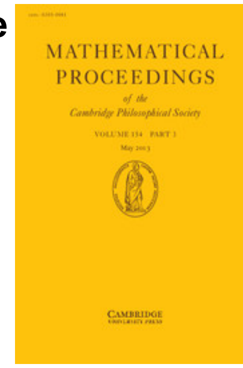

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A. R. D. Mathias

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Unsound ordinals

BY A. R. D. MATHIAS
Peterhouse, Cambridge

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Abstract. An ordinal is termed *unsound* if it has subsets A_n $(n \in \omega)$ such that uncountably many ordinals are realised as order types of sets of the form $\bigcup \{A_n | n \in a\}$ where $a \subseteq \omega$. It is shown that if ω_1 is regular and $\aleph_1 = 2^{\aleph_0}$ then the least unsound ordinal is exactly $\omega_1^{\omega+2}$ but that if ω_1 is regular and $\aleph_1 \nleq 2^{\aleph_0}$, the least unsound ordinal, assuming one exists, is at least $\omega_1^{\omega+\omega+1}$. Arguments due to Kechris and Woodin are presented showing that under the axiom of determinacy there is an unsound ordinal less than ω_2. The relation between unsound ordinals and ideals on ω is explored. The paper closes with a list of open problems.

0. *Introduction*

Definition 0·0. Let $B = \langle B_n | n \in \omega \rangle$ be a sequence of sets of ordinals, and for $a \subseteq \omega$ write $B[a]$ for $\{B_n | n \in a\}$ and $\tau_B(a)$ for the order type of $\bigcup \{B_n | n \in a\}$. The set $\{\tau_B(a) | a \subseteq \omega\}$ will be called the *spektron* of B, and denoted by $\sigma\pi(B)$. A sequence of subsets of a set X will be called an (X)-*sequence*. An ordinal η is *sound* if every (η)-sequence has countable spektron; in the contrary case, η is termed *unsound*.

The present paper is addressed to a question raised by Woodin in September 1982, which in this terminology runs: Is there an unsound ordinal?

Definition 0·1. The ordinal $\omega_1^{\omega+1}$ will play a pivotal rôle in our investigations: we shall denote this ordinal throughout the paper by θ.

Definition 0·2. The letter ζ will denote the term 'the least unsound ordinal'.

As we have been unable to prove in ZF alone that there is an unsound ordinal, '$\zeta \geqslant \eta$' should be read as 'every ordinal less than η is sound' while '$\zeta < \eta$' and '$\zeta = \eta$' are interpreted to imply that an unsound ordinal exists.

In Section 1 we assume that ω_1 is regular, and show that $\zeta \geqslant \omega_1^{\omega+2}$.

In Section 2 we show that if there is an uncountable well-ordered set of reals, then $\zeta = \omega_1^{\omega+2}$. However, in Section 3 we show that if ω_1 is regular but every well-orderable set of reals is countable, then $\zeta \geqslant \omega_1^{\omega+\omega+1}$.

In Section 4, by their authors' permission, a lemma of Kechris and arguments of Woodin are presented showing that assuming the axiom of determinacy – the context in which Woodin was interested – $\zeta < \omega_2$. Section 5 relates unsound ordinals to ideals on ω, and Section 6 lists some open problems.

Our set-theoretic notation is largely standard. We denote the empty set by 0. We write otp(A), where A is a set of ordinals, for its order type, which will be an ordinal. Thus $\tau_B(a) = \text{otp}(\bigcup B[a])$. An *aleph* is an infinite initial ordinal. For X, T sets of ordinals, we say T is unbounded or cofinal in X if $T \cap X$ is. cf(η) is the cofinality of η. $[\kappa, \lambda)$ is the half-open interval $\{\nu | \kappa \leqslant \nu < \lambda\}$ of ordinals. The notation $\lambda = \Sigma\{A_\nu | \nu < \zeta\}$ is used to mean that $\lambda = \bigcup\{A_\nu | \nu < \zeta\}$, and moreover that whenever $\nu < \nu' < \zeta$,

$\xi \in A_\nu$ and $\xi' \in A_{\nu'}$ we have $\xi < \xi'$, so that the A_ν's are pairwise disjoint convex subsets of λ. If $f: X \to Y$, $X_1 \subseteq X$ and $Y_1 \subseteq Y$, we write $f[X_1]$ for $\{f(x) | x \in X_1\}$ and $f^{-1}[Y_1]$ for $\{x | f(x) \in Y_1\}$. If $Y_1 = \{y\}$, we write $f^{-1}\{y\}$ instead of $f^{-1}[\{y\}]$. The notation $B[a]$ used in 0·0 is thus a special case of this notation. *Power* (X) is the power set of X, $\{Y | Y \subseteq X\}$.

Definition 0·3. An ordinal ζ is *indecomposable* if ($\xi < \zeta$ and $\eta < \zeta$) implies $\xi + \eta < \zeta$. We shall use the result (cf. Bachmann[**1**], pp. 84, 68) that the indecomposable ordinals are precisely those of the form ω^α for some α.

Definition 0·4. Let $\pi: X \cong Y$ be an order-isomorphism between two sets, X and Y, of ordinals. If A is an (X)-sequence, the (Y)-sequence B obtained by setting

$$B(n) = \pi[A(n)] \quad \text{for} \quad n \in \omega$$

will be called the *copy* of A by π. The name B of the new sequence will often be introduced by a phrase such as 'copy A to B by π' or 'copy X to Y and each A_n to B_n'.

Our underlying set theory is Zermelo–Fraenkel, without the axiom of choice. All assumptions are given in the statements of Theorems, but standing assumptions may be omitted from the statements of Lemmata and Propositions. The end of a proof is signalled by ❙.

In many sections of this paper we shall assume the well-known consequence of the axiom of choice that ω_1 is regular. It will be convenient to present in this section two theorems of ZF alone.

LEMMA 0·0. *The closure of a countable set A of ordinals is countable.*

Proof. Let $\pi: \xi \cong A$ enumerate A in increasing order: so $\xi < \omega_1$. Define

$$\rho: \xi + 1 \to \mathrm{cl}(A) \quad \text{by} \quad \rho(n) = \pi(n)$$

for $n < \omega$, $\rho(\lambda) = \bigcup \{\pi(\nu) | \nu < \lambda\}$ for $\omega \leqslant \lambda \leqslant \xi$. Then ρ is onto $\mathrm{cl}(A)$: for $\eta \in A$ implies $\eta = \pi(\nu)$ say, which implies $\eta = \rho(\nu+1)$ if $\nu \geqslant \omega$, or $\eta = \rho(\nu)$ if $\nu \leqslant \omega$: and if

$$\eta \in \mathrm{cl}(A) \backslash A, \quad \text{then} \quad \eta = \rho(\bigcup \{\nu | \pi(\nu) < \eta\}). \quad ❙$$

LEMMA 0·1. *ω_1 is regular if and only if the union of every countable family of countable sets of ordinals is countable.*

Proof. Suppose that ω_1 is regular and let each A_n $(n \in \omega)$ be a countable set of ordinals: let ξ_n be the order type of A_n. Each $\xi_n < \omega_1$, so defining inductively $\zeta_0 = \xi_0$, $\zeta_{n+1} = \zeta_n + \xi_n$, each $\zeta_n < \omega_1$, and $\zeta =_{\mathrm{df}} \bigcup \{\zeta_n | n \in \omega\}$ is less than ω_1. But clearly there is a surjection of ζ onto $\bigcup \{A_n | n \in \omega\}$, which is therefore countable.

The converse is trivial. ❙

1. *The soundness of θ*

Throughout this section we assume that ω_1 is regular.

Definition 1·0. An ordinal η is *solid* if whenever $\eta = \bigcup \{B_n | n \in \omega\}$, one of the sets B_n has order type η. η is *hollow* otherwise.

Examples 1·1. 0 and 1 are solid; every other countable ordinal is hollow; ω_1, being regular, is solid.

LEMMA 1·2. *If κ is solid then $\kappa . \omega_1$ is solid.*

Proof. Let $\kappa . \omega_1 = \Sigma \{A_\nu | \nu < \omega_1\}$, where each A_ν is of order type κ. Suppose

$$\kappa . \omega_1 = \bigcup \{B_n | n \in \omega\}.$$

For each ν, $A_\nu = \bigcup\{A_\nu \cap B_n | n \in \omega\}$, so, since κ is solid, there is an n with

$$\operatorname{otp}(A_\nu \cap B_n) = \kappa.$$

By the regularity of ω_1, there is a p such that for uncountably many different ν's, $\operatorname{otp}(A_\nu \cap B_p) = \kappa$. Then $\operatorname{otp}(B_p) = \kappa . \omega_1$. ▎

Since ω_1 is solid, iteration of Lemma 1·2 immediately establishes

LEMMA 1·3. *Each ω_1^k for $k \in [1, \omega)$ is solid.* ▎

LEMMA 1·4. *If κ is solid and, for each $i < \omega$, B_i is a set of ordinals with* $\operatorname{otp}(B_i) < \kappa$, *then* $\operatorname{otp}(\bigcup\{B_i | i < \omega\} < \kappa$.

Proof. If not, let λ be the supremum of the first κ elements of $\bigcup\{B_i | i < \omega\}$. Set $C_i = B_i \cap \lambda$, and copy $\bigcup_i C_i$ to κ and each C_i to D_i. Then by the solidity of κ, some D_i has order type κ. ▎

LEMMA 1·5. *ω_1^ω is hollow.*

Proof. Set $B_0 = \omega_1$ and for $n \geqslant 1$, $B_n = [\omega_1^n, \omega_1^{n+1})$. ▎

LEMMA 1·6. *Every countable ordinal is sound.*

Proof. For B a (κ)-sequence, $\sigma\pi(B) \subseteq \kappa + 1$, and is countable if κ is. ▎

LEMMA 1·7. *ω_1 is sound.*

Proof. Let $(B_n)_n$ be an (ω_1)-sequence, and let $a \subseteq \omega$. If $\operatorname{otp}(B_n) = \omega_1$ for some $n \in a$, $\tau_B(a) = \omega_1$. Otherwise $a \subseteq a_0 =_{\mathrm{df}} \{n | \operatorname{otp}(B_n) < \omega_1\}$, and $\tau_B(a) \leqslant \tau_B(a_0)$, but $\tau_B(a_0) < \omega_1$ by the regularity of ω_1. ▎

LEMMA 1·8. *If $\lambda < \kappa$ and κ is sound then λ is sound.* ▎

LEMMA 1·9. *If ξ and η are sound, so is $\xi + \eta$.*

Proof. Let $(B_n)_n$ be an $(\xi + \eta)$-sequence. Put $C_n = \xi \cap B_n$, and $D_n = \{\nu < \eta | \xi + \nu \in B_n\}$. Then $\sigma\pi(B) = \{\tau_C(a) + \tau_D(a) | a \subseteq \omega\}$, which is countable since $\sigma\pi(C)$ and $\sigma\pi(D)$ are. ▎

LEMMA 1·10. *If* $\operatorname{cf}(\rho) = \omega$ *and every $\eta < \rho$ is sound, so is ρ.*

Proof. Let $(\rho_k)_k$ be an increasing sequence with supremum ρ, and suppose that $(B_n)_n$ is a (ρ)-sequence with uncountable spektron. Let λ be the supremum of the first ω_1 elements of the spektron of B, and set $T = \sigma\pi(B) \cap \lambda$. Note that $\operatorname{cf}(\lambda) = \omega_1$ and $\sup T = \lambda$.

Now put $C^k = (B_n \cap \rho_k)_n$ and $T^k = \lambda \cap \sigma\pi(C^k)$.

Each T^k is countable, as ρ_k is sound, and so by Lemma 0·6 $\bigcup\{T^k | k \in \omega\}$ is countable. Its supremum, ξ, say, is therefore less than λ, which has cofinality ω_1. But each $\eta \in T$ is a supremum of members of $\bigcup\{T^k | k \in \omega\}$, and so is less than or equal to ξ, a contradiction. ▎

Suppose now that there is an unsound ordinal less than ω_2. Then ζ, the least such, is not a successor, by Lemma 1·9, and so by Lemma 1·10, $\operatorname{cf}(\zeta) = \omega_1$.

LEMMA 1·11. *For some $\beta \geqslant 1$, $\zeta = \omega_1^\beta$.*

Proof. $\zeta \geqslant \omega_1$. By Lemma 1·9, ζ is indecomposable, and so of the form ω^α for some $\alpha \geqslant \omega_1$. Let $\alpha = \omega_1\beta + \gamma$, where $\beta \geqslant 1$ and $\gamma < \omega_1$. Thus $\zeta = \omega^{\omega_1\beta} . \omega^\gamma = \omega_1^\beta . \omega^\gamma$. If $\gamma > 0$, ω^γ would be a countable limit ordinal, and thus, $\operatorname{cf}(\zeta) = \omega$, contradicting Lemma 1·10. Thus $\gamma = 0$, and the Lemma follows. ▎

LEMMA 1·12. *ζ is hollow.*

Proof. Let $(A_n)_n$ be a (ζ)-sequence with uncountable spektron. Discard those A_n with $\operatorname{otp}(A_n) = \zeta$: that is, define $B_n = A_n$ if $\operatorname{otp} A_n < \zeta$, $B_n = 0$, the empty set, otherwise. Then $\sigma\pi(B)$ is still uncountable, and so, by the minimality of ζ, $\tau_B(\omega) = \zeta$. Copy $\bigcup B[\omega]$ to ζ and each B_k to C_k. Then $\zeta = \bigcup\{C_k | k \in \omega\}$ and each C_k is of type less than ζ. ❙

LEMMA 1·13. $\zeta \geqslant \theta$.

Proof. By 1·11, 1·3, and 1·10. ❙

The following discussion takes place in *ZF* alone.

Let κ be a regular aleph, η be an ordinal of cofinality κ, and G a cofinal subset of η. Write $\xi = \operatorname{otp}(G)$, so that $\operatorname{cf}(\xi) = \kappa$, and, by a method going back to Cantor and expounded in Sierpiński [9], chapter XIV, section 19, theorem 3, express ξ as a sum of powers of κ:

$$\xi = \kappa^{\zeta_0}\lambda_0 + \kappa^{\zeta_1}\lambda_1 + \ldots + \kappa^{\zeta_m}\lambda_m$$

where each $\lambda_i \in [1, \kappa)$ and $\zeta_0 > \zeta_1 > \zeta_2 > \ldots > \zeta_m$. Note that λ_m cannot be a limit ordinal and ζ_m cannot equal 0, as otherwise $\operatorname{cf}(\xi)$ would not equal κ. Put $\lambda_m = \lambda + 1$: then for some $\rho < \xi$, $\xi = \rho + \kappa^{\zeta_m}$, so $\kappa = \operatorname{cf}(\xi) = \operatorname{cf}(\kappa^{\zeta_m})$, so ζ_m is a successor ordinal or a limit ordinal of cofinality κ. In particular, if $\eta = \kappa^{\omega+1}$, $\zeta_m = \omega + 1$or some $k \in [1, \omega)$. These remarks lead to the following

LEMMA 1·14. *Let κ be a regular aleph and let $G \subseteq \kappa^{\omega+1}$. Then exactly one of the following holds:*

(i) *G is of order type $\kappa^{\omega+1}$;*

(ii) *there is a $\xi < \kappa^{\omega+1}$ and a $k \in [1, \omega)$ such that $G \backslash \xi$ is of order type κ^k;*

(iii) *there is a $\xi < \kappa^{\omega+1}$ such that $G \backslash \xi$ is empty.*

Proof. Part (iii) covers the case when G is bounded below $\kappa^{\omega+1}$. When G is unbounded in $\kappa^{\omega+1}$, the foregoing discussion shows that (i) or (ii) holds. ❙

Assume again that ω_1 is regular. With Lemma 1·14 for $\kappa = \omega_1$ and $\kappa^{\omega+1} = \theta$ in mind, we make

Definition 1·15. For $G \subseteq \theta$ with $\operatorname{otp}(G) < \theta$, set $\xi(G)$ = the least $\xi < \theta$ such that 1·14 (ii) or 1·14 (iii) holds, and call $G \backslash \xi(G)$ the *tail* of G. Set $k(G) = k$ if the tail has order type ω_1^k (case (ii)) and $= 0$ if the tail of G is empty (case (iii)).

THEOREM 1·16. *If ω_1 is regular, $\omega_1^{\omega+1}$ is sound.*

Proof. Let $(A_n)_n$ be a (θ)-sequence with uncountable spektron. As in 1·12, we may suppose that each A_n is of order type less than θ, since those A_n of order type θ make no contribution to the uncountability of $\sigma\pi(A)$. Let $\xi = \sup\{\xi(A_n) | n \in \omega\}$. $\xi < \theta$ as $\operatorname{cf}(\theta) = \omega_1$. Put $B_n = A_n \backslash \xi$ and $k(n) = k(A_n)$. Then each B_n is either empty or of order type $\omega_1^{k(n)}$. The spektron of the sequence $(A_n \cap \xi)_n$ is countable, as every ordinal less than θ is sound, so $\sigma\pi((B_n)_n)$ is uncountable, by arguments similar to those of 1·9.

Now let $a \subseteq \omega$. Two cases arise: if $\sup\{k(n) | n \in a\} = \omega$, then by Lemma 1·14, $\operatorname{otp}(\bigcup B[a]) = \theta$, but if $\sup\{k(n) | n \in \omega\} = m < \omega$, then by Lemma 1·4,

$$\operatorname{otp}(\bigcup B[a]) = \omega_1^m.$$

So $\sigma\pi(B) \subseteq \{\omega_1^k | k \in \omega\} \cup \{\theta\}$, and is thus countable after all. ∎

2. *The unsoundness of $\omega_1^{\omega+2}$, provided $\aleph_1 \leqslant 2^{\aleph_0}$*

We shall utilize the paradox of Milner and Rado [9], theorem 5, that for each aleph κ, every ordinal ξ less than κ^+ is expressible as the disjoint union of sets A_k ($k \in [1,\omega)$) with $\mathrm{otp}(A_k) \leqslant \kappa^k$. With an eye to applications in § 5, we prove sharpened versions of their results: the central ideas in the proof of Proposition 2·10 are theirs.

Definition 2·0. A *decomposition* of η is a sequence of disjoint sets A_n with $\bigcup A[\omega] = \eta$.

In the following, let κ be an aleph. The prefix 'κ-', included in the definitions for greater precision, will be omitted whenever the context permits.

Definition 2·1. A *Milner–Rado κ-decomposition* of $\eta \in [\kappa, \kappa^+)$ is a decomposition $(A_n)_n$ of η with $A_0 = 0$ and $\mathrm{otp}(A_n) \leqslant \kappa^n$ for $n \in [1,\omega)$.

Definition 2·2. A *strong κ-decomposition* of $\eta \in [\kappa, \kappa^+)$ is a Milner–Rado decomposition $(A_n)_n$ of η such that for all $a \in [\omega]^\omega$, $\mathrm{otp}(\bigcup A[a]) = \eta$.

Note that for a trivial reason a successor ordinal cannot have a strong decomposition.

Definition 2·3. A *κ-superdecomposition* of $\eta \in [\kappa, \kappa^+)$ is a decomposition $(A_n)_n$ such that $A_0 = 0$, $\mathrm{otp}\, A_n = \kappa^n$ for all $n \in [1,\omega)$, and for all $(B_n)_n$ with each $B_n \subseteq A_n$ and $\mathrm{otp}\, B_n = \mathrm{otp}\, A_n$ and all $a \in [\omega]^\omega$, $\mathrm{otp}(\bigcup B[a]) = \eta$.

Thus every superdecomposition is strong.

We first determine, for regular κ, those $\eta \in [\kappa, \kappa^+)$ which admit superdecompositions.

PROPOSITION 2·4. *Let κ be an aleph. Then*

(i) *κ^ω has a superdecomposition.*

(ii) *$\kappa^{\omega+1}$ has a strong decomposition which if κ is regular is a superdecomposition.*

(iii) *If κ is regular, the only ordinals in $[\kappa, \kappa^+)$ with a superdecomposition are κ^ω and $\kappa^{\omega+1}$.*

Proof. (i) Set $J_0 = 0$, $J_1 = \kappa$, $J_n = [\kappa^{n-1}, \kappa^n)$ for $n \geqslant 2$. This is plainly a superdecomposition of κ^ω, and will be called the *canonical* one.

(ii) Write $\kappa^{\omega+1} = \Sigma'_{\nu<\kappa} I_\nu$ where each I_ν is of order type κ^ω. Let $(J_n)_n$ be the canonical superdecomposition of κ^ω, and write each I_ν as $\Sigma_n J_{\nu,n}$ by copying κ^ω to I_ν and thereby J_n to $J_{\nu,n}$. Set $A_0 = 0$ and $A_n = \bigcup_{\nu<\kappa} J_{\nu,n-1}$ for $n > 0$.

Then $A_1 = 0$ and $\mathrm{otp}\, A_n = \kappa^n$ for $n > 1$, and the A_n's are disjoint. If $a \in [\omega]^\omega$, each $I_\nu \cap \bigcup A[a]$ has order type κ^ω, by (i), so $\mathrm{otp}(\bigcup A[a]) = \kappa^{\omega+1}$. Thus $(A_n)_n$ is a strong decomposition, which we shall also call *canonical*.

Now suppose that κ is regular, $a \in [\omega]^\omega$, $B_n \subseteq A_n$ and $\mathrm{otp}\, B_n = \mathrm{otp}\, A_n$. Then for each n, B_n is cofinal in A_n and A_n is cofinal in $\kappa^{\omega+1}$, so $\sup \bigcup B[a] = \kappa^{\omega+1}$. Since a is infinite, $\mathrm{otp}(\bigcup B[a] \setminus \xi) \geqslant \kappa^\omega$ for each $\xi < \kappa^{\omega+1}$; by Lemma 1·14, $\mathrm{otp}(\bigcup B[a]) = \kappa^{\omega+1}$.

In the proof of (iii) we shall need the following

LEMMA 2·5. *Let κ be a regular aleph, $k \in [1,\omega)$, and $H = \Sigma_{\nu<\kappa} I_\nu$, where each I_ν is of order type less than κ^k. Then* $\mathrm{otp}(H) \leqslant \kappa^k$.

Proof. For $k = 1$, this follows from the regularity of κ. For $k = m+1$, pick for each $\nu < \kappa$ ξ_ν minimal so that $\mathrm{otp}\, I_\nu < \kappa^m . \xi_\nu$. Then each $\xi_\nu < \kappa$, so (by the case $k = 1$) $\Sigma_{\nu<\kappa} \xi_\nu \leqslant \kappa$ and $\mathrm{otp}(H) \leqslant \kappa^m . \Sigma \xi_\nu \leqslant \kappa^{m+1}$, as required. ∎

Proof of 2·4 (iii). Suppose κ regular, let $(A_n)_n$ be a superdecomposition of $\eta \in [\kappa, \kappa^+)$, and for $\xi \leqslant \eta$ set $a_\xi = \{n | \sup A_n < \xi\}$. η must be a limit ordinal, and for all $\xi < \eta$, a_ξ is finite, as $\mathrm{otp}(\bigcup A[a_\xi]) \leqslant \xi$.

If a_η is infinite, $\mathrm{otp}(\bigcup A[a_\eta]) = \eta$, so $\mathrm{cf}(\eta) = \omega$. We may now pick increasing sequences $k_n \in a_\eta$, $\eta_n < \eta$, with $\sup_n \eta_n = \eta$ and $\eta_n = \sup A_{k_n}$. If we now put

$$B_{k_n} = A_{k_n} \setminus \eta_{n-1} \quad \text{and} \quad b = \{k_n | n \in \omega\},$$

we find $\mathrm{otp}(\bigcup B[b]) = \kappa^\omega$, so $\eta = \kappa^\omega$.

If a_η is finite, then for $n \notin a_\eta$, $\mathrm{cf}(\eta) = \mathrm{cf}(A_n) = \kappa$. Write $\eta = \Sigma_{\nu<\kappa} I_\nu$ in any fashion with each I_ν non-empty. Suppose $n \notin a_\eta$. Then $\mathrm{otp}(A_n \setminus \xi) = \kappa^n$ for each $\xi < \eta$ so by Lemma 2·5, $\{\nu | \mathrm{otp}(A_n \cap I_\nu) \geqslant \kappa^{n-1}\}$ is cofinal in κ. Armed with this fact and exploiting the regularity of κ, we may define $f: \kappa \to \omega \setminus a_\eta$ such that for all $n \notin a_\eta$,

$$\{\nu < \kappa | \mathrm{otp}(A_n \cap I_\nu) \geqslant \kappa^{n-1} \text{ and } f(\nu) = n\}$$

is cofinal in κ. Now set for $n \notin a_\eta$, $B_n = \bigcup\{A_n \cap I_\nu | f(\nu) = n\}$. Then $\mathrm{otp}(B_n) = \kappa^n = \mathrm{otp}(A_n)$, so $\eta = \mathrm{otp}(\bigcup B[\omega \setminus a_\eta])$; but by Lemma 1·14, $\mathrm{otp}(\bigcup B[\omega \setminus a_\eta]) = \kappa^{\omega+1}$. ∎

We shall now show that for any aleph κ, every limit ordinal $\lambda \in [\kappa, \kappa^+)$ has a strong decomposition. For each λ that will be proved by an induction from κ to λ.

LEMMA 2·6. *Let $\xi < \kappa^+$. Then there is a function f which assigns to each limit ordinal $\lambda \leqslant \xi$ a closed cofinal subset $f(\lambda)$ of λ with* $\mathrm{otp}(f(\lambda)) \leqslant \kappa$.

Proof. With AC the lemma is obvious. If AC fails, let $Q \subseteq \kappa \times \kappa$ code ξ, so that in $L[Q]$, $\xi < \kappa^+$. As AC is true in $L[Q]$ and the form of the conclusion is absolute, the lemma now follows. ∎

LEMMA 2·7. *Suppose $G = \Sigma_{\nu<\mu} I_\nu$, where $\mu \leqslant \kappa$ and each* $\mathrm{otp}(I_\nu) \in [\kappa, \kappa^+)$. *Suppose that for each $\nu < \mu$, $(A_{\nu,n})_n$ is a strong κ-decomposition of I_ν. Set $B_0 = 0$, $B_{n+1} = \bigcup\{A_{\nu,n} | \nu < \mu\}$. Then $(B_n)_n$ is a strong κ-decomposition of G.*

Proof. $A_0 = A_1 = 0$. $\mathrm{otp}\, A_{n+1} \leqslant \kappa^n . \mu \leqslant \kappa^{n+1}$. The A_n's are disjoint.

Let $a \in [\omega]^\omega$. Then $(\bigcup B[a]) \cap I_\nu = \bigcup\{A_{\nu,n} | n \in a\}$, which is of order type equal to that of I_ν, as $(A_{\nu,n})_n$ is a strong decomposition. Hence

$$\mathrm{otp}(\bigcup A[a]) = \Sigma\, \mathrm{otp}(I_\nu) = \mathrm{otp}\, G. \quad ∎$$

LEMMA 2·8. *Let $\xi \in [1, \kappa^+)$. There is a function that assigns to each $\eta \in [1, \xi)$ a strong κ-decomposition of κ^η.*

Proof. Fix $\xi < \kappa^+$, and let f be a function, as in 2·6, which assigns to each limit ordinal $\lambda \leqslant \xi$ a closed cofinal subset $f(\lambda)$ of λ with $\mathrm{otp}(f(\lambda)) \leqslant \kappa$.

We first define by induction on $\eta \leqslant \xi$ a strong decomposition $A^\eta = (A^\eta_n)_n$ of κ^η.

For $\eta \in [1, \omega)$ take $A^\eta_n = 0$ for $n < \eta$ and $A^\eta_{\eta+k} = \{\lambda + k | \lambda \text{ a limit ordinal less than } \kappa^\eta\}$ for $k > 0$.

Take A^ω to be the canonical superdecomposition of κ^ω.

For $\eta = \mu + 1$, write $\kappa^\eta = \Sigma_{\nu<\kappa} I_\nu$ where each I_ν is of order type κ^μ, and strongly decompose each I_ν by copying κ^μ to I_ν and $(A^\mu_n)_n$ to $(A_{\nu,n})_n$. Set $A^\eta_0 = 0$,

$$A^\eta_{k+1} = \bigcup\{A_{\nu,k} | \nu < k\}.$$

In particular, $A^{\omega+1}$ will be the canonical strong decomposition of 2·4 (ii).

For η a limit ordinal greater than ω, let η_ν $(\nu < \rho \leqslant \kappa)$ be the closed sequence cofinal in η yielded by f, let $A^{(\nu)}$ be the strong decomposition of the interval $[\kappa^{\eta_\nu}, \kappa^{\eta_{\nu+1}})$ copied from $A^{\eta_{\nu+1}}$ when that interval is copied from $\kappa^{\eta_{\nu+1}}$, and set

$$A^{\eta}_{n+1} = \bigcup \{A^{(\nu)}_{n} | \nu < \rho\} \quad \text{and} \quad A^{\eta}_{0} = 0:$$

then, as before, each $\operatorname{otp}(A^{\eta}_{n+1}) \leqslant \kappa^{n+1}$.

Let $a \in [\omega]^{\omega}$. That for $\eta \in [1, \xi]$, $\operatorname{otp}(\bigcup A^{\eta}[a]) = \kappa^{\eta}$ is trivial for $\eta \leqslant \omega$ and follows from Lemma 2·7 for $\eta \geqslant \omega + 1$. ∎

LEMMA 2·9. *There is a function assigning to each limit ordinal λ less than κ a decomposition of it into ω pieces each of order type λ.*

Proof. Let A^0 be a decomposition of ω into ω infinite pieces. For any limit λ less than κ, write $\lambda = \omega . \eta$, copy A^0 to each $[\omega . \nu, \omega . (\nu + 1))$ as $A_{(\nu)}$ and take

$$A^{\lambda}_{n} = \bigcup_{\nu < \eta} A_{(\nu), n}. \quad ∎$$

Note that the above is in a trivial sense a strong κ-decomposition of λ.

PROPOSITION 2·10. *Let $\xi < \kappa^{+}$. There is a function assigning to each limit ordinal in $[\kappa, \xi)$ a strong decomposition thereof.*

Proof. Lemma 2·8 yields a strong decomposition of each κ^{η} $(\eta \in [1, \xi))$. Any limit ordinal $\lambda \in [\kappa, \kappa^{\xi})$ is of the form

$$\kappa^{\eta_0}\pi_0 + \kappa^{\eta_1}\pi_1 + \ldots + \kappa^{\eta_k}\pi_k$$

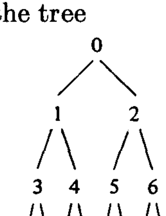
where $\eta_0 > \eta_1 > \ldots > \eta_k$ and each $\pi_i < \kappa$, and so a strong decomposition of it can be built up using Lemma 2·8 and, if $\eta_k = 0$, Lemma 2·9. ∎

Remark 2·11. Proposition 2·10 is a theorem of *ZF*. The assertion that there is a function defined on κ^{+} assigning to each $\zeta \in [\kappa, \kappa^{+})$ a Milner–Rado decomposition thereof is equivalent in *ZF* to the statement that there is a function defined on κ^{+} and assigning to each limit ordinal less than κ^{+} a cofinal subset of order type at most κ; implies, in *ZF*, that $\kappa^{+} \leqslant 2^{\kappa}$; and may therefore be unprovable in *ZF*, since its falsehood is (semantically) equiconsistent with *AC* plus the existence of a strong inaccessible greater than κ: cf. the discussion of Church's alternatives in Jech [**7**], chapter 11, section 4, problems 23 and 24.

2·10 immediately yields the original Milner–Rado theorem that every ordinal in $[\kappa, \kappa^{+})$ has a Milner–Rado κ-decomposition. Of the foregoing discussion, that corollary is all we need for the next theorem; the rest will be applied in Section 5.

THEOREM 2·12. *If $\aleph_1 \leqslant 2^{\aleph_0}$, $\omega_1^{\omega+2}$ is unsound.*

Proof. There is a perfect set of pairwise almost disjoint infinite subsets of ω, for example the set of paths through the tree



So if $\aleph_1 \leqslant 2^{\aleph_0}$, there is a sequence $(a_\nu | \nu < \omega_1)$ of pairwise almost disjoint infinite subsets of ω. For each $\nu < \omega_1$, let $\tilde{a}_\nu \colon \omega \leftrightarrow a_\nu$ enumerate a_ν in increasing order.

For $\nu < \omega_1$, put $I^{1+\nu} = [\theta.2^\nu, \theta.2^{\nu+1})$, and $I^0 = \theta$. Then $\operatorname{otp}(I^\nu) = \theta.2^\nu$, and $\omega_1^{\omega+2} = \Sigma_{\nu<\omega_1} I^\nu$.

Let $A^\nu = (A_k^\nu)_k$ be a Milner–Rado decomposition of I^ν.

Now set $B_n^\nu = 0$ for $n \notin a_\nu$; and for $n \in a_\nu$, set $B_n^\nu = A_k^\nu$, where $n = \tilde{a}_\nu(k)$.

Finally, put
$$B_n = \bigcup \{B_n^\nu | \nu < \omega_1\} \quad \text{and} \quad C^\nu = \bigcup \{B_n | n \in a_\nu\}.$$
$C^\nu \supseteq I^\nu$, so $\operatorname{otp}(C^\nu) \geqslant \theta.2^\nu$. For $\nu \neq \mu$,
$$C^\nu \cap I^\mu = \bigcup \{B_n^\mu | n \in a_\nu \cap a_\mu\};$$
as $a_\nu \cap a_\mu$ is finite, $\operatorname{otp}(C^\nu \cap I^\mu) < \omega_1^\omega$. Hence for each $\nu \in [2, \omega_1)$,
$$\theta.2^\nu \leqslant \operatorname{otp}(C^\nu) \leqslant \omega_1^\omega.\nu + \theta.2^\nu + \omega_1^\omega.\omega_1 \leqslant \theta(1+2^\nu+1) < \theta.2^{\nu+1} \leqslant \operatorname{otp}(C^{\nu+1}).$$

Thus $\{\tau_B(a_\nu)|\nu < \omega_1\}$ is uncountable, and $\omega_1^{\omega+2}$ is unsound. ∎

Remark 2·13. If $\kappa \leqslant 2^{\aleph_0}$, then $\kappa^{\omega+2}$ is κ-unsound in the sense that there is a $(\kappa^{\omega+2})$-sequence with spektron of order type at least κ.

3. *The soundness of $\omega_1^{\omega+\omega}$, provided $\aleph_1 \nleqslant 2^{\aleph_0}$*

Our goal in this section is the following

THEOREM 3·0. *Suppose that ω_1 is regular and that every well-orderable set of reals is countable. Then the least unsound ordinal is at least $\omega_1^{\omega+\omega+1}$.*

We assume throughout the section that ω_1 is regular and that $\aleph_1 \nleqslant 2^{\aleph_0}$. As before, we write θ for $\omega_1^{\omega+1}$ and, supposing that there is an unsound ordinal, ζ for the least such. From work in previous sections, we know that ζ is at least $\theta.\omega_1$, and that the theorem will be established if we can prove that for each $k \in [1, \omega)$, $\theta.\omega_1^k$ is sound.

We begin by making various reductions which will illustrate the use made of our hypothesis that $\aleph_1 \nleqslant 2^{\aleph_0}$. We shall then in an apparent digression prove a proposition about ω-colourings of ω_1^k: this proposition will enable us to establish the soundness of $\theta.\omega_1^k$ by direct calculation of spektra.

LEMMA 3·1. *Let ξ be an unsound ordinal. Then there is a (ξ)-sequence B with uncountable spektron and $B_n \cap B_m$ empty for $n < m < \omega$.*

Proof. Let A be a (ξ)-sequence with uncountable spektron. For $\nu < \xi$, let
$$a_\nu = \{n | \nu \in A_n\}.$$
Then $\{a_\nu | \nu < \xi\}$ is a well-orderable set of reals and hence countable: enumerate it as $\{b_i | i < \omega\}$ and put $B_i = \{\nu | a_\nu = b_i\}$. Then for $i \neq j$, $B_i \cap B_j$ is empty. For fixed n and i, and arbitrary $\mu, \nu \in B_i$,
$$\nu \in A_n \leftrightarrow n \in a_\nu = b_i = a_\mu \leftrightarrow \mu \in A_n,$$
so that $B_i \subseteq A_n$ or $B_i \cap A_n = 0$. Thus each A_n is a union of some B_i's, so $\sigma\pi(B)$ contains $\sigma\pi(A)$ and is hence uncountable. ∎

Remark 3·2. In previous sections we have not assumed that the sequences considered are of pairwise disjoint sets. Lemma 3·1 shows that there is no loss of generality in doing so. Though the proof given here assumes that $\aleph_1 \nleqslant 2^{\aleph_0}$, the lemma is also true if $\aleph_1 \leqslant 2^{\aleph_0}$, provided ω_1 is regular, since then we know that $\zeta = \omega_1^{\omega+2}$ and our construction in the last section is of such a disjoint sequence. That sequence also possesses the properties described in the next lemma.

Definition 3·3. A *solid* set of ordinals is one of order type ω_1^m for some $m \in [1, \omega)$. If the exact value of m is to be specified the set will be called *m-solid.*

LEMMA 3·4. *If ζ is the least unsound ordinal, there is a (ζ)-sequence, of solid and pairwise disjoint sets, with uncountable spektron.*

Proof. Let B be as in Lemma 3·1. Using 2·10, we may assign to each B_n a Milner–Rado decomposition $(B_{n,m})_m$ thereof with $\operatorname{otp}(B_{n,m}) \leqslant \omega_1^m$. Enumerating the double sequence $(B_{n,m})_{n,m}$ as a single sequence $(D_i)_i$ we obtain a (ζ)-sequence D of pairwise disjoint sets, each of order type less than ω_1^ω. We may now apply the discussion of 1·14 to find for each i an ordinal $\xi_i < \zeta$ such that $D_i \setminus \xi_i$ is either empty or cofinal in ζ and of order type ω_1^k for some $k = k_i \in [1, \omega)$. Then $\xi =_{\mathrm{df}} \bigcup_i \xi_i$ is also less than ζ and therefore sound, so $\sigma\pi((D_i \cap \xi)_i)$ is countable, and therefore by arguments in the Proof of 1·9, $\operatorname{otp}(\zeta \setminus \xi) = \zeta$ and $\sigma\pi((D_i \setminus \xi)_i)$ is uncountable. By copying $(D_i \setminus \xi)_i$ from $\zeta \setminus \xi$ to ζ, we obtain a (ζ)-sequence as desired. ▌

An important ingredient in making further reductions is Toulmin's notion of a shuffle.

Definition 3·5. An ordinal η is called a *shuffle* of ordinals ρ and σ if η is the union of two disjoint sets of order type ρ and σ respectively.

PROPOSITION 3·6 (Toulmin[13], p. 184). (i) *No indecomposable ordinal is a shuffle of two smaller ordinals.*

(ii) *Only finitely many ordinals are shuffles of a prescribed pair of ordinals.*

The significance of Toulmin's theorem for our enquiry is the following

PROPOSITION 3·7. *A shuffle of two sound ordinals is sound.*

Proof. Let η be the disjoint union of two sets R, of order type ρ, and S, of order type σ, and let A be an (η)-sequence. Define an (R)-sequence, B, and an (S)-sequence, C, by $B_n = R \cap A_n$, $C_n = S \cap A_n$. By the soundness of ρ and σ, the spektra of both B and C are countable. But every ordinal in $\sigma\pi(A)$ is a shuffle of an ordinal in $\sigma\pi(B)$ and an ordinal in $\sigma\pi(C)$: by 3·6 and 0·1 $\sigma\pi(A)$ is countable. ▌

That yields a second (and better) proof of 1·9.

The contrapositive of Proposition 3·7 yields the following extremely useful

LEMMA 3·8. *Suppose that the (ζ)-sequence A has uncountable spektron and that ζ, the least unsound ordinal, is the disjoint union of two sets, H and T, with* $\operatorname{otp}(H) < \zeta$. *Then $\sigma\pi((A_n \cap T)_n)$ is uncountable.* ▌

We turn to a discussion of subsets and partitions of ω_1^k for $k \in [1, \omega)$.

Definition 3·9. Let X be an m-solid set. For any $k \in [1, m]$, X can be written uniquely as $\Sigma_{\nu<\kappa} I_\nu^k$, where $\kappa = \omega_1^{m-k}$ and each I_ν^k is k-solid. The sets I_ν^k will be called the *k-blocks* of X. For $\eta \in X$ and $k \in [1, m]$, the unique k-block of X containing η will be denoted by $X(k, \eta)$. For $k = m$, of course, the k-block of η is X.

LEMMA 3·10. *Let $1 \leqslant k \leqslant l \leqslant m$, and let X be m-solid.*

(i) *For Y an l-block of X, and Z a k-block of X, either $Z \subseteq Y$, when Z will be a k-block of Y, or $Z \cap Y = 0$.*

(ii) *For a given k-block Z of X, there is exactly one l-block Y with $Z \subseteq Y$.*

(iii) *The k-blocks of an l-block of X are precisely the k-blocks of X included in the given l-block.*

Proof. Special case: Let X be a convex m-solid set of ordinals, and enumerate it monotonically as $\{\xi_\nu | \nu < \omega_1^m\}$. Define equivalence relations $\approx_k (k \in [1, m])$ on X by

$$\nu \approx_k \mu \quad \text{iff} \quad (\nu \in [\mu, \mu + \omega_1^k) \quad \text{or} \quad \mu \in [\nu, \nu + \omega_1^k)).$$

Then the k-blocks of X are precisely the $\approx_k$-equivalence classes, the first elements of the $\approx_k$-classes being the ξ_ν for ν a multiple of ω_1^k. The lemma now follows from elementary facts about nested equivalence relations.

The *General case*, for X not convex, follows from the special case by copying. ∎

Definition 3·11. Let T be a set of ordinals and X an m-solid set. We define the relation 'T is *pervasive* in X' or 'T *pervades* X' by induction on m. For $m = 1$, T is *pervasive* in X if for some $\xi \in X$, $T \cap X = X \setminus \xi$. For $m = n+1$, write $X = \Sigma\{Y_\nu | \nu < \omega_1\}$, where each Y_ν is an n-solid interval, so that the Y_ν's are the n-blocks of X. Then T is *pervasive* in X if there is a $\xi < \omega_1$ such that for $\nu < \xi$, $Y_\nu \cap T = 0$ and for $\xi \leqslant \nu < \omega_1$, T pervades Y_ν.

LEMMA 3·12. *Let T pervade the m-solid set X. Then*

(i) *if Y is a k-block of X for some $k \in [1, m]$, T is either disjoint from or pervasive in Y;*
(ii) *$X \setminus T$ is of order type less than ω_1^m;*
(iii) *$T \cap X$ is m-solid;*
(iv) *if $\nu \in X \cap T$, $X(1, \nu) \setminus \nu \subseteq T$;*
(v) *if X is an interval and $\nu \in X \cap T$, $[\nu, \nu + \omega_1) \subseteq T$;*
(vi) *for each ν in $X \cap T$, T pervades $X \setminus \nu$.*

Proof. (i) For fixed k by induction on $m \geqslant k$. For $m = k$, the result is trivial; for $m > k$, let Y be a k-block of X, and let Z be the $(m-1)$-block of X with $Y \subseteq Z$. By definition, either $Z \cap T = 0$ or T is pervasive in Z; but then, since by 3·9, Y is a k-block of Z, the induction implies that T is either disjoint from or pervasive in Y.

(ii) By induction on m: true for $m = 1$, as $X \setminus T$ is then countable. For $m = n+1$, let Y_ν $(\nu < \omega_1)$ be the n-blocks of X. For some $\xi < \omega_1$, T is pervasive in each Y_ν, and

$$X \setminus T = \Sigma\{Y_\nu | \nu < \xi\} + \Sigma\{Y_\nu \setminus T | \xi \leqslant \nu < \omega_1\}.$$

By induction each $\text{otp}(Y_\nu \setminus T)$ is less than ω_1^n, so by 2·5, $\text{otp}(\Sigma\{Y_\nu \setminus T | \xi \leqslant \nu < \omega_1\}) \leqslant \omega_1^n$. Hence $\text{otp}(X \setminus T) \leqslant \omega_1^n . (\xi + 1) < \omega_1^m$, as required.

(iii) Immediate from (ii).

(iv) Put $Y = X(1, \nu)$. The $\nu \in Y \cap T$ so by (i) T is pervasive in Y, a 1-solid set; for some $\xi \in Y$, $T \cap Y = Y \setminus \xi$, so $\xi \leqslant \nu$ and $Y \setminus \nu \subseteq Y \setminus \xi \subseteq T$.

(v) By (iv), since when X is an m-solid interval and $\nu \in X$, $X(1, \nu) \setminus \nu = [\nu, \nu + \omega_1)$.

(vi) For $m = 1$, (vi) follows from (iv). Now suppose $m = n+1$, and that we have proved (vi) for $m = n$. Let Z_μ $(\mu < \omega_1)$ be the n-blocks of X, so $X = \Sigma\{Z_\mu | \mu < \omega_1\}$, and let $\nu \in Z_\rho$. As T pervades X, there is a $\xi < \omega_1$ such that for all $\mu < \xi$, $Z_\mu \cap T = 0$ (so $\xi \leqslant \rho$), and for all $\mu \in [\xi, \omega_1]$, T is pervasive in Z_μ.

The n-blocks of $X \setminus \nu$ are $Z_\rho \setminus \nu$, which T pervades by (vi) for the case $m = n$, and the $Z_\mu (\mu > \rho)$, which T pervades as $\xi \leqslant \rho$. Hence T pervades $X \setminus \nu$. ∎

LEMMA 3·13. *If X is m-solid, $l \in [1, m]$, $\nu \in X$, then $X(l, \nu) \setminus \nu$ comprises the next ω_1^l elements of X after and including ν.*

Proof. By the characterization in the proof of 3·10 of blocks in terms of the equivalence relations $\approx_k$. ∎

LEMMA 3·14. *Let $T \subseteq X$, $\nu \in T$, $l \in [1, m]$, and let T pervade X. Then*

(i) $T(l,\nu)\backslash\nu = (X(l,\nu) \cap T)\backslash\nu$.
(ii) $T(l,\nu) = T \cap X(l,\nu)$.
(iii) $T(l,\nu)$ *pervades* $X(l,\nu)$.
(iv) $T(l,\nu)\backslash\nu$ *pervades* $X(l,\nu)\backslash\nu$.
(v) *For* $l = 1$, $T(1,\nu)\backslash\nu = X(1,\nu)\backslash\nu$.

Proof. (i) T is pervasive in $X(l,\nu)$, so $T \cap X(l,\nu)$ is l-solid, as is $T(l,\nu)$. The result now follows from 3·13 and the fact that $T \subseteq X$.

(ii) In view of (i) we need only prove that $T(l,\nu) \cap \nu = T \cap X(l,\nu) \cap \nu$. But for $\eta \in T \cap \nu$, these statements are equivalent: $\eta \in T(l,\nu)$; $\nu \in T(l,\eta)$; $\nu \in (T(l,\eta)\backslash\eta)$ (as $\eta < \nu$); $\nu \in (X(l,\eta) \cap T)$ (by (i)); $\eta \in X(l,\nu)$.

(iii) From (ii) and 3·12(i).

(iv) From (ii) and 3·12(vi).

(v) From (i) and 3·12(iv). ▎

Definition 3.15. Suppose that X is an m-solid set, and that ϕ_0 is a function whose domain includes X and whose values lie in a countable set R_0. We define for $k \in [1, m]$ countable sets $R_{X,k}$ and functions $\phi_{X,k}$: $X \to R_{X,k}$ by induction on k as follows:

Set $\phi_{X,1}(\nu) =_{\mathrm{df}} \{r_0 \in R_0 | \phi_0^{-1}\{r_0\}$ is unbounded in $X(1,\nu)\}$.

$\{\phi_{X,1}(\nu) | \nu \in X\}$ is a well-orderable set of subsets of the countable set R_0: it is therefore countable. Call it $R_{X,1}$. Note that $\phi_{X,1}$ is constant on each 1-block of X.

Set $\phi_{X,2}(\nu) =_{\mathrm{df}} \{r_1 \in R_{X,1} | \phi_{X,1}^{-1}\{r_1\}$ is unbounded in $X(2,\nu)\}$.

The range of $\phi_{X,2}$ is again a countable set; call it $R_{X,2}$. Note that $\phi_{X,2}$ is constant on each 2-block of X.

Repeat for all $k \leqslant m$. Thus set $\phi_{X,k+1}(\nu) =_{\mathrm{df}} \{r_k \in R_{X,k} | \phi_{X,k}^{-1}\{r_k\}$ is unbounded in $X(k+1,\nu)\}$.

LEMMA 3·16. *Let $\nu \in T$, $k \geqslant 1$, and $r \in R_{X,k}$, and put $P = \phi_{X,k}^{-1}\{r\}$. Then P is unbounded in $X(k+1,\nu)$ iff P is unbounded in $T(k+1,\nu)$.*

Proof. 'if': because $T(k+1,\nu) \subseteq X(k+1,\nu)$.

'only if': note that $\phi_{X,k}$ is constant on each k-block in $X(k+1,\nu)$, and that T, and hence $T(k+1,\nu)$, meets all except countably many of those. ▎

LEMMA 3·17. *Let T be a pervasive subset of the m-solid set X, and let ϕ_0 be a function defined on X with values in a countable set R_0. Then for all $\nu \in T$ and all $k \leqslant m$,*

$$\phi_{X,k}(\nu) = \phi_{T,k}(\nu).$$

Proof. $k = 1$: Let $\nu \in T$. By 3·14(v), $X(1,\nu)\backslash\nu = T(1,\nu)\backslash\nu$, so

$$\begin{aligned} r_0 \in \phi_{X,1}(\nu) \quad &\text{iff} \quad \phi_0^{-1} \text{ is unbounded in } X(1,\nu) \\ &\text{iff} \quad \phi_0^{-1} \text{ is unbounded in } T(1,\nu) \\ &\text{iff} \quad r_0 \in \phi_{T,1}(\nu). \end{aligned}$$

Thus $\phi_{X,1}(\nu) = \phi_{T,1}(\nu)$.

The case $k+1$ follows by applying 3·16 to the inductive hypothesis that for all $\nu \in T$, $\phi_{X,k}(\nu) = \phi_{T,k}(\nu)$. ▎

Where no ambiguity will result, we write ϕ_k instead of $\phi_{X,k}$.

PROPOSITION 3·18. *Let $k \in [1, \omega)$, X a k-solid set, and ϕ_0 a map from X to a countable set R_0. Then, effectively from X and ϕ_0, a subset T of X can be found that is pervasive in X and such that for $\nu \in T$,*

$$\phi_0(\nu) \in \phi_1(\nu) \in \ldots \in \phi_k(\nu).$$

Proof. By induction on k. For $k = 1$, let ξ be the least element of X such that for all $\nu \in X \setminus \xi$, $\phi_0(\nu) \in \phi_1(\nu)$: such a ξ exists since ω_1 is regular and R_0 is countable. Set $T = X \setminus \xi$ in this case.

Suppose $k = m+1$, and write $X = \Sigma\{Y_\rho | \rho < \omega_1\}$, where each Y_ρ is m-solid. By the effectivity of this construction we may find simultaneously for all $\rho < \omega_1$ sets $T_\rho \subseteq Y_\rho$ which are pervasive in Y_ρ and such that for all $\rho < \omega_1$ and $\nu \in T_\rho$,

$$\phi_0(\nu) \in \phi_1(\nu) \in \ldots \in \phi_m(\nu).$$

ϕ_m will be constant on each T_ρ, with value $\psi(\rho)$ say. The set of values $\{\psi(\rho) | \rho < \omega_1\}$ will be countable, and hence we can pick $\sigma < \omega_1$ minimal such that for each $\rho \in [\sigma, \omega_1)$, $\bigcup\{\rho' < \omega_1 | \psi(\rho') = \psi(\rho)\} = \omega_1$. Then $\{\psi(\rho) | \sigma \leqslant \rho < \omega_1\}$ is actually $\phi_k(\nu)$ for any $\nu \in X$, so if we set $T = \bigcup\{T_\rho | \sigma \leqslant \rho < \omega_1\}$, T will pervade X and we shall have

$$\phi_0(\nu) \in \phi_1(\nu) \in \ldots \in \phi_m(\nu) \in \phi_k(\nu)$$

for all $\nu \in T$, as required. ▎

We are now ready to prove Theorem 3·0. Fix $k \geqslant 1$ such that for $l < k$, $\theta . \omega_1^l$ is sound. We shall show that $\theta . \omega_1^k$ is sound.

By Lemma 3·4 we may suppose that $\zeta = \theta . \omega_1^k$ and that A is a $(\theta . \omega_1^k)$-sequence, with uncountable spektron, of pairwise disjoint sets each solid and cofinal in ζ, and aim for a contradiction.

Write $\zeta = \Sigma\{I_\nu | \nu < \omega_1^k\}$ where each I_ν is of order type θ. For each ν let $\xi_\nu \in I_\nu$ be minimal such that for each n, $A_n \cap I_\nu \setminus \xi_\nu$ is either empty or solid and cofinal in I_ν. Then $\bigcup\{I_\nu \cap \xi_\nu | \nu < \omega_1^k\}$ is of order type less than ζ, and hence, by Lemma 3·8 $\sigma\pi((A_n \cap S)_n)$ is uncountable, where $S = \bigcup\{I_\nu \setminus \xi_\nu | \nu < \omega_1^k\}$. Copy S to ζ, $I_\nu \setminus \xi_\nu$ to I_ν and each $A_n \cap S$ to B_n. Our problem is thus reduced to proving the following:

(3·19) Let $B\colon \omega \to Power(\zeta)$ be a decomposition of ζ into pairwise disjoint sets, each solid and cofinal in ζ, and, writing $\zeta = \Sigma\{I_\nu | \nu < \omega_1^k\}$, each $B_n \cap I_\nu$ either empty or solid and cofinal in I_ν. Then $\sigma\pi(B)$ is countable.

For such a sequence B, define functions $f_\nu\colon \omega \to \omega$ for $\nu < \omega_1^k$ thus:

$$(3\cdot 20) \qquad \begin{aligned} f_\nu(n) &= 0 \quad \text{if} \quad B_n \cap I_\nu \quad \text{is empty,} \\ &= p \quad \text{if} \quad B_n \cap I_\nu \quad \text{is } p\text{-solid.} \end{aligned}$$

Put $R_0 = \{f_\nu | \nu < \omega_1^k\}$. R_0, being a well-orderable set of 'reals', is countable. Define $\phi_0(\nu) = f_\nu$ for $\nu < \omega_1^k$, and define ϕ_l for $1 \leqslant l \leqslant k$ as above. By Proposition 3·18 we may find T pervasive in ω_1^k such that for $\nu \in T$, $\phi_0(\nu) \in \phi_1(\nu) \in \ldots \in \phi_k(\nu)$.

Set $X = \bigcup\{I_\nu | \nu \notin T\}$ and $Y = \bigcup\{I_\nu | \nu \in T\}$. Since $\text{otp}(\omega_1^k \setminus T) < \omega_1^k$, $\text{otp}(X) < \zeta$, and so we need only show that $\sigma\pi((B_n \cap Y)_n)$ is countable.

Copy Y to ζ, T to ω_1^k, and $B_n \cap Y$ to C_n. Then C has all the properties ascribed to B in the hypothesis of (3·19); moreover if we set $g_\nu(n) = 0$ if $C_n \cap I_\nu$ is empty and $g_\nu(n) = p$ if $C_n \cap I_\nu$ is p-solid, $Q_0 = \{g_\nu | \nu < \omega_1^k\}$, a countable set, $\psi_0(\nu) = g_\nu$ for $\nu < \omega_1^k$ and define countable sets Q_l and functions ψ_l with image Q_l, for $1 \leqslant l \leqslant k$, by the process set out in 3·15, then by Lemma 3·17, we shall have for all ν,

(3·21) $$\psi_0(\nu) \in \psi_1(\nu) \in \psi_2(\nu) \in \ldots \in \psi_k(\nu).$$

(3·22) For $a \subseteq \omega$, define $h_0(\nu, a) = \sup\{g_\nu(n) | n \in a\}$, and for $1 \leqslant l \leqslant k$, define $h_l(\nu, a) = \sup\{h_0(\rho, a) | \nu \leqslant \rho < \nu + \omega_1^l\}$.

Furthermore, write $\tau(a)$ for $\tau_C(a) = \operatorname{otp} \bigcup C[a]$, and $\tau_\nu(a)$ for $\operatorname{otp}(I_\nu \cap \bigcup C[a])$. Note that h_l is constant on each l-block, by (3·21).

Each $h_l(\nu, a) \leqslant \omega$. We shall show that $\tau(a)$ may be computed from the functions h_l and will be one of the countably many possibilities 0, ω_1^{k+1}, ω_1^{k+2}, $\omega_1^{k+3}, \ldots, \omega_1^{\omega+1}$, $\omega_1^{\omega+2}, \ldots, \omega_1^{\omega+1+k}$, which contradiction will complete the proof of Theorem 3·0.

LEMMA 3·23. *Let $m \in [1, \omega)$ and let $X = \Sigma\{X_\nu | \nu < \omega_1^m\}$, where each X_ν is of order type $\leqslant \theta$. Suppose further that* $\operatorname{otp}\{\nu | \operatorname{otp}(X_\nu) = \theta\} = \omega_1^m$. *Then* $\operatorname{otp} X = \theta . \omega_1^m$. ∎

LEMMA 3·24. *Suppose that $X = \Sigma\{X_\nu | \nu < \omega_1\}$, where each X_ν is of order type $< \omega_1^\omega$, but for all $\nu < \omega_1$ and all $p < \omega$, there is a ρ in $[\nu, \omega_1)$ with* $\operatorname{otp}(X_\rho) \geqslant \omega_1^p$. *Then* $\operatorname{otp}(X) = \theta$.

Proof. Evidently $\operatorname{otp} X \leqslant \omega_1^\omega . \omega_1 = \theta$. Lemma 1·14 implies that $\operatorname{otp} X \geqslant \theta$. ∎

If for some $\nu < \omega_1^k$, $h_0(\nu, a) = \omega$, $\tau_\nu(a) = \theta$ by 3·24; by 3·21 $\{\rho | g_\rho = g_\nu\}$ is k-solid, and so $\tau(a) = \theta . \omega_1^k$.

If for some l in $[1, k]$, there is a $\nu < \omega_1^k$ with $h_l(\nu, a) = \omega$, but $h_{l-1}(\nu, a) < \omega$ for all $\nu < \omega_1^k$, then $\tau(a) = \theta . \omega_1^{k-l}$: in particular, if $l = k$, $\tau(a) = \theta$.

To see that, fix l with $h_l(\nu, a) = \omega$ for some ν, but with each $h_{l-1}(\mu) < \omega$. For $E \subseteq \omega_1^k$, write J_E for $\bigcup\{I_\nu | \nu \in E\}$.

Now for E any $(l-1)$-block (or, if $l = 1$, E any $\{\xi\} \subseteq \omega_1^k$), $\operatorname{otp}((\bigcup C[a]) \cap J_E) < \omega_1^\omega$, since h_{l-1} is everywhere finite; so for each l-block F, $\operatorname{otp}(J_F \cap (\bigcup C[a])) \leqslant \theta$. For an l-block F on which $h_l = \omega$, $(\bigcup C[a]) \cap J_F$ has order type θ, by 3·24. (3·21) tells us that the set of such l-blocks is of order type ω_1^{k-l}. By 3·23, $\tau(a) = \theta . \omega_1^{k-l}$.

Suppose that $h_k(0, a) = p \in [1, \omega)$. Then for some $\nu < \omega_1^k$ and $n \in a$, $g_\nu(n) = p$, so $\operatorname{otp} C_n = \omega_1^{p+k}$ (by 3·21), and for all other $q \in a$, $\operatorname{otp} C_q \leqslant \omega_1^{p+k}$; so by solidity, $\tau(a) = \omega_1^{p+k}$.

Finally if $h_k(0, a) = 0$, then $\tau(a) = 0$. ∎

Remark 3·25. The cases $\tau(a) = \omega_1^r$ for $0 \leqslant r \leqslant k$ do not occur because of the pruning carried out before the final argument.

4. *Determinacy and unsoundness*

The question of the existence of unsound ordinals was raised by Woodin in connection with a then unsolved problem in the study of the axiom of determinacy. Subsequently this problem was solved by Kechris as Theorem 4·2 below, and led, by a further reflection argument due to Woodin, to the following

THEOREM 4·0. *The axiom of determinacy implies that there is an unsound ordinal less than ω_2.*

By kind permission of the authors, the proof of 4·0 is now presented. For definitions of unexplained terms and for background material Guaspari [**5**], Moschovakis [**10**], and the three Cabal volumes [**2**], [**3**], and [**4**] are suggested.

LEMMA 4·1 (Kechris, using ideas of Martin and Steel). *Assume* $(\mathbf{\Sigma}^1_2 \cup \mathbf{\Pi}^1_2)$*-determinacy.*

Then there is a Π^1_3 norm $\psi\colon B \to \boldsymbol{\delta}^1_3$ on a complete Π^1_3 set B and an ordinal $\xi_0 < \boldsymbol{\delta}^1_3$ such that for any $\boldsymbol{\Sigma}^1_3$ set $S \subseteq B$ there is a $\boldsymbol{\Sigma}^1_1$ set $S^* \subseteq B$ with

$$\{\psi(x)|x \in S^*\}\backslash\xi_0 = \{\psi(x)|x \in S\}\backslash\xi_0.$$

Proof. Let A be a Σ^1_2 set such that putting

$$A' = \{\alpha \in {}^\omega\omega|\exists n\,(\alpha)_n \in A\}, \quad A'' = \{(x,\alpha)|x(\alpha) \in A'\}$$

(where here x is viewed as a strategy for David, the second player, in the game where Goliath, the first player, plays α, David plays β, and David wins iff $\beta \in A'$), and $B = \{x|\forall\alpha(x,\alpha) \in A''\}$, B is a complete Π^1_3 set.

Let $|\ |\colon A \to \omega_1$ be a Σ^1_2 norm on A. Define $\rho\colon A'' \to \omega_1$ by $\rho(x,\alpha) = \min_{n\in\omega}|(x(\alpha))_n|$: then ρ is a Σ^1_2 norm on A''.

Finally let ψ be the Π^1_3 norm on B obtained from ρ as in the usual proof, using $\boldsymbol{\Delta}^1_2$ determinacy, of $PWO(\Pi^1_3)$: so for $y \in B$,

$$x \in B \quad \text{and} \quad x \leqslant_\psi y \quad \text{iff} \quad \exists\tau\,\forall\alpha(x,\alpha) \leqslant_\rho (y,[\alpha]^*\tau)$$

and

$$\text{iff} \quad \forall\sigma\,\exists\beta(x,\sigma^*[\beta]) \leqslant_\rho (y,\beta).$$

Now let $S \subseteq B$ be $\boldsymbol{\Sigma}^1_3$: suppose $S(z) \leftrightarrow \exists\alpha\,P(z,\alpha)$, where P is $\boldsymbol{\Pi}^1_2$. By Wadge's Lemma ([**2**], page 152), for which in this instance $\boldsymbol{\Sigma}^1_2 \cup \boldsymbol{\Pi}^1_2$ determinacy suffices, either there is an x^* such that $\forall z,\alpha(P(z,\alpha) \leftrightarrow \text{not } A(x^*(z,\alpha))$ or there is a y^* such that

$$\forall\beta(P(y^*(\beta)) \leftrightarrow A(\beta)):$$

but the second alternative is impossible since A, being Σ^1_2 complete, cannot be $\boldsymbol{\Pi}^1_2$.

Fix such an x^* and define x so that $(x(z,\alpha,y))_0 = x^*(z,\alpha)$ and

$$(x(z,\alpha,y))_{n+1} = (z(y))_n.$$

Then if $(z,\alpha) \notin P$, $x^*(z,\alpha) \in A$; while if $(z,\alpha) \in P$, $z \in S \subseteq B$, so $\exists n(z(y))_n \in A$. So for all z,α,y there is an n with $(x(z,\alpha,y))_n \in A$; and so $x \in B$.

Now for each z,α define $x_{z,\alpha}(y) = x(z,\alpha,y)$, and put $S^* = \{x_{z,\alpha}|z,\alpha \in \mathbb{R}\}$. Evidently S^* is $\boldsymbol{\Sigma}^1_1$ and is a subset of B.

If $(z,\alpha) \in P$, $x^*(z,\alpha) \notin A$, so $(x_{z,\alpha}(y))_0$ is irrelevant to the computation of $\psi(x_{z,\alpha})$, but as $(x_{z,\alpha}(y))_{n+1} = (z(y))_n$ for all n, we easily obtain $\psi(x_{z,\alpha}) = \psi(z)$.

If $(z,\alpha) \notin P$, then $x^*(z,\alpha) \in A$: but then $\psi(x_{z,\alpha}) \leqslant \psi(t)$ for any t such that

$$\forall n\,\forall y(t(y))_n = x^*(z,\alpha),$$

since for any y,

$$\min_n|(t(y))_n| = |x^*(z,\alpha)| \geqslant \min|(x(z,\alpha,y))_n|.$$

So if we put $X = \{t|\exists\delta \in A\,\forall y\,\forall n(t(y))_n = \delta\}$, notice that $X \subseteq B$, $X \in \boldsymbol{\Sigma}^1_2$ and so

$$\bigcup\{\psi(t)|t \in X\} < \xi_0 \quad \text{for some} \quad \xi_0 < \boldsymbol{\delta}^1_3$$

(since otherwise B could be expressed in $\boldsymbol{\Sigma}^1_3$ form as $\{z|\exists x \in X\,z \leqslant_\psi x\}$) and remark that since the definition of X is independent of S, so is ξ_0, we shall complete the proof that B and ψ are as required. ∎

THEOREM 4·2 (Kechris). *Assume the axiom of determinacy. Then for any map ϕ of a set W of reals onto $\boldsymbol{\delta}^1_3$, there is a compact subset J of W with $\{\phi(x)|x \in J\}$ uncountable.*

Proof. Let $\phi\colon W \twoheadrightarrow \boldsymbol{\delta}^1_3$ and let $\psi\colon B \twoheadrightarrow \boldsymbol{\delta}^1_3$ be as in 4·1. The method of proof of the Coding Lemma (Moschovakis [**10**], page 426) yields an $S \in \boldsymbol{\Sigma}^1_3$ such that

$$x \in B \to [S_x \neq 0 \quad \text{and} \quad \forall w \in S_x(w \in W \quad \text{and} \quad \psi(x) = \phi(w))].$$

Put $D = \{\langle x, w\rangle | x \in B \text{ and } w \in S_x\}$. $D \notin \mathbf{\Sigma}^1_3$, since $\forall x(x \in B \text{ iff } \exists w \langle x, w\rangle \in D)$ and $B \notin \mathbf{\Sigma}^1_3$, so by Wadge's Lemma, there is a continuous function F such that for all a, $a \in B$ iff $F(a) \in D$.

For $\xi < \boldsymbol{\delta}^1_3$ set $B_\xi = \{x \in B | \psi(x) < \xi\}$ and $D_\xi = \{\langle x, w\rangle | x \in B_\xi \text{ and } w \in S_x\}$. As ψ is a Π^1_3 norm, each $B_\xi \in \mathbf{\Delta}^1_3$ and each $D_\xi \in \mathbf{\Sigma}^1_3$. Now note two facts:

$$\forall \xi < \boldsymbol{\delta}^1_3 \exists \eta < \boldsymbol{\delta}^1_3 \{a | F(a) \in D_\xi\} \subseteq B_\eta; \tag{4·3}$$

$$\forall \xi < \boldsymbol{\delta}^1_3 \exists \eta < \boldsymbol{\delta}^1_3 \{F(a) | a \in B_\xi\} \subseteq D_\eta. \tag{4·4}$$

(4·3) holds because $\{a | F(a) \in D_\xi\}$ is a $\mathbf{\Sigma}^1_3$ subset of B and hence bounded in the norm ψ as $B \notin \mathbf{\Sigma}^1_3$; (4·4) holds because $\{F(a) | a \in B_\xi\}$ is a $\mathbf{\Sigma}^1_3$ subset of D, and were it not contained in some D_η the following would be a $\mathbf{\Sigma}^1_3$ definition of B:

$$x \in B \quad \text{iff} \quad \exists a \in B_\xi (x \leqslant_\psi (F(a))_0).$$

Let ξ_0 be the ordinal $< \boldsymbol{\delta}^1_3$ obtained in 4·1. From (4·3) and (4·4) the set

$$\{\xi < \boldsymbol{\delta}^1_3 | F^{-1}[D_\xi] = B_\xi \quad \text{and} \quad \xi > \xi_0\}$$

is closed unbounded in $\boldsymbol{\delta}^1_3$: let $\eta_\nu (\nu < \omega_1)$ enumerate the first ω_1 elements of that set, and put $\eta = \sup\{\eta_\nu | \nu < \omega_1\}$.

By 4·1, let C be a $\mathbf{\Sigma}^1_1$ subset of B_η with $\sup\{\psi(x) | x \in C\} = \eta$. Now consider the game where Goliath plays $x \in {}^\omega\omega$, David plays $w \in {}^\omega 2$ (this restriction is the source of the promised compactness) and David wins iff $(x \in C \rightarrow (w \in WO \text{ and } \psi(x) < \eta_{|w|}))$, where WO is the set of codes of well-orderings. Since WO is not $\mathbf{\Sigma}^1_1$, David cannot win, so let σ be a winning strategy for Goliath, and put $K = \sigma[{}^\omega 2]$. Then K is compact and $\{\psi(x) | x \in K\}$ is cofinal in η.

Put $J = \{w | \exists x \langle x, w\rangle \in F[K]\}$. Then J is compact: $K \subseteq C \subseteq B_\eta = F^{-1}[D_\eta]$, so $F[K] \subseteq D_\eta$, so by choice of S and definition of D_η, $J \subseteq \{w | \phi(w) < \eta\}$. But $\{\phi(w) | w \in J\}$ must be unbounded in η, since otherwise there would be an $\eta_\nu < \eta$ with

$$K \subseteq B_{\eta_\nu} = F^{-1}[D_{\eta_\nu}];$$

so ϕ takes uncountably many values on the compact set J. ▮

COROLLARY 4·5 (Woodin). *Under AD*, $\zeta < \boldsymbol{\delta}^1_3$.

Proof. Write $\kappa = \aleph_\omega$, and let X be a complete Σ^1_3 set and $T \subseteq {}^{<\omega}(\omega \times \kappa)$ a tree on κ, closed under shortening, such that for all reals α,

$$\alpha \in X \quad \text{iff} \quad \exists f \in {}^\omega\kappa \forall n (\alpha \restriction n, f \restriction n) \in T.$$

Thus $\alpha \notin X$ iff $T(\alpha)$ is well-founded, where $T(\alpha) = \{s \in {}^{<\omega}\kappa | (\alpha \restriction lh(s), s) \in T\}$.

Let W be the complement of X, and define $\phi(\alpha)$ for $\alpha \in W$ to be the order type of $T(\alpha)$ under the Kleene–Brouwer ordering $<_{KB}$. For each $\eta < \boldsymbol{\delta}^1_3$, $\{\alpha | \phi(\alpha) < \eta\}$ is in $\mathbf{\Delta}^1_3$, by Sierpiński's equations (Moschovakis [**10**], page 94, theorem 2F·1), and so the image of ϕ is cofinal in $\boldsymbol{\delta}^1_3$. The composition of ϕ with a suitable collapsing function is thus onto $\boldsymbol{\delta}^1_3$, and so by 4·2 there is a perfect set $S \subseteq {}^{<\omega}\omega$ such that $\{\phi(\alpha) | \alpha \in [S]\}$ is uncountable. (Here $[S] =_{\text{df}} \{\alpha | \forall n \alpha \restriction n \in S\}$.) Now put

$$U = \{(s, t) | s \in {}^{<\omega}\omega,\ t \in {}^{<\omega}\kappa,\ lh(s) = lh(t),\ (s, t) \in T \quad \text{and} \quad s \in S\}.$$

Under $<_{KB}$, U is well-ordered, since $[S] \cap p[T] = 0$: so let $\pi: U \simeq \xi$ be the isomorphism with an ordinal, ξ. $\xi < \aleph_{\omega+1} = \boldsymbol{\delta}^1_3$ since $\text{card}(U) \leqslant \aleph_\omega$.

For $s \in S$, put $A_s = \{\pi((s,t)) | lh(s) = lh(t) \text{ and } (s,t) \in U\}$. For $\alpha \in [S]$,

$$\phi(\alpha) = \text{otp}(\{t | (\alpha \restriction lh(t), t) \in T\}, <_{KB}) = \text{otp}(\bigcup \{A_{\alpha \restriction n} | n \in \omega\})$$

since the map $t \mapsto (\alpha \restriction lh(t), t)$ is $<_{KB}$-preserving. Hence $\sigma\pi(\{A_s | s \in S\})$ contains $\phi[[S]]$ and is therefore uncountable; thus ξ is unsound. ∎

The following reflection argument, due to Woodin, improves the bound on ζ from $\boldsymbol{\delta}^1_3$ to ω_2, assuming AD, and thus completes the proof of Theorem 4·0.

PROPOSITION 4·6. $(AD + DC)$ (Woodin) $\zeta < \omega_2$.

Proof. By the last proposition we know that there is an unsound ordinal, η say. Let $S = (S_n)_n$ be an (η)-sequence with uncountable spektron. Pick an increasing sequence $\langle \lambda_\nu | \nu < \omega_1 \rangle$ of ordinals such that for each $\nu < \omega_1$, there is a set $a \subseteq \omega$ with $\text{otp}(\bigcup S[a]) = \lambda_\nu$.

Consider the following integer game. Goliath plays x, David plays y, and David wins iff x does not code a countable ordinal or if x codes γ then y codes a sequence $\langle a_\nu | \nu < \delta \rangle$ of subsets of ω, where $\delta > \gamma$ and $\text{otp}(\bigcup S[a_\nu]) = \lambda_\nu$ for each $\nu < \delta$.

Standard arguments reveal that, granting the determinacy of the game, David must have a winning strategy, τ say.

Consider the model $L[\tau, S]$. Suppose g is a code of $\nu < \omega_1$ that is generic over $L[\tau, S]$. Then by the nature of τ, in the generic extension $L[\tau, S][g]$ there is a set $a \subseteq \omega$ such that $\text{otp}(\bigcup S[a]) = \lambda_\nu$.

Let κ be a cardinal in $L[\tau, S]$ with $\kappa > \eta$. Choose an elementary submodel M of $L_\kappa[\tau, S]$ with $\tau \in M$, $\omega_1 \subseteq M$, $S \in M$, and M of cardinality less than ω_2. Let π: $M \cong N$ be the map collapsing M to a transitive set N. Put $A_n = \pi(S_n)$, $\mu_\nu = \pi(\lambda_\nu)$, and let ξ be the height of N. Note that $A_n \subseteq \xi$, that $\langle \mu_\nu | \nu < \omega_1 \rangle$ is an increasing sequence, and that $\xi < \omega_2$.

We assert that $A = (A_n)_n$ witnesses the unsoundness of ξ. To see that, note first that by AD, ω_1 is strongly inaccessible in $L[\tau, S]$. Hence N and $L[\tau, S]$ have the same bounded subsets of ω_1. Suppose g codes an ordinal ν less than ω_1 and that g is generic over N. Then g is generic over $L[\tau, S]$ so that in fact $N[g]$ is an elementary submodel of $L_\kappa[\tau, S][g]$. Hence in $N[g]$ there is a set $a \subseteq \omega$ with $\text{otp}(\bigcup A[a]) = \mu_\nu$.

Finally, since ω_1 is strongly inaccessible in N, for every $\nu < \omega_1$ there is a code of ν generic over N.

Thus A has uncountable spektron, and $\zeta \leqslant \xi < \omega_2$. ∎

5. *Unsound ideals*

5·0. Suppose that η is indecomposable and that the (η)-sequence A partitions η into non-empty disjoint sets A_n each of order type less than η. Define

$$I_A = \{x \subseteq \omega | \tau_A(x) < \eta\}.$$

Then by the indecomposability of η, I_A is an ideal on ω containing all finite sets.

5·1. A partition A as above is evidently interdefinable with a surjection α: $\eta \twoheadrightarrow \omega$ where the pre-image of each $\{n\}$, i.e. A_n, is a set of order type less than η. If ψ: $\omega \twoheadrightarrow \omega$ is a surjection, then the composition $\psi \circ \alpha$: $\eta \twoheadrightarrow \omega$ will be a surjection, giving rise to an (η)-sequence B, where $B_k = \bigcup \{A_n | \psi(n) = k\}$. We write $B = \psi_* A$, and call B the *projection* of A by ψ.

5·2. If I is an ideal on ω containing all finite sets and ψ: $\omega \twoheadrightarrow \omega$ is a surjection with

$\psi^{-1}\{n\} \in I$ for each n, then $\psi_* I =_{\mathrm{df}} \{a \subseteq \omega | \psi^{-1}[a] \in I\}$ is also an ideal on ω containing all finite sets.

LEMMA 5·3. *Let η be indecomposable, A an (η)-sequence of non-empty disjoint sets, each of order type less than η, and $\psi: \omega \twoheadrightarrow \omega$ a surjection with each $\psi^{-1}\{n\} \in I_A$. Then*

$$I_{\psi_* A} = \psi_* I_A.$$

Proof.

$$\begin{aligned} x \in I_{\psi_* A} \quad &\text{iff} \quad \operatorname{otp} \bigcup \{\bigcup \{A(n) | \psi(n) = k\} | k \in x\} < \eta \\ &\text{iff} \quad \operatorname{otp} \bigcup \{A(n) | \psi(n) \in x\} < \eta \\ &\text{iff} \quad \psi^{-1}[x] \in I_A \\ &\text{iff} \quad x \in \psi_* I_A. \quad \blacksquare \end{aligned}$$

5·4. If A and η are as above, and $\sigma\pi(A)$ is uncountable, so that η is unsound, then $\{x \subseteq \omega | \sigma\pi((A_n)_{n \in x})$ is countable$\}$ is an ideal on ω containing all finite sets: call it J_A. Note that if η is the *least unsound* ordinal, $I_A \subseteq J_A$. If $\psi: \omega \twoheadrightarrow \omega$ is a surjection, then $\psi_* J_A \subseteq J_{\psi_* A}$; whether the reverse inclusion holds depends, as we shall see, on circumstances.

5·5. In [8] the author defined the notion of a *feeble* filter on ω; we may call an ideal *feeble* if its dual filter is, and the definition then runs: an ideal I on ω containing all finite subsets thereof is *feeble* if there is a surjection $\psi: \omega \twoheadrightarrow \omega$ with each pre-image $\psi^{-1}\{n\}$ finite (call such ψ *surfinjections*) and such that $\psi_* I$ is the ideal Fin of all finite subsets of ω. The author proved that if $\omega \to (\omega)^\omega$, every ideal is feeble; Talagrand and independently Jalali-Naini ([6], chapter I, 5·2·4 and 5·2·6) showed that an ideal is feeble if and only if, viewed as a subset of Cantor space ${}^\omega 2$ it has the property of Baire, so in models constructed by Solovay, using an inaccessible, and Shelah, without, every ideal is feeble.

5·6. Suppose therefore that I_A is feeble and that $\psi: \omega \twoheadrightarrow \omega$ is a surfinjection with $\psi_* I_A = Fin$. Put $B = \psi_* A$: then as $I_B = I_{\psi_* A} = \psi_* I_A = Fin$, B has countable spektron, since for any infinite x, $\tau_B(x) = \eta$ (as $x \notin I_B$). Note that in this case $\omega \in J_B$, which is therefore improper and not equal to $\psi_* J_A$. We have thus proved the following

PROPOSITION 5·7. *Suppose that all sets of reals have the property of Baire or that $\omega \to (\omega)^\omega$. Let η be any indecomposable ordinal and A a decomposition of η into sets of order type less than η. Then there is a partition of ω into finite sets s_k such that, setting $B_k = \bigcup A[s_k]$, the sequence $(B_k)_k$ has countable spektron.*

The above shows that when AC fails in certain familiar ways, though there still may be unsound ordinals (which there will be, for example, in Shelah's [11] model of 'all sets of reals have the property of Baire' in which $(\omega_1)_L = \omega_1$), every sequence with uncountable spektron projects by a finite-to-one function to one with countable spektron.

A related illustration of the instability of unsoundness is the following

PROPOSITION 5·8. *Suppose that $\aleph_1 \nleq 2^{\aleph_0}$, ω_1 is regular, $k \in \omega$, and that A is a decomposition of $\omega_1^{\omega+k}$ with each A_n of order type less than $\omega_1^{\omega+k}$. Then*

$$\exists x \in [\omega]^\omega \, \forall z \in [x]^\omega \operatorname{otp} \bigcup A[z] = \omega_1^{\omega+k}.$$

Proof. For $k = 0$, pick n_i increasing with $\operatorname{otp}(A_{n_i}) > \omega_1^i$, and set $x = \{n_i | i \in \omega\}$.

For $k \geqslant 1$, let $(J_n)_n$ be the superdecomposition of $\omega_1^{\omega+1}$ constructed in 2·4 (ii): so $\operatorname{otp}(J_n) = \omega_1^n$ for $n \geqslant 1$ and whenever $K_n \subseteq J_n$, $\operatorname{otp}(K_n) = \operatorname{otp}(J_n)$ and $a \in [\omega]^\omega$,

$$\operatorname{otp}(\bigcup K[a]) = \omega_1^{\omega+1}.$$

Now for $k = 1$, suppose $\omega_1^{\omega+1} = \dot{\bigcup}_n A_n$ as above: by the solidity of the J_n's we may define a function $f\colon \omega \to \omega$ by setting $f(n) =$ the least l such that $\operatorname{otp}(J_n \cap A_l) = \operatorname{otp}(J_n)$. Then if the image of f is finite, there will be $p \in \omega$ and $a \in [\omega]^\omega$ such that for all $n \in a$, $f(n) = p$: but then $\operatorname{otp}(A_p) \geqslant \operatorname{otp}(A_p \cap \bigcup J[a]) = \operatorname{otp}(\bigcup \{A_p \cap J_n | n \in a\}) \geqslant \omega_1^{\omega+1}$, contradicting our hypothesis on the $\operatorname{otp}(A_n)$'s. So the image of f is infinite: call it x.

Now for $z \in [x]^\omega$, put $b = f^{-1}[z]$. Then $\bigcup A[z] = \bigcup \{A_{f(n)} | n \in b\} \supseteq \bigcup \{A_{f(n)} \cap J_n | n \in b\}$: but each $A_{f(n)} \cap J_n$ has order type that of J_n, so by the property of $(J_n)_n$,

$$\operatorname{otp} \bigcup A[z] = \omega_1^{\omega+1}.$$

For $k > 1$, write $\omega_1^{\omega+k} = \Sigma_{\nu<\lambda} I_\nu$ where $\lambda = \omega_1^{k-1}$ and each I_ν is of order type $\omega_1^{\omega+1}$. Decompose I_ν as $\bigcup_{n<\omega} J_n^\nu$ by copying $(J_n)_n$. Define $f_\nu(n) =$ the least l with

$$\operatorname{otp}(A_l \cap J_n^\nu) = \operatorname{otp}(J_n^\nu),$$

and using the hypothesis that $\aleph_1 \nleq 2^{\aleph_0}$ and the solidity of λ find $f\colon \omega \to \omega$ such that $N =_{\mathrm{df}} \{\nu | f_\nu = f\}$ has order type λ.

As before, f cannot have finite image: if it did, then for some $p \in \omega$ and $a \in [\omega]^\omega$, $\forall n \in a (f(n) = p)$. But then the above reasoning can be repeated to show that for all $\nu \in N$, $\operatorname{otp}(A_p \cap I_\nu) = \omega_1^{\omega+1}$ and so $\operatorname{otp}(A_p) = \omega_1^{\omega+k}$, contrary to hypothesis.

So put $x =$ image of f: thus $x \in [\omega]^\omega$. For $z \in [x]^\omega$, put $b = f^{-1}[z]$ and note that for each $\nu \in N$, $(\bigcup A[z]) \cap I^\nu$ contains $\bigcup \{A_{f(n)} \cap J_n^\nu | n \in b\}$ and so is of order type $\omega_1^{\omega+1}$; and hence $\operatorname{otp}(\bigcup A[z]) = \omega_1^{\omega+k}$, as required. ∎

The last two propositions suggest that when AC fails in specified ways, each ordinal is 'nearly sound'. The next theorem, the last of this section, shows that by assuming the continuum hypothesis we may improve previous work to arrange for $\omega_1^{\omega+2}$ to be resoundingly unsound.

THEOREM 5·9. *Assume that* $2^{\aleph_0} = \aleph_1$. *Then there is an* $(\omega_1^{\omega+2})$*-sequence with*

$$\operatorname{otp}(A_n) \leqslant \omega_1^n$$

for each n such that for every $\psi\colon \omega \twoheadrightarrow \omega$ *with each* $\operatorname{otp}(\bigcup A[\psi^{-1}\{n\}]) < \omega_1^{\omega+2}$, $\psi_* A$ *also has uncountable spektron: moreover* $I_A = J_A$ *and is a prime ideal.*

Proof. Let U be a non-principal ultrafilter generated by a sequence $\langle b_\nu | \nu < \omega_1 \rangle$ where for $\nu < \rho < \omega_1$, $b_\rho \backslash b_\nu$ is finite and $b_\nu \backslash b_\rho$ is infinite. U will always be a p-point, and might in addition be a Ramsey ultrafilter.

For a and $b \in [\omega]^\omega$, we shall say that a *hits* b if $a \cap b$ is infinite, and we shall say that $a \subseteq b \pmod{Fin}$ if $a \backslash b$ is finite.

Set $c_\nu = b_\nu \backslash b_{\nu+1}$. Then no $c_\nu \in U$, and for $\nu \neq \rho$, $c_\nu \cap c_\rho$ is finite.

If $x \in U$, then for some $\mu < \omega_1$, $b_\mu \subseteq x \pmod{Fin}$, so for all $\nu \in [\mu, \omega_1)$, $x \supseteq b_\nu \supseteq c_\nu$ $(\mathrm{mod}\, Fin)$: i.e., x hits all but countably many c_ν's.

If $x \notin U$, then for some μ, $b_\mu \subseteq \omega \backslash x \pmod{Fin}$, so for all $v \in [\mu, \omega_1)$, $x \cap c_\nu$ is finite, so x hits only countably many c_ν's.

Thus $U = \{x | x \text{ hits uncountably many } c_\nu\text{'s}\}$.

Now write $\omega_1^{\omega+2}$ as $\Sigma\{I_\nu|\nu < \omega_1\}$, where $I_\nu = [\theta.2^\nu, \theta.2^{\nu+1})$, so I_ν has order type $\theta.2^\nu$. Modify a strong decomposition of I_ν to obtain a sequence $G_\nu = (G_{\nu,n})_n$ with

$$G_{\nu,0} = G_{\nu,1} = 0, \quad \operatorname{otp}(G_{\nu,n}) \leqslant \omega_1^{n-1} \quad \text{for} \quad n \geqslant 2, \quad G_{\nu,n} = 0 \quad \text{for} \quad n \notin c_\nu,$$

and for each $x \in [c_\nu]^\omega$, $\operatorname{otp}(\bigcup G_\nu[x]) = \operatorname{otp}(I_\nu)$.

Set $A_n = \bigcup\{G_{\nu,n}|\nu < \omega_1\}$.

Evidently $\operatorname{otp}(A_n) \leqslant \omega_1^n$. Let $x \in [\omega]^\omega$, and write $X = \bigcup A[x]$. Let $\nu < \omega_1$: then $X \cap I_\nu = \bigcup\{G_{\nu,n}|n \in x \cap c_\nu\}$, so if x hits c_ν, $\operatorname{otp}(X \cap I_\nu) = \operatorname{otp}(I_\nu)$, while if $x \cap c_\nu$ is finite, $\operatorname{otp}(X \cap I_\nu) \leqslant \omega_1^{k-1} < \omega_1^\omega$, where $k = \max(x \cap c_\nu)$.

Thus if x hits uncountably many c_ν's, $\operatorname{otp}(X) = \zeta$, whereas if $\rho =_{\mathrm{df}} \sup\{\nu|x \text{ hits } c_\nu\}$ is less than ω_1,

$$\theta.2\rho \leqslant \operatorname{otp} X \leqslant \theta.(2^\rho + 1) < \theta.2^{\rho+1}.$$

Hence $U = \{x|\tau_A(x) = \zeta\} = \mathit{Power}(\omega)\setminus I_A$, and

$$U = \{x|\tau_A[[x]^\omega] \text{ is uncountable}\} = \mathit{Power}(\omega)\setminus J_A,$$

so that $I_A = J_A$.

Suppose now that $\psi: \omega \twoheadrightarrow \omega$ is a surjection with each $\operatorname{otp}(\bigcup A[\psi^{-1}\{n\}]) < \zeta$, so that each $\psi^{-1}\{n\}$ is in I_A. Write $B = \psi_* A$. Then $I_B = \psi_* I_A$, which is a non-principal prime ideal. We shall show that B has uncountable spektron, which will imply that J_B is proper: as $I_B \subseteq J_B$ and I_B is prime, it will follow that $I_B = J_B$, so that in this case $\psi_* J_A = J_{\psi_* A}$.

Pick ν_n $(n \in \omega)$ with $b_{\nu_n} \cap \psi^{-1}\{n\}$ finite, and set $\mu = \sup_{n\in\omega} \nu_n$; $\mu < \omega_1$, and $\psi \restriction b_\mu$ is finite-to-one.

We shall show that for all $\nu \geqslant \mu$ there is an $a \in \psi_* I_A$ $(= I_B)$ such that $\psi^{-1}[a]$ hits c_ν: from this it follows that

$$\theta.2^\nu \leqslant \tau_A(\psi^{-1}[a]) = \tau_B(a) < \zeta,$$

and therefore as ν was arbitrary in $[\mu, \omega_1)$ and $\operatorname{cf}(\zeta) = \omega_1$, $\sigma\pi(B)$ is uncountable, as required.

So let $\nu \in [\mu, \omega_1)$, and put $a_0 = \psi[c_\nu]$. a_0 will be an infinite subset of ω as $c_\nu \subseteq b_\mu$ $(\operatorname{mod} \mathit{Fin})$ and $\psi \restriction b_\mu$ is finite-to-one. Divide a_0 into two infinite pieces a_1 and a_2: $\psi^{-1}[a_1]$ and $\psi^{-1}[a_2]$ are disjoint and therefore not both in U. Choose $a \in \{a_1, a_2\}$ with $\psi^{-1}[a] \notin U$. $\psi^{-1}[a]$ hits c_ν by construction, and $a \in \psi_* I_A$. ∎

6. *Open problems*

Several problems concerning unsound ordinals remain unsolved, which we list in this final section.

The one that comes immediately to mind is

Problem 6·0. Is it consistent with (say) $ZF + DC + \aleph_1 \not\leqslant 2^{\aleph_0}$ that every ordinal is sound?

A natural candidate for a model of 'all ordinals are sound' is Solovay's model for 'all sets of reals are Lebesgue measurable'. Since the reflection argument at the end of Section 4 relies heavily on AD, it is natural to ask

Problem 6·1. Is it provable in (say) $ZF + DC$ that if there is an unsound ordinal then there is one less than ω_2?

If it is provable outright that there is an unsound ordinal less than ω_2, how large is the least one? The work in Section 3 on raising the sound barrier may be susceptible of improvement: an exploration of the obstacles to further progress suggests the next two questions.

Problem 6·2. If $\aleph_1 \nleqslant 2^{\aleph_0}$, is $\omega_1^{\omega+\omega+1}$ sound?

For any (ζ)-sequence A the map τ_A: $Power(\omega) \to \zeta + 1$ which assigns to each $a \subseteq \omega$ the order type of $\bigcup A[a]$ induces a prewellordering $\leqslant_A$ of $Power(\omega)$. The discussion of Section 3 establishes the following

PROPOSITION 6·3. *If* $\aleph_1 \nleqslant 2^{\aleph_0}$ *and* $\eta < \omega_1^{\omega+\omega+1}$, *then for any* (η)-*sequence* A, η *can be partitioned as* $H \dot{\cup} T$ *so that* otp $(H) < \zeta$ *and, writing* A' *for* $(A_n \cap T)_n$, $\leqslant_{A'}$ *is Borel.* ∎

This suggests a way to answer 6·2 affirmatively. If that succeeds, it is likely to establish the soundness of every ordinal less than $\omega_1^{\omega_1}$, at which new difficulties appear:

Problem 6·4. If $\aleph_1 \nleqslant 2^{\aleph_0}$, is $\omega_1^{\omega_1}$ sound?

If Problem 6·4 has an affirmative answer, then one begins to speculate how far one can go before either an unsound ordinal is reached or the axiom of determinacy is refuted.

On a more mundane level, one can generalize the result of Section 2 to show that if $\aleph_2 \leqslant 2^{\aleph_0}$, there is an (ω_3)-sequence with spektron of cardinality $\aleph_2$, so that there exists what may be termed an ω_2-unsound ordinal.

Problem 6·5 (Woodin). How strong is the theory $ZF + DC +$ all sets of reals are Lebesgue measurable + there is an ω_2-unsound ordinal?

We have generally assumed that ω_1 is regular. For completeness we ask a general and a particular question:

Problem 6·6. If ω_1 is singular, is it unsound?

Problem 6·7. Is ω_1 unsound in Levy's model in which cf$(\omega_1) = \omega$?

Two questions arising from the discussion of Section 4 should be listed:

Problem 6·8 (Solovay). Consider the map ϕ that assigns to each real of the form $\alpha^{\#}$ the ordinal that is the cardinal successor in $L[\alpha]$ of the true ω_1. Is there a perfect set of sharps on which the image of ϕ is uncountable?

Problem 6·9 (Kechris). Does 4·2 remain true when $\boldsymbol{\delta}_3^1$ is replaced by $\aleph_\omega$ or (better still) $\aleph_2$?

Finally, a problem relevant to Section 5:

Problem 6·10. Does $\omega \to (\omega)^\omega$ hold in Shelah's model [**11**] for 'all sets of reals have the property of Baire'?

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