# A BARREN EXTENSION

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ABSTRACT. It is shown that provided $\omega \to (\omega)^\omega$, a well-known Boolean extension adds no new sets of ordinals. Under an additional assumption, the same extension preserves all strong partition cardinals. This fact elucidates the role of the hypothesis $V = L[R]$ in the Kechris-Woodin characterization of the axiom of determinacy.

## §0. INTRODUCTION.

Let $B = \text{Power}(\omega)/\text{Fin}$ be the quotient of the Boolean algebra of all subsets of $\omega$ by the ideal Fin of finite sets, and $P = [\omega]^\omega/\text{Fin}$ the set of non-zero elements of $B$, with the induced partial ordering. We shall study the Boolean extension that results from using $P$ as a notion of forcing: with a famous theorem [5] about the existence under AC of $(\omega_1,\omega_1{}^*)$ gaps in $P$ in mind, we shall call this the Hausdorff extension.

Our underlying set theory is Zermelo-Fraenkel. We shall be working in contexts where the full axiom of choice is false; at times, we shall use DC, the axiom of dependent choices, or DCR, its weaker form, which states that if $Q$ is a relation on Power($\omega$) such that $\forall p \exists q\, Q(p,q)$, then there is a map $f:\omega \to \text{Power}(\omega)$ such that $\forall n\, Q(f(n),f(n+1))$. Our notation of Boolean extensions and forcing follows that described in Mathias [15], 3.7 and 3.8. We write $[\kappa]^\lambda$ for the set of subset of $\kappa$ of order type $\lambda$, and follow Supercontinuity [8] in our notation of partition relations.

It has long been known that under DCR the Hausdorff extension adds no new sets of integers: what it does add is a Ramsey ultrafilter on $\omega$. (Cf [15], Theorem 4.2). For a recent discussion under AC of $P$, see Dordal [3].

Since without AC there are difficulties in the simultaneous choice of representatives of equivalence classes, it will be convenient to take as our forcing conditions infinite subsets $p,q$ of $\omega$, with the understanding that if $p$ and $q$ have finite symmetric difference, written $p \simeq q$, they force the same statements, and that $p$ is a stronger condition than $q$ if $p \setminus q$ is finite.

In section 1 we prove in the theory $ZF + \omega \to (\omega)^{\omega}$ that the Hausdorff extension is barren in the sense that every map in the extension from an ordinal into the ground model lies in the ground model: in particular, no new sets of ordinals are added. We characterize this latter property in terms of the Galvin-Prikry notion [6] of completely Ramsey families.

In section 2, we consider three set-theoretic principles, called LU, LSU and EP. LU and LSU are the weak and strong uniformisation principles discussed in Mathias [16], where it is shown that LSU is equivalent in ZF + DCR to $\omega \to (\omega)^{\omega}$ + LU, and that LSU is true if AD holds and $V = L[R]$ or if ADR holds, or if V is Solovay's model for "all sets of reals are Lebesgue measurable". EP, which we derive in this section from LSU in the theory ZF + DCR, is, to use the topological terminology of Ellentuck [4], the assertion that the intersection of any well-ordered collection of co-meagre sets is co-meagre.

In section 3, we prove in the theory ZF + LU + EP that the Hausdorff extension preserves every partition property of the form $\kappa \to (\kappa)^{\lambda}{}_{\mu}$, where $\kappa > \omega$, $2 \le \mu < \kappa$, and $0 < \lambda = \omega\lambda \le \kappa$.

Finally, in section 4, we comment on the implications of the results of section 3 for the recent characterization [12] of AD in $L[R]$ by Kechris and Woodin, and discuss some problems related to our work: reference to this discussion is made in the text by the string [PROB].

Assumptions are given in full in the statements of theorems, but may be omitted in Lemmata and Propositions when the flow of the narrative demans it. The end of a proof is signalled by $\dashv$.

## §1. THE BARRENNESS OF THE HAUSDORFF EXTENSION.

THEOREM 1.0. Let M be a transitive model of $ZF + \omega \to (\omega)^{\omega}$ and N its Hausdorff extension. Then M and N have the same sets of ordinals; moreover every sequence in N of elements of M lies in M.

PROOF. It will be sufficient to prove in the theory $ZF + \omega \to (\omega)^{\omega}$ that if $p_0 \Vdash f:\hat{\zeta} \to \hat{V}$, then for some $q_0 \le p_0$, $q_0 \Vdash f \in \hat{V}$.

Fix then such $p_0$, $f$, $\zeta$ and suppose that no such $q_0$ exists. Then for each $p \in [p_0]^\omega$ there will be at least one ordinal $\xi < \zeta$ for which no $x \in V$ exists with $p \Vdash f(\hat{\xi}) \equiv \hat{x}$: define $\phi(p)$ to be the least such $\xi$.

For $p \subseteq \omega$, define $p_\ell \subseteq \omega$, $p_r \subseteq \omega$ by writing $\tilde{q}$ for the monotonic enumeration of $q \subseteq \omega$ and setting $\tilde{p}_\ell(n) = \tilde{p}(2n)$, $\tilde{p}_r(n) = \tilde{p}(2n+1)$.

Now define a partition $\pi:[p_0]^\omega \to 3$ by setting $\pi(p) = 0$, 1 or 2 according as $\phi(p_\ell)$ is less than, equal to, or greater than $\phi(p_r)$, and let $P \in [p_0]^\omega$ be homogeneous for $\pi$. Notice that $\pi(P) = 1$: for putting $k = \min P$ and $Q = P \setminus \{k\}$, we have $Q_\ell = P_r$, $Q_r = P_\ell \setminus \{k\}$; $P_\ell$ and $P_\ell \setminus \{k\}$ force the same statements, and so have the same value under $\phi$; thus if $\pi(P) = 0$, $\pi(Q) = 2$; if 2, 0; but $\pi(P) = \pi(Q)$ by homogeneity, so $\pi(P)$ can only equal 1.

Put now $\nu = \phi(P)$. By definition of $\phi$ there are $q, r \in [P]^\omega$ and $x, y \in V$ with $x \neq y$, $q \Vdash f(\hat{\nu}) \equiv \hat{x}$ and $r \Vdash f(\hat{\nu}) \equiv \hat{y}$. Define $s \in [P]^\omega$ by choosing the first three elements of $s$ from $q$, the fourth from $r$, then three from $q$, then one from $r$, and so on, so that $s_\ell \subseteq q$, $(s_r)_\ell \subseteq q$, $(s_r)_r \subseteq r$.

Since $s_r$ is compatible with both $q$ and $r$, and extends $P$, $\phi(s_r) = \nu$. Since $s_\ell \subseteq q$, $\phi(s_\ell) \geq \phi(q)$; but $\phi(q) > \nu$, since $q \subseteq P$ and $q$ forces the value $x$ for $f(\nu)$. So $\phi(s_\ell) > \phi(s_r)$ and thus, $\pi(s) = 2 \neq 1 = \pi(P) = \pi(s)$, an evident absurdity. $\dashv$

As in Happy Families [15], for $s \in [\omega]^{<\omega}$, we write $|s|$ for $\sup\{n+1 \mid n \in s\}$, and for $p \in [\omega]^\omega$ with $|s| \leq \cap p$, we write $[s,p]$ for $\{x \in [\omega]^\omega \mid s \subseteq x \subseteq s \cup p\}$. Thus $[0,p] = [p]^\omega$.

<u>DEFINITION 1.1.</u> A subset $A$ of $[\omega]^\omega$ will be called invariant if ($p \in A$ and $p \approx p'$) always implies $p' \in A$; similarly a function $F$ defined on (an invariant subset of) $[\omega]^\omega$ will be called invariant if ($p \in$ domain(F) and $p \approx p'$) always implies ($p' \in$ domain(F) and $F(p) = F(p')$).

<u>DEFINITION 1.2.</u> Following Galvin and Prikry [6] we call a subset $P$ of $[\omega]^\omega$ completely Ramsey, or CR, if for all $\langle s,p \rangle$ there is a $q \in [p]^\omega$ with either $[s,q] \subseteq P$ or $[s,q] \cap P = 0$.

The CR sets are ~~chosen~~ those in $\underset{\sim}{C}_H$ in the notation of [15], 1.3. The statement $\omega \to (\omega)^\omega$ is equivalent to the assertion that all subsets of $[\omega]^\omega$ are completely Ramsey.

<u>DEFINITION 1.3.</u> We shall call a subset $P$ of $[\omega]^\omega$ $CR^+$ if for all $\langle s,p \rangle$ there is a $q \in [p]^\omega$ with $[s,q] \subseteq P$, and $CR^-$ if for all $\langle s,p \rangle$ there is a $q \in [p]^\omega$ with $[s,q] \cap P = 0$.

The $CR^-$ sets are those in $\underset{\sim}{I}_H$ in [15], 1.4, and are the meagre sets in the topology defined by Ellentuck in [4].

1.4 In a manner familiar to readers of [15], we shall want relativized versions of these concepts: we shall for example say that $P$ is $CR^+$ on $[s,p]$ if for all $<t,q>$ with $s \subseteq t$ and $(t \setminus s) \cup q \subseteq p$, there is an $r$ in $[q]^\omega$ with $[t,r] \subseteq P$.

We are now in a position to reformulate a weaker form of the conclusion of 1.0 as

PROPOSITION 1.5 (ZF) The following are equivalent:

(1.6) The Hausdorff extension adds no new sets of ordinals.

(1.7) Let $\zeta$ be an ordinal and $<C_\nu | \nu < \zeta>$ a sequence of invariant CR subsets of $[\omega]^\omega$. Then there is a $p \in [\omega]^\omega$ such that for each $\nu < \zeta$, $C_\nu$ is, relative to $[p]^\omega$, either $CR^+$ or $CR^-$.

PROOF. (1.6) → (1.7) : Fix $<C_\nu | \nu < \zeta>$ and let $G$ be the generic filter in Power($\omega$) added by the Hausdorff extension. In $V[G]$ put

$$B = \{\nu < \zeta \mid \exists p \in G(C_\nu, \text{ relative to } [p]^\omega, \text{ is } CR^+)\}$$

By (1.6), there is an $A \subseteq \zeta$, and a $q$ such that $q \Vdash B \equiv \hat{A}$. We shall show that for all $\nu$ in $A$, $C_\nu$ is $CR^+$ relative to $[q]^\omega$, and that for all $\nu$ not in $A$, $C_\nu$ is $CR^-$ relative to $[q]^\omega$, by proving the contrapositives of these statements.

Let $\nu < \zeta$. If $C_\nu$ is not $CR^+$ relative to $[q]^\omega$, then since $C_\nu$ is CR there is $[s,r] \subseteq [0,q]$ with $[s,r] \cap C_\nu$ empty, so by the invariance of $C_\nu$, $[r]^\omega \cap C_\nu$ is empty, and so $r \Vdash \hat{\nu} \notin B$; hence $q \not\Vdash \hat{\nu} \in B$, so $q \not\Vdash \hat{\nu} \in \hat{A}$, and thus $\nu \notin A$.

If $C_\nu$ is not $CR^-$ relative to $[q]^\omega$, there is $[s,r] \subseteq [0,q]$ with $[s,r] \subseteq C_\nu$, so by invariance $[r]^\omega \subseteq C_\nu$, and so $r \Vdash \hat{\nu} \in B$; hence $q \not\Vdash \hat{\nu} \notin B$, so $q \not\Vdash \hat{\nu} \notin \hat{A}$, and so $\nu \in A$.

(1.7) → (1.6): Suppose $\Vdash B \subseteq \hat{\zeta}$. Set $C_\nu = \{q \mid q \Vdash \hat{\nu} \in B\}$. Then each $C_\nu$ is invariant. Notice that $C_\nu$ is $CR^+$ relative to $[p]^\omega$ iff $p \Vdash \hat{\nu} \in B$, and $C_\nu$ is $CR^-$ relative to $[p]^\omega$ iff $p \Vdash \hat{\nu} \notin B$: so a $p$ satisfying the conclusion of (1.7) will force $B \in \hat{V}$. ⊣

REMARK 1.8. (1.7) is false without the hypothesis of invariance: for $n \in \omega$, let $C_n = \{p \mid n \le \inf p\}$.

1.9 The conclusion of Theorem 1.0 may be derived from certain square-bracket partition relations: write $\omega \to [\omega]^\omega \lambda$, where $2 \le \lambda \le \omega$, to mean that for any $\psi:[\omega]^{\omega} \to \lambda$ there is an $x \in [\omega]^{\omega}$ such that for some $\nu < \lambda$ and all $y \in [x]^{\omega}$, $\psi(y) \neq \nu$. Kleinberg has observed that from a sequence $\langle \pi_n \mid 2 \le n < \omega \rangle$ of partitions of $[\omega]^{\omega}$, where for each $n$, $\pi_n$ is a counterexample to $\omega \to [\omega]^{\omega}_n$, a counterexample to $\omega \to [\omega]^{\omega}\omega$ may be constructed: thus assuming DC, or at least AC for countable families, $\omega \to [\omega]^{\omega}_{\omega}$ implies that for some $n < \omega$, $\omega \to [\omega]^{\omega}n{+}2$. It is not known whether $\omega \to (\omega)^{\omega}$ can be derived from any $\omega \to [\omega]^{\omega}n$ for $n \ge 3$ [PROB] but the conclusion of theorem 1.0 may be derived as follows.

Suppose $\omega \to [\omega]^{\omega}n$. Let $\beta$ be a prime number very much larger than $n$, and for $p \in [\omega]^{\omega}$, $0 \le i < \beta$, let $(p)_i$ be $\{\tilde{p}(m\beta+i) \mid m \in \omega\}$. Thus if $q = p\setminus\{\tilde{p}(0)\}$, $(q)_i = (p)_{i+1}$, $(q)_{\beta-1} = (p)_0\setminus\{\tilde{p}(0)\}$.

Let $p_0$, $f$, $\zeta$, $\phi$ be as in the proof of Theorem 1.0. Set

$$\psi(p) = \{i \mid i < \beta \text{ and } \forall j{<}\beta\, \phi((p)_j) \le \phi((p)_i)\}.$$

By (repeated) application of $\omega \to [\omega]^{\omega}n$, a $\overline{p} \in [p_0]^{\omega}$ may be found for which $\psi[[\overline{p}]^{\omega}]$ is of cardinality less than $n$. But then $\phi((p)_0) = \phi((p)_1) = \ldots = \phi((p)_{\beta-1})$ for any $p \in [\overline{p}]^{\omega}$, since if $\phi((p)_j) < \phi((p)_i)$ for some $i,j$ less than $\beta$, by the primality of $\beta$ and the invariance of $\phi$, the $\beta$ values

$\psi(p)$, $\psi(\tilde{p}\setminus\{p(0)\})$, $\psi(\tilde{p}\setminus\{\tilde{p}(0),\tilde{p}(1)\})$, .. , $\psi(p\setminus\{\tilde{p}(0),\tilde{p}(1),\ldots,\tilde{p}(\beta-1)\})$ will be distinct.

Now put $\nu = \phi(\overline{p})$. As before, there are $q, r \in [\overline{p}]^{\omega}$, $x, y \in V$, with $q \Vdash f(\hat{\nu}) \equiv \hat{x}$, $r \Vdash f(\hat{\nu}) \equiv \hat{y}$, and $x \neq y$. Find $s \subseteq q \cup r$ so that $(s)_0 \cap q$ is infinite, $(s)_0 \cap r$ is infinite, $(s)_1 \subseteq q$. Then $\phi((s)_0) = \nu$ while $\phi((s)_1) \ge \phi(q) > \nu$, contradicting the property of $\overline{p}$ established in the previous paragraph. $\dashv$

REMARK 1.10. It is a theorem of Vopěnka and Balcar [21] that if two transitive models, of which at least one is known to satisfy the axiom of choice, have the same sets of ordinals, they coincide. That they might differ if neither satisfies AC is due to Jech [9]: for an extension of that result, see Monro [17].

## 2. THE LARGENESS OF THE INTERSECTION OF $CR^+$ FAMILIES.

DEFINITION 2.0. A function $F:[\omega]^\omega \to \text{Power}(\omega)$ is called strongly continuous on $[s,p]$ if there is a tree $\langle t_u | u \in [p]^{<\omega}\rangle$ of finite subsets of $\omega$ such that for all $q \in [p]^\omega$ and all $k \in q$,

$$F(q)\cap(k+1) = t_{q\cap(k+1)}.$$

DEFINITION 2.1. A relation $R \subseteq [\omega]^\omega \times \text{Power}(\omega)$ is (strongly) uniformised on $[s,p]$ if there is some (strongly continuous) function $F$ such that for all $q \in [s,p]$, $R(q,F(q))$.

DEFINITION 2.2. LU (LSU) is the assertion that for any relation $R$ as above, such that $\forall p \exists y\, R(p,y)$, $\{x | R$ is (strongly) uniformised on $[0,x]\}$ is $CR^+$.

THEOREM 2.3. (ZF + DCR + LSU) Let $\theta$ be any ordinal and $\langle C_\nu | \nu < \theta\rangle$ a sequence of $CR^+$ families. Then $\bigcap_{\nu<\theta} C_\nu$ is $CR^+$.

PROOF. For $\theta$ countable, this is a theorem of ZF + DCR, and is due to Galvin and Prikry [6]. A proof may be given by the methods of Happy Families ([15]; see in particular Proposition 1.10).

Let us therefore suppose the theorem true for all sequences of length less than $\theta$, and that it fails at $\theta$ for the sequence $\langle C_\nu | \nu < \theta\rangle$. By the minimality of $\theta$ and the first sentence of this proof, $\theta$ must be of uncountable cofinality.

Now put $D_\nu =_{df} \{p \in [\omega]^\omega | [p]^\omega \subseteq \bigcap_{\mu<\nu} C_\mu\}$. Then by Proposition 2.8 of Happy Families (for the case $A = [\omega]^\omega$, $B = [\omega]^\omega \setminus (\bigcap_{\mu<\nu} C_\mu))$, $D_\nu$ is $CR^+$ for each $\nu < \theta$, $D_\mu \supseteq D_\nu$ for $\mu < \nu < \theta$, $q \subseteq p \in D_\nu$ implies $q \in D_\nu$, and, by relativizing to some $[s,S]$ if necessary, we may further assume that $\bigcap_{\nu<\theta} D_\nu$ is empty.

Thus we may define $\chi(p)$, for $p \in [\omega]^\omega$, to be the least $\nu < \theta$ with $p \notin D_\nu$. The function $\chi:[\omega]^\omega \to \theta$ will have these properties:

$$q \subseteq p \to \chi(p) \leq \chi(q),$$

$$\forall \nu<\theta \quad \forall \langle s,q\rangle \; \exists p \in [s,q] \; (\chi(p) > \nu).$$

Define now $\psi(p) = \bigcup\{\chi(q) | q \approx p\}$ and $\phi(p) = \bigcap\{\chi(q) | q \approx p\}$. We assert that

$$\forall\, q \in [\omega]^\omega \exists p \in [q]^\omega (\psi(q) < \phi(p)).$$

To see that, put $\nu = \psi(q)$, construct sequences $n_0 < n_1 < n_2 < \ldots$ and $q = p_0 \supseteq p_1 \supseteq p_2 \supseteq \ldots$ such that for each $i, n_i = \min p_i$, and for each

$s \subseteq n_i+1$, $[s, p_{i+1}] \subseteq D_\nu$, and set $p = \{n_i | i < \omega\}$.

If $p' \approx p$, there is an $n_k$ such that $p'\backslash n_k = p\backslash n_k$, so $p' \in [p' \cap (n_k+1), p_{k+1}]$, so $p' \in D_\nu$, and $\chi(p') > \nu$; consequently $\phi(p) > \nu$.

By LSU, there is a $\bar{p} \in [\omega]^\omega$ and a strongly continuous function $F:[\bar{p}]^\omega \to [\bar{p}]^\omega$ such that

$$\forall p \in [\bar{p}]^\omega (F(p) \subseteq p \quad \text{and} \quad \phi(F(p)) > \psi(p)).$$

Define the relation $R$ on $[\bar{p}]^\omega$ by setting $R(p,q)$ iff $\exists q' \approx q\ \exists p' \approx p (F(p') = q')$: $R$ will be well-founded since if $R(p,q)$,

$$\chi(q) \geq \phi(q) = \phi(q') > \phi(F(p')) > \psi(p') = \psi(p) \geq \chi(p).$$

Since $F$ is continuous, $R$ is Borel, and so by the Kunen-Martin theorem (Moschovakis [18], page 99, Theorem 2G.2; see also page 114, footnote 12), if we define $\rho_R(q) = \cup\{\rho_R(p)+1 | p \in [\bar{p}]^\omega$ and $F(p) = q\}$, then for some $\eta$ less than $\omega_1$ and all $q \in [\bar{p}]^\omega$, $\rho_R(q) < \eta$.

Notice that for any $\langle s,p \rangle$ with $s \cup p \subseteq \bar{p}$, there is a $q \in [s,p]$ with $\rho_R(q) > \rho_R(s \cup p)$. To see that, take $q = s \cup (F(p) \backslash |s|)$: then $s \cup p \approx p$ and $q \approx F(p)$, so $R(s \cup p, q)$, and so $\rho_R(q) > \rho_R(s \cup p)$.

Thus if for $\zeta < \eta$ we set $E_\zeta = \{p \in [\bar{p}]^\omega | \rho_R(p) = \zeta\}$, each $E_\zeta$ will be $CR^-$ relative to $[0,\bar{p}]$. But then since $\eta$ is countable $\bigcup_{\zeta<\eta} E_\zeta$ is $CR^-$, relative to $[0,\bar{p}]$; but that is absurd, as $[0,\bar{p}] = \bigcup_{\zeta<\eta} E_\zeta$. ⊣

DEFINITION 2.4. EP is the statement of Theorem 2.3, that the intersection of any well-ordered collection of $CR^+$ sets is $CR^+$.

PROPOSITION 2.5 (ZF) EP implies (1.7).

PROOF. If $\langle C_\nu | \nu < \zeta \rangle$ is as in (1.7), let $D_\nu = \{x | \forall s \subseteq \cap x\ ([s,x] \subseteq C_\nu$ or $[s,x] \cap C_\nu = 0\}$. Then each $D_\nu$ is $CR^+$: by EP, $\bigcap_{\nu<\zeta} D_\nu$ is not empty. Let $P$ be a member: then $P$ satisfies the conclusion of (1.7). ⊣

## §3. THE PERSISTENCE OF PARTITION PROPERTIES.

PROPOSITION 3.0 (ZF + EP) Let $\kappa$ be an ordinal and $\Phi:[\omega]^\omega \to [\kappa]^\kappa$ an invariant function. Then $\Phi$ is constant on some $[p]^\omega$.

PROOF. For $\nu < \kappa$, put $C_\nu = \{p | \nu \in \Phi(p)\}$. Then each $C_\nu$ is invariant, so by 1.7, which by 2.5 follows from EP, there is a $p \in [\omega]^\omega$ such that, relative to $[p]^\omega$, each $C_\nu$ is either $CR^+$ or $CR^-$ : put

$D_\nu = \{q \in [p]^\omega | [q]^\omega \subseteq C_\nu\}$ in the first case and $= \{q \in [p]^\omega | [q]^\omega \cap C_\nu = 0\}$ in the second. Then each $D_\nu$ is $CR^+$ on $[p]^\omega$.

Let $p$ be in the intersection, non-empty by EP, of $\{D_\nu | \nu < \kappa\}$. Then for each $q \in [p]^\omega$ and $\nu < \kappa$, $\nu \in \Phi(q)$ iff $\nu \in \Phi(p)$, since

$$\nu \in \Phi(p) \to p \in C_\nu \to D_\nu \subseteq C_\nu \to q \in C_\nu \to \nu \in \phi(q), \text{ and}$$
$$\nu \notin \Phi(p) \to p \notin C_\nu \to D_\nu \cap C_\nu = 0 \to q \notin C_\nu \to \nu \notin \Phi(q). \dashv$$

REMARK 3.1. The above Proposition may fail if $\Phi$ is not required to be invariant : for example, if $\Phi$ is an injection.

PROPOSITION 3.2 (ZF + LU) Suppose that $0 < \lambda = \omega\lambda \leq \kappa$, $2 \leq \mu < \kappa$, $\kappa \to (\kappa)^\lambda \mu$, that there is a surjection $\psi : [\omega]^\omega \to [\kappa]^\kappa$ and that $\langle \pi_p | p \in [\omega]^\omega \rangle$ is a collection of partitions $\pi_p : [\kappa]^\lambda \to \mu$. Then there is a $p^* \in [\omega]^\omega$ and an invariant function $\Phi : [p^*]^\omega \to [\kappa]^\kappa$ such that for each $p \in [p^*]^\omega$, $\Phi(p)$ is homogeneous for $\pi_p$.

PROOF. For each $p$, define $\rho_p : [\kappa]^\lambda \to \mu$ by

$$\rho_p(x) = \pi_p({}_\omega x),$$

where, as in Supercontinuity [8], ${}_\omega x = \{\bigcup_{n \in \omega} \tilde{x}(\omega\zeta + n) | \zeta < \lambda\}$: note that as $\omega\lambda = \lambda$, ${}_\omega x$ is in $[\kappa]^\lambda$ whenever $x$ is. Set

$$H_p = \{q \in [\omega]^\omega | \psi(q) \text{ is homogeneous for } \rho_p\}.$$

By LU there is a $p^*$ and an $F : [p^*]^\omega \to [\omega]^\omega$ such that for all $q \in [p^*]^\omega$, $\psi(F(q))$ is homogeneous for $\rho_p$. Write $B(q) = \psi(F(q))$.

For each $q \in [p^*]^\omega$, define $C(q) \in [\kappa]^\kappa$ thus: let $C(q)(0)$ be the least ordinal greater than all $B(q')(0)$ for $q' \approx q$; let $C(q)(\nu)$ be the least ordinal $\xi$ such that setting $\eta = \cup\{C(q)(\nu') | \nu' < \nu\}$, the interval $[\eta, \xi)$ contains at least one element of each $B(q')$ for $q' \approx q$.

The regularity of $\kappa$ ensures the soundness of this definition. Note that this definition does not rely on an enumeration of any $\{q' | q' \approx q\}$, and therefore that $C(q) = C(q')$ whenever $q \approx q'$ : so that $C$ is invariant.

Now put $\Phi(q) = {}_\omega C(q)$. Then $\Phi$ is invariant; any $x \in [\Phi(q)]^\lambda$ is of the form ${}_\omega y$ for some $y \in [B(q)]^\lambda$, and hence $\pi_q(x) = \pi_q({}_\omega y) = \rho_q(y)$, which, by the homogeneity of $B(q)$, is independent of $y$ and therefore of $x$.

Hence $\Phi(q)$ is homogeneous for $\pi_q$ as required. $\dashv$

THEOREM 3.3 (ZF + EP + LU) Suppose $0 < \lambda = \omega\lambda \leq \kappa$, $2 \leq \mu < \kappa$, $\kappa \to (\kappa)^{\lambda}\mu$, and that there is a surjection $\psi : [\omega]^{\omega} \twoheadrightarrow [\kappa]^{\kappa}$. Then in the Hausdorff extension, $\kappa \to (\kappa)^{\lambda}\mu$.

PROOF. Note first that by 2.5 no new subsets of $\kappa$ are added; consequently $[\kappa]^{\lambda}$ is the same whether interpreted in the ground model or in the extension, and thus may be written without ambiguity.

Note secondly that $\kappa \to (\kappa)^{\lambda}\mu+1$.

Suppose $p_0 \Vdash f : [\hat{\kappa}]^{\hat{\lambda}} \to \hat{\mu}$. We shall find a $p_2 \in [p_0]^{\omega}$ and an $A \in [\kappa]^{\kappa}$ such that $p_2 \Vdash f$ is constant on $[\hat{A}]^{\hat{\lambda}}$.

For each $p \in [p_0]^{\omega}$, define a partition $\pi_p : [\kappa]^{\lambda} \to \mu+1$ by

$$\pi_p(A) = \zeta \quad \text{if} \quad p \Vdash f(\hat{A}) \equiv \hat{\zeta}$$

$$\pi_p(A) = \mu \quad \text{otherwise.}$$

By 3.2, there is a $p_1 \in [p_0]^{\omega}$ and an invariant function $\Phi : [p_1]^{\omega} \to [\kappa]^{\kappa}$ such that for each $p \in [p_1]^{\omega}$, $\Phi(p)$ is homogeneous for $\pi_p$. By 3.1 there is some $p_2 \in [p_1]^{\omega}$ with $\Phi$ constant on $[p_2]^{\omega}$. Set $A = \Phi(p_2)$. We assert that $p_2 \Vdash \hat{A}$ is homogeneous for $f$.

If not, there will be $D, E \in [A]^{\lambda}$, $q \in [p_2]^{\omega}$, $\xi < \zeta < \mu$ with $q \Vdash (f(\hat{D}) \equiv \hat{\xi}$ and $f(\hat{E}) \equiv \hat{\zeta})$: so $\pi_q(D) = \xi$, $\pi_q(E) = \zeta$, and thus $A$ is inhomogeneous for $\pi_q$. But $A = \Phi(p_2) = \Phi(q)$ which is homogeneous for $\pi_q$. ⊣

## §4. PARTITION CARDINALS WITHOUT DETERMINACY.

An important ordinal in the study of AD is $\theta$, the least ordinal $> 0$ onto which Power($\omega$) cannot be mapped.

LEMMA 4.0. If the Hausdorff extension is barren, then $\theta$ is the same whether calculated in the ground model or in the extension.

PROOF. Suppose $p \Vdash f : \widehat{[\omega]^{\omega}} \twoheadrightarrow \hat{\theta}$. For each pair $(q,r)$ with $q \setminus p$ finite, put $\psi(q,r) = \xi$ if $q \Vdash f(\hat{r}) \equiv \hat{\xi}$, and $\psi(q,r) = 0$ otherwise. Then $\psi$ is a surjection of $[\omega]^{\omega} \times [\omega]^{\omega}$ onto $\theta$, a contradiction. ⊣

PROPOSITION 4.1 (AD + V = L[R]) If $0 < \omega\lambda = \lambda \leq \kappa$, $2 \leq \mu < \kappa$ and $\kappa \to (\kappa)^{\lambda}\mu$, then $\kappa \to (\kappa)^{\lambda}\mu$ in the Hausdorff extension.

PROOF. By a theorem of Kleinberg [14], $\kappa$ is measurable, so by arguments to be found in [13], $\kappa < \theta$; hence, by a theorem of Moschovakis

([18], 7D.19, page 442), there is a surjection $\psi:[\omega]^\omega \to\!\!\to [\kappa]^\kappa$.

DC is provable in ZF + AD + V = L[R], by Kechris [10]; so is LSU, by Theorem 2.2 of [16]; by Theorem 2.3 above, EP then holds.

Thus all the hypothesis of 3.3 hold, so the conclusion follows. $\dashv$

THEOREM 4.2 If AD is consistent, so is DC +

$\forall\lambda<\theta\ \exists\ \kappa(\lambda \leq \kappa < \theta$ and $\kappa$ is a strong partition cardinal) + "there is a Ramsey ultrafilter on $\omega$".

PROOF. If AD holds, it stays true in L[R], as strategies are reals. Again, by [10], DC holds, so DC holds in the Hausdorff extension. As $\omega \to (\omega)^\omega$, (cf Theorem 2.2 of [16]), the extension is barren. By the results of Kechris, Kleinberg, Moschovakis and Woodin [11], there are arbitrarily large strong partition cardinals below $\theta$ in the ground model, so by 4.0 and 4.1, the same holds in the extension. A remark in the introduction completes the proof. $\dashv$

REMARK 4.3. The significance of 4.2 is that as the existence of an ultrafilter on $\omega$ implies the failure of AD, the hypothesis V = L[R] is an essential ingredient in the Kechris-Woodin [12] derivation of AD from the existence of arbitrary large strong partition cardinals below $\theta$.

REMARK 4.4. The arguments of section 1 generalize with little change to obtain a new proof of Henle's result [7] that Spector forcing [20] at a strong partition cardinal $\kappa$ is barren.

REMARK 4.5. If AD holds and V = L[R], then $\omega_1$ and $\omega_2$ are measurable, and $\omega \to (\omega)^\omega$, so in the Hausdorff extension, there is a Ramsey ultrafilter on $\omega$, and $\omega_1$ and $\omega_2$ are still measurable: we may say that there are in this model two and a half contiguous measurables.

A model for that statement may also be found assuming something presumably much weaker than Con(AD): by an unpublished result of Woodin, a model of ZFC in which $\kappa$ is $\lambda$-supercompact and $\lambda > \kappa$ is measurable admits a Boolean extension in which $\kappa = \omega_1$, $\lambda = \omega_2$, $\kappa$ and $\lambda$ are still measurable, and $\omega \to (\omega)^\omega$; in the Hausdorff extension of that model, there will be a Ramsey ultrafilter on $\omega$, while the measures on $\kappa$ and $\lambda$ will remain measures as no new subsets of either will be added.

Several open problems are related to [16] and the present work:

PROBLEM 4.6. Can $\omega \to (\omega)^\omega$ be deduced from $\omega \to [\omega]^\omega n$ for any $n \geq 3$?

PROBLEM 4.7. Does AD imply that $\omega_1$ is huge? Is there a huge measure on $[\omega_2]^{\omega_1}$?

PROBLEM 4.8. Is it a theorem of ZF + DC + $\aleph_1 \not\leq 2^{\aleph_0}$ or of ZF + DC + $\omega \to (\omega)^\omega$ that there are no MAD families coded on $\omega$?

PROBLEM 4.9. Is there a $k \in \omega$ such that it is a theorem of ZF + DC that there cannot exist $k$ contiguous measurables?

By a result of Kechris, if AD is consistent, $k$ must be greater than three. $k$ might be quite small, though: for limitations on contiguous large cardinals, see Apter [1] and Bull [2].

PROBLEM 4.10. Is EP a theorem of ZF + DC + $\omega \to (\omega)^\omega$ ?

The least $\theta$ for which 2.3 fails must be measurable.

PROBLEM 4.11. If $\kappa$ and $\lambda$ are strong partition cardinals with $\kappa < \lambda$, will $\kappa$ remain strong in the Spector extension for $\lambda$, or vice versa?

The similarities between strong partition cardinals and $\omega$, when $\omega \to (\omega)^\omega$, which are studied in [8], and the result mentioned in 4.4 above, suggest that Spector forcing should preserve something more than plain measurability. But there are limits: the analogue of 2.3, when $\omega$ is replaced by a strong partition cardinal $\kappa$, fails for $\theta = \kappa^+$.

PROBLEM 4.12. Does ADR imply that every set of reals is Souslin?

A theorem of Woodin states that ADR is provable in ZF + AD + "every set of reals in Souslin".

PROBLEM 4.13. How strong is the theory ZF + DC + V = L[R] + "$\theta$ is a regular limit cardinal"? Does it prove the existence of $\alpha^{\#}$ for every real $\alpha$?

The challenge in this question is to get $\theta$ a limit cardinal: it is a result of the Cabal that ZF + DC + V = L[R] proves that $\theta$ is regular. In [13], Kechris and Woodin show that if AD + V = L[R] holds, then $\theta$ cannot be weakly compact.

PROBLEM 4.14. It follows from the results of [10], that Con($\omega_1$ is a strong partition cardinal) follows from, e.g., Con(there are arbitrarily large strong partition cardinals), but the proof goes via determinacy. Is there a direct proof?

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