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# THE ORDER EXTENSION PRINCIPLE

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Let $O$ be the sentence "Every set has a total ordering", $OE$ the sentence "Every partial ordering can be extended to a total ordering".

THEOREM. *Let $\mathfrak{M}$ be a countable standard model of $ZF + V = L$. Then $\mathfrak{M}$ can be extended to a countable standard model $\mathfrak{N}$ of $ZF + O + \neg\, OE$.*

COROLLARY (PROOF BY HANDWAVING).

$$\mathrm{Con}\,(ZF) \to \mathrm{Con}\,(ZF + O + \neg\, OE).$$

PROOF OF THE THEOREM. A theorem of Jońsson [**3**] shows that there is in $\mathfrak{M}$ a partially ordered set $\langle I, R\rangle$ satisfying the following conditions in $\mathfrak{M}$.

P1. Every finite partial ordering can be embedded as a submodel of $\langle I, R\rangle$.

P2. Every isomorphism of finite submodels of $\langle I, R\rangle$ can be extended to an automorphism of $\langle I, R\rangle$.

I specify that $R$ is irreflexive; that is, that for no $i \in I$ is $\langle i, i\rangle \in R$. I write $i \|_R j$ for "neither $i\, R\, j$ nor $j\, R\, i$", and emphasize that if $\langle J, S\rangle$ is a submodel of $\langle I, R\rangle$, and $i \|_S j$ then $i \|_R j$.

Construct in $\mathfrak{M}$ a ramified language $\mathfrak{L}$ with the usual limited quantifiers $\forall_\alpha$, limited comprehension operators $E_\alpha$, variables, brackets, et cetera. $\mathfrak{L}$ has two predicate symbols $\in$ and $\equiv$, a name $\boldsymbol{u}$ for each $u \in \mathfrak{M}$, an individual constant $\dot{a}_i$ for each $i \in I$ (so that the set $\{\langle \dot{a}_i, i\rangle : i \in I\} \in \mathfrak{M}$), and two further constants $\dot{A}$ and $\dot{<}$. The wffs and limited comprehension terms (henceforth "nouns") of $\mathfrak{L}$ are defined as usual, with the restriction that if $E_\alpha \times \mathfrak{A}$ is a noun and $\dot{A}$ or $\dot{<}$ occurs in $\mathfrak{M}$ then $\alpha > \omega$. For details of the forcing method, the reader is referred to the expository article by Dr. R. B. Jensen [**2**]. In the rest of these notes, ramification indices will be omitted. $s, t, \ldots$ will be used as variables for nouns.

A *condition* is a finite consistent set of wffs of the form $\boldsymbol{n} \in \dot{a}_i$ or $\neg\, \boldsymbol{n} \in \dot{a}_i$ $(i \in I, n \in \omega)$. $P, P', Q, \ldots$ are variables for conditions.

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Define a weak forcing relation $\Vdash$ by

$$P \Vdash \boldsymbol{n} \in \dot{a}_i \leftrightarrow_{\mathrm{Df}} \boldsymbol{n} \in \dot{a}_i \in P;$$

$$P \Vdash s \in \dot{A} \leftrightarrow_{\mathrm{Df}} \forall P' \supseteq P \exists Q \supseteq P' \exists i \in I Q \Vdash s \in \dot{a}_i;$$

$$P \Vdash s < t \leftrightarrow_{\mathrm{Df}} \forall P' \supseteq P \exists Q \supseteq P' \exists i, j \in I$$

such that $i R j$ and $Q \Vdash s \equiv \dot{a}_i \wedge t \equiv \dot{a}_j$ so that, for $i, j \in I$,

$$i R j \rightarrow O \Vdash \dot{a}_i \dot{<} \dot{a}_j, \quad i \|_R j \rightarrow O \Vdash \dot{a}_i \| \dot{a}_j \quad \text{and} \quad O \Vdash \dot{a}_i \in \dot{A}.$$

Obtain a complete sequence of conditions and thereby an interpretation $\Omega$ of the nouns of the language $\mathfrak{L}$, which defines the model $\mathfrak{N}$.

Write $a_i = \Omega(\dot{a}_i)$, $A = \Omega(\dot{A})$, $< \, = \Omega(\dot{<})$. Then $a_i \subseteq \omega$, all $i$; $A = \{a_i : i \in I\}$; and $<$ is a partial ordering of $A$.

Lemma 1. *$<$ cannot be extended in $\mathfrak{N}$ to a total ordering of $A$.*

Let $G \in \mathfrak{M}$ be the group of all automorphisms $\in \mathfrak{M}$ of $\langle I, R \rangle$. For $\tau \in G$, define $\tau(P)$, $\tau(\mathfrak{A})$ for conditions $P$ and $\mathfrak{L}$-sentences $\mathfrak{A}$ to be the condition or wff obtained by replacing each occurrence of $\dot{a}_i$ by $\dot{a}_{\tau(i)}$, for each $i \in I$.

*S*-Lemma. $P \Vdash \mathfrak{A} \leftrightarrow \tau(P) \Vdash \tau(\mathfrak{A})$.

*R*-Lemma. *Suppose $P \Vdash \mathfrak{A}$, and $\boldsymbol{c} = (\dot{a}_1, \ldots, \dot{a}_n)$ is the set of $\dot{a}_i$ occurring in $\mathfrak{A}$. Let $P|\boldsymbol{c}$ be the set of conditions in $P$ in which $\dot{a}_1$ or $\cdots$ or $\dot{a}_n$ occurs. Then $P|\boldsymbol{c} \Vdash \mathfrak{A}$.*

Proof. Let $\boldsymbol{d}$ be the set of $\dot{a}_i$ occurring in $P$ but not in $\mathfrak{A}$. Write $Q = P \mid \boldsymbol{c}$, and suppose $\sim Q \Vdash \mathfrak{A}$. Then $\exists Q' \supseteq Q(Q' \Vdash \neg\mathfrak{A})$; this $Q'$ mentions (names of) reals in $\boldsymbol{c}$, and also others, say those in the finite set $\boldsymbol{e}$. $\boldsymbol{c} \cap \boldsymbol{d} = \boldsymbol{c} \cap \boldsymbol{e} = O$. By properties P1, P2 of $\langle I, R \rangle$, there is a $\tau \in G$ with $\dot{a} \in \boldsymbol{c} \rightarrow \tau(\dot{a}) = \dot{a}$; $\dot{a} \in \boldsymbol{e} \rightarrow \tau(\dot{a}) \notin \boldsymbol{c} \cup \boldsymbol{d}$. Then $\tau(Q') \Vdash \neg\mathfrak{A}$; $\tau(Q') \cup P$ is a condition but $\tau(Q') \cup P \Vdash \mathfrak{A} \wedge \neg\mathfrak{A}$—a contradiction. (Regard $\tau$ as acting on $\{\dot{a}_i : i \in I\}$ by defining $\tau(\dot{a}_i) = \dot{a}_{\tau(i)}$.)

Proof of Lemma 1. Suppose $\Delta$ is a noun such that $\Omega(\Delta)$ is a total ordering of $A$ extending $<$. Let $\mathfrak{A}(\Delta)$ be the $\mathfrak{L}$-sentence "$\Delta$ is a total ordering of $\dot{A}$ extending $\dot{<}$." Let $\boldsymbol{c}$ be the set of $\dot{a}_i$ occurring in $\Delta$. Pick $\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4, \dot{a}_5, \dot{a}_6$ so that $O \Vdash \dot{a}_4 \dot{<} \dot{a}_6$; for $i, j = 1, \ldots, 6$, $i < j$, $\langle i, j \rangle \neq \langle 4, 6 \rangle$, $O \Vdash \dot{a}_i \| \dot{a}_j$; for $\dot{a} \in \boldsymbol{c}$, $i = 1, \ldots, 6$, $O \Vdash \dot{a} \| a_i$; and $\mathfrak{N} = \dot{a}_1 \Delta \dot{a}_2 \wedge \dot{a}_2 \Delta \dot{a}_3$. Then there is a condition $P$ with $P \Vdash \mathfrak{A}(\Delta) \wedge \dot{a}_1 \Delta \dot{a}_2 \wedge \dot{a}_2 \Delta \dot{a}_3$.

By the *R*-Lemma, we may assume that $P$ mentions only reals in $\boldsymbol{c} \cup \{\dot{a}_1, \dot{a}_2, \dot{a}_3\}$. Set

$$P_1(\boldsymbol{c}, \dot{a}_1, \dot{a}_2) = P|\boldsymbol{c} \cup \{\dot{a}_1, \dot{a}_2\},$$

$$P_2(\boldsymbol{c}, \dot{a}_2, \dot{a}_3) = P|\boldsymbol{c} \cup \{\dot{a}_2, \dot{a}_3\}.$$

Then $P_1(\boldsymbol{c}, \dot{a}_1, \dot{a}_2) \Vdash \mathfrak{A}(\Delta) \wedge \dot{a}_1 \Delta \dot{a}_2$. There is a $\tau_1 \in G$ with $\dot{a} \in \boldsymbol{c} \rightarrow \tau_1(\dot{a}) = \dot{a}$; $\tau_1(\dot{a}_1) = \dot{a}_6$ and $\tau(\dot{a}_2) = \dot{a}_5$. By the *S*-Lemma,

$$(*) \qquad Q_1 =_{\mathrm{Df}} P_1(\boldsymbol{c}, \dot{a}_6, \dot{a}_5) \Vdash \mathfrak{A}(\Delta) \wedge \dot{a}_6 \Delta \dot{a}_5.$$

There is a $\tau_2 \in G$ with $\dot{a} \in \boldsymbol{c} \to \tau_2(\dot{a}) = \dot{a}$, and which sends $\dot{a}_2$ to $\dot{a}_5$ and $\dot{a}_3$ to $\dot{a}_4$. Applying $\tau_2$,

$$Q_2 =_{\mathrm{Df}} P_2(\boldsymbol{c}, \dot{a}_5, \dot{a}_4) \Vdash \mathfrak{A}(\Delta) \wedge \dot{a}_5 \,\Delta\, \dot{a}_4.$$

Now $O \Vdash \dot{a}_4 \dot{<} \dot{a}_6$; $Q_1 \cup Q_2$ is consistent and so a condition. But

$$Q_1 \cup Q_2 \Vdash \mathfrak{A}(\Delta) \wedge \dot{a}_4 < \dot{a}_6 \wedge \dot{a}_6 \,\Delta\, \dot{a}_5 \wedge \dot{a}_5 \,\Delta\, \dot{a}_4,$$

which is a contradiction; Lemma 1 is thus proved.

LEMMA 2. *$O$ is true in $\mathfrak{N}$.*

The argument is an adaptation of that of Mostowski [4] which he used to show the truth of $O$ in a model with urelements. The fundamental step is the following:

LEMMA. *Let $t(\boldsymbol{c}, \boldsymbol{d})$, $t'(\boldsymbol{c}, \boldsymbol{e})$ be nouns mentioning only reals in $\boldsymbol{c} \cup \boldsymbol{d}$, $\boldsymbol{c} \cup \boldsymbol{e}$ respectively, where $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ are finite disjoint subsets of $\{\dot{a}_i : i \in I\}$. Suppose that $\mathfrak{N} \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t'(\boldsymbol{c}, \boldsymbol{e})$. Then there is a noun $t''(\boldsymbol{c})$, mentioning only the reals in $\boldsymbol{c}$, with $\mathfrak{N} \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t''(\boldsymbol{c})$.*

PROOF. Let $P = P(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e})$ be a condition in the complete sequence defining $\mathfrak{N}$ such that

$$P \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t'(\boldsymbol{c}, \boldsymbol{e}).$$

A noun $t''(\boldsymbol{c})$ will be found for which

$$P \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t''(\boldsymbol{c}).$$

It will be enough to consider the case when $\boldsymbol{e}$ contains the single real $\dot{a}_0$; the general case follows by induction.

Let $\dot{a}_1, \dot{a}_2, \dot{a}_3$ be reals not in $\boldsymbol{c} \cup \boldsymbol{d} \cup \{\dot{a}_0\}$ such that $O \Vdash \dot{a}_1 \dot{<} \dot{a}_0 \wedge \dot{a}_2 \parallel \dot{a}_0 \wedge \dot{a}_0 \dot{<} \dot{a}_3$ and for $i = 1, 2, 3$ there is a $\tau_i \in G$ with $\tau_i(\dot{a}_0) = \dot{a}_i$, and $a \in \boldsymbol{c} \cup \boldsymbol{d} \to \tau(a) = a$. (This last clause will be abbreviated as $\tau_i : \boldsymbol{c}, \boldsymbol{d}, \dot{a}_0 \cong \boldsymbol{c}, \boldsymbol{d}, \dot{a}_i$.)

Write $P = P_0(\boldsymbol{c}) \cup P_1(\boldsymbol{d}) \cup P_2(\dot{a}_0)$. Then applying $\tau_i$ $(i = 1, 2, 3)$,

$$P_0(\boldsymbol{c}) \cup P_1(\boldsymbol{d}) \cup P_2(\dot{a}_i) \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t'(\boldsymbol{c}, \dot{a}_i);$$

$$P \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t'(\boldsymbol{c}, \dot{a}_0);$$

so $P \cup P_2(\dot{a}_i) \Vdash t(\boldsymbol{c}, \boldsymbol{d}) \equiv t'(\boldsymbol{c}, \dot{a}_0) \equiv t'(\boldsymbol{c}, \dot{a}_i)$, and by the $R$-Lemma

(*) $$P_0(\boldsymbol{c}) \cup P_2(\dot{a}_0) \cup P_2(\dot{a}_i) \Vdash t'(\boldsymbol{c}, \dot{a}_0) \equiv t'(\boldsymbol{c}, \dot{a}_i).$$

Let $\bar{P}_2(x)$ denote the conjunction of the sentences in $\bar{P}_2(\dot{a}_0)$ with $\dot{a}_0$ replaced throughout by the $\mathfrak{L}$-variable $x$. Let $D(\boldsymbol{c}, x)$ denote the result of replacing $\dot{a}_0$ by $x$ in the conjunction of the sentences in the diagram of the partial ordering $\boldsymbol{c}, \dot{a}_0$. Set $Q =_{\mathrm{Df}} P_0(\boldsymbol{c}) \cup P_2(\dot{a}_0)$. Note that $Q \subseteq P$. I assert that

(**) $$Q \Vdash \forall x (x \in \dot{A} \wedge \bar{P}_2(x) \wedge D(\boldsymbol{c}, x) : \to t'(\boldsymbol{c}, \dot{a}_0) \equiv t'(\boldsymbol{c}, x)).$$

Let $s$ be a noun. I have to show that if $Q' \supseteq Q$ and $Q \Vdash s \in \dot{A} \wedge \bar{P}_2(s) \wedge D(\boldsymbol{c}, s)$ then there is a $Q'' \supseteq Q'$ with $Q'' \Vdash t'(\boldsymbol{c}, \dot{a}_0) \equiv t'(\boldsymbol{c}, s)$.

Now given such a $Q'$, there is a $Q'' \supseteq Q'$, and a $j \in I$ with $Q'' \Vdash s \equiv \dot{a}_j$. Pick $i = 1, 2$ or $3$ so that $\dot{a}_0, \dot{a}_j \cong \dot{a}_0, \dot{a}_i$. There is a $\tau \in G$ so that, for this choice of $i$, $\tau : c, \dot{a}_i, \dot{a}_0 \cong c, \dot{a}_j, \dot{a}_0$; for $Q' \subseteq Q''$ and so $Q'' \Vdash s \equiv \dot{a}_j \wedge \bar{P}_2(\dot{a}_j) \wedge D(c, \dot{a}_j)$; and therefore $O \Vdash D(c, \dot{a}_j)$ and so $c, \dot{a}_j \cong c, \dot{a}_0$. But $c, \dot{a}_0 \cong c, \dot{a}_i$; by the choice of $i$, $c, \dot{a}_j, \dot{a}_0 \cong c, \dot{a}_i, \dot{a}_0$, and so such a $\tau$ exists. Further, as $Q'' \Vdash \bar{P}_2(\dot{a}_j)$, $P_2(\dot{a}_j) \subseteq Q''$, and so $P_0(c) \cup P_2(\dot{a}_0) \cup P_2(\dot{a}_j) \subseteq Q''$; applying $\tau$ to (*),

$$Q'' \Vdash t'(c, \dot{a}_0) \equiv t'(c, \dot{a}_j).$$

As $Q'' \Vdash \dot{a}_j \equiv s$, $Q'' \Vdash t'(c, \dot{a}_0) \equiv t'(c, s)$, as required.

The noun $t''(c)$ can now be constructed. Suppose $t'(c, \dot{a}_0)$ is $Ey\mathfrak{A}(c, \dot{a}_0, y)$. Then set

$$t''(c) =_{\mathrm{Df}} Ey\, \exists x(x \in \dot{A} \wedge \bar{P}_2(x) \wedge D(c, x) \wedge \mathfrak{A}(c, x, y)).$$

By (**), $Q \Vdash t'(c, \dot{a}_0) \equiv t''(c)$. As $Q \subseteq P$, and everything forced by $P$ is true in $\mathfrak{N}$, the Lemma is proved.

Now set up in $\mathfrak{N}$ a ramified language $\mathfrak{L}^*$ with a name $a^*$ for each $a \in A$ (so that $\{\langle a^*, a\rangle : a \in A\} \in \mathfrak{L}$), names $A^*$, $<^*$ for $A$, $<$; names $u^*$ for each $u < \mathfrak{M}$ (possible as $\mathfrak{M} = L_{\mathfrak{L}}$ and is therefore definable in $\mathfrak{N}$) and variables and quantifiers and the rest, as in $\mathfrak{L}$. Define an interpretation $\Omega^*$ for nouns of $\mathfrak{L}^*$, by setting

$$\Omega^*(a^*) = a, \quad \text{for each } a \in A;$$

$$\Omega^*(A^*) = A; \quad \Omega^*(<^*) = < \quad \text{and} \quad \Omega^*(u^*) = u;$$

and then extending to all nouns of $\mathfrak{L}^*$ *so that* $\Omega^*$ *is* $\mathfrak{N}$*-definable*. There is a clear 1-1 correspondence * between nouns $t$ of $\mathfrak{L}$ and nouns $t^*$ of $\mathfrak{L}^*$ so that $\Omega^*(t^*) = \Omega(t)$ for every $t$. Now define supp* $(x)$, for $x \in \mathfrak{N}$, as the finite subset (call it $\boldsymbol{d}$) of $A$ of minimal cardinality such that there is a term $t^*(\boldsymbol{d})$ mentioning only the names $a^*$ for $a \in \boldsymbol{d}$ with $\Omega^*(t^*) = x$. By the Lemma, supp* is always well-defined. Further, supp* *is* $\mathfrak{N}$*-definable*. For each finite subset $\boldsymbol{d}$, let $V_{\boldsymbol{d}} = \{x \in \mathfrak{N} : \text{supp}^*(x) = \boldsymbol{d}\}$. Each $V_{\boldsymbol{d}}$ has an $\mathfrak{N}$-definable well-ordering. Well-order the nouns of $\mathfrak{L}^*$ mentioning only the reals in $\boldsymbol{d}$, and use the interpretation $\Omega^*$ to induce a well-ordering of $V_{\boldsymbol{d}}$. Each $\boldsymbol{d}$ is a finite subset of $A$, which is ordered, being a subset of the real line. The set of such $\boldsymbol{d}$ can be ordered by the usual lexicographical method.

This argument shows that more than Lemma 2 holds in $\mathfrak{N}$. Observe that $x \in \mathfrak{N}$ is well-orderable (in $\mathfrak{N}$) iff there is a $\boldsymbol{d}$ with $\forall y(y \in x \to \text{supp}^*(y) \subseteq \boldsymbol{d})$.

Then there is an $\mathfrak{N}$-definable function which assigns to each set $x$ in $\mathfrak{N}$ an ordering, which is a well-ordering whenever $x$ has a well-ordering in $\mathfrak{N}$. Thus the axiom of choice for families of well-ordered sets holds in $\mathfrak{N}$.

Coda. Professor Scott asked at the end of my oration whether there was a general method for translating Fraenkel-Mostowski independence proofs into Cohen ones. Jech and Sochor have a method for certain classes of existential statements [**1**], and the papers of Pincus ([**5**], [**6**]) contain an extensive discussion of the question. On p. 740 of [**6**], my observation that the statement that the power set of every well-ordered set can be well-ordered, which is equivalent in ZF to the axiom of choice, is true in at least one Fraenkel-Mostowski model in which choice fails is misquoted. Kripke suggested considering instead the statement that every linearly ordered set can be well-ordered, which implies the other, but which avoids the suggestion of a hierarchy contained in the words "power set."

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