 \mathfrak{x}_{1 \epsilon \mathring{a}_1} \ldots \mathcal{Q}_n \mathfrak{x}_{n \epsilon \mathring{a}_n} \vartheta$$

where $\vartheta \in S_\varphi$ but may contain other names besides those shown. n is not greater than k.

We consider the tree T of all finite sequences $\langle c_0, \ldots c_\ell \rangle$ of members of P_φ where $\ell \leqslant n$ and for each i, $c_i \in a_i$. Provably in MW, T is a set. We write $\emptyset$ for the empty sequence.

We define for each $t \in T$ a sentence φ_t by recursion on the length of t.

Let $\varphi_\emptyset = \varphi^*$.

Once we have defined φ_t then for $c \in a_{\ell h(t)}$ we define $\varphi_{t \frown \langle c \rangle}$ to be $\mathrm{Subst}(\varphi_t, \mathfrak{x}_{\ell h(t)}, \mathring{c})$.

Let $T_\varphi = \{\varphi_t \mid t \in T\}$, a tree of sentences.

Let g_φ be the evaluation defined on S_φ in Step 3. Extend it to T_φ by a reverse induction: if $\ell h(t) = n+1$, φ_t will be a member of S_φ, and so $g_\varphi(\varphi_t)$ has been defined in Step 3. If $g_\varphi(\varphi_u)$ has been defined for all $u \in T$ of length $\ell h(t)+1$, then define

$$g_\varphi(\varphi_t) = \begin{cases} \sup\{g(\varphi_{t \frown \langle c \rangle}) \mid c \in a_{\ell h(t)}\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigvee \\ \inf\{g(\varphi_{t \frown \langle c \rangle}) \mid c \in a_{\ell h(t)}\} & \text{if } \mathcal{Q}_{\ell h(t)} \text{ is } \bigwedge \end{cases}$$

So $\overset{0}{\models} \varphi \iff g_\varphi(\varphi_\emptyset) = 1$.

We have finally to observe that as M models MW, then for φ a $\dot{\Delta}_0$ sentence of $\mathcal{L}_M$, all the above sets and functions, in particular P_φ, S_φ, T_φ and g_φ are in M; so the desired Σ_1 formula simply says that there exist sets and functions which obey the rules imposed on them and which lead to the evaluation of φ. ⊣ (9·1)

The same argument with very few changes will give a less laborious proof of the result proved in section 10 of [M3]:

9·2 Theorem *The truth predicate $\models_u \varphi$, for u a transitive set and φ an arbitrary sentence of $\mathcal{L}_u$, is Δ_1^{MW}.*

Proof : Immediate from Theorem 9·0, since the process of replacing each unbounded quantifier $\bigwedge \mathfrak{x}$ or $\bigvee \mathfrak{x}$ by the corresponding bounded quantifier $\bigwedge \mathfrak{x}_{\epsilon \mathring{u}}$ or $\bigvee \mathfrak{x}_{\epsilon \mathring{u}}$ is primitive recursive. ⊣ (9·2)

9·3 Remark A similar argument shows that the predicate $\models_A \varphi$ is Δ_1^{MW}, where φ is a sentence of some language (not necessarily the language of sets) and A is a small internal structure for that language, where the functions and relations of the language are all coded in the usual way by sets.

Notes and acknowledgments

The first author's interest in the problems addressed in this paper and its sequel was fired in 1987 by Stanley's call, in his review [Stan] of Devlin's treatise [De], for a development of constructibility that would meet Devlin's unachieved aims. Subsequently, in Barcelona in the mid 90s, the author became greatly interested in the problem of finding the weakest system of set theory that will support a recognisable theory of forcing. Over the following decade he accumulated assorted observations about weak systems, which during the set theory year, 2003-4, at the CRM at Barcelona, grew into a coherent apparatus [M3] for addressing the problems with Devlin's book, and which sowed the seeds of the theory of rudimentary recursion and the sense that the search for the correct minimal theory for a development of forcing was getting warm.

He began a series of draft papers, called rudrec or fifo, with a draft number; rudrec4, dated October 2004, gives the definition of *rudimentarily recursive* and the beginning of a discussion of forcing in that context, and asks for an example of a function that, in today's terminology, is gentle but not rud rec.

In February 2006, the author was encouraged by a correspondence [S] with Dana Scott who had found the Gandy–Jensen theory of rudimentary functions useful in the study of certain problems in formal geometry, and who was relieved to find that [M3] had, as he put it, "rescued" Devlin's book.

Gradually the theory of rudimentary recursion matured; by November 2007 the idea of what is now called a provident set was there, and a scenario for a proof that a generic extension of a provident set would be provident. That scenario ran into difficulties but the proof was saved by the idea of construction from a "dynamic" predicate: one defined by simultaneous recursion with a particular strict continuous progress, as in [M5, Definition 8·5]. Without this notion, the proof as it stood would have needed the ground model to be limit provident, not just provident.

Progressively more mature versions of this material were presented in the ERMIT seminar in Réunion, where they benefitted from the comments of Dr Olivier Esser; in talks in January 2008 at University College, London and at Oberwolfach, following which the term "provident" was adopted; in a talk in May 2008 at Leeds; and in October 2008 at Lisbon, in talks based on rudrec36 and fifo27. At the Zermelo centenary meeting at Brussels in late October 2008, the complementary theme of the inadequacy of Zermelo's sytem for forcing was discussed, as was the compensatory use of the passage to the provident closure or to the lune [M4].

In March 2009 copies of rudrec39 and fifo29 were sent to an Editor of *Fundamenta Mathematicæ*, who declined the first as being on a topic unsuited to his pages but asked for formal submission of the second, on forcing over provident sets, to be made: accordingly on May 5, 2009, fifo31 [M6] was submitted, with rudrec41 [M5] attached for the assistance of the referee.

In July 2009 the material was presented in condensed form in talks at Oxford and at Bedlewo, and an extended abstract prepared for the website of the latter meeting.

The first author received the referee's conditionally favourable report from *Fundamenta* on July 5th 2010, a week or so after receiving the kind invitation of Professor Martin Hyland to give a course of twenty four lectures to a graduate and post-doctoral audience at Cambridge (U.K.) in the Michaelmas Term, 2010. He accepted with pleasure this invaluable opportunity of testing in detail his approach to constructibility and forcing *via* weak systems, rudimentary recursion and provident sets.

In the Cambridge audience was the second author, who promptly found the counterexample given in Proposition 4·5 to the question whether the composition of two rud rec functions is rud rec, and who went on to prove Proposition 4·8 and Theorem 4·9. The elegance and strength of his notion of a gentle function have subsequently been confirmed in his Theorem 4·19; and comparison with [M5] will show how his ideas have interacted with those of the first author, in some cases, such as those of "dynamic predicate" and "function of uniform affine growth", supplanting them, and in others, clarifying and developing them and where necessary giving them appropriate concrete form and generality.

Besides those mentioned above, the first author thanks Carlos Montenegro in Bogotà, and in Barcelona James Harris and the seminar of Joan Bagaria, for their patience in listening to immature versions of these ideas; and for their steadfast encouragement in his study of weak systems, Kai Hauser, Ronald Jensen and Colin McLarty.

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