# The Stallings–Swan Theorem

Reading group on Ends of Groups

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The goal of these notes is to explain the statement and give a proof of the following:

**Theorem 0.1** (Stallings '68 [3])**.** *Let $G$ be a finitely generated group of cohomological dimension at most $1$. Then $G$ is free.*

The proof will rely on the following results from the previous talks:

**Theorem 0.2** (Stallings '68 [3])**.** *Let $G$ be a finitely generated group.*

1. *$G$ has more than one end if and only if it splits over a finite subgroup.*
2. *$G$ has two ends if and only if it is virtually $\mathbb{Z}$.*

We will use the cohomological definition of ends:

**Definition 0.3.** Let $G$ be an infinite, finitely generated group. Then the number of ends of $G$ is $\dim_{\mathbb{F}_2} H^1(G; \mathbb{F}_2 G) + 1$.

The main reference is Stallings' original paper [3], as well as notes taken from lectures on group cohomology by Gareth Wilkes and Brita Nucinkis.

## 1 Group cohomology via projective resolutions.

We start by recalling the definition of group cohomology. We will first need some algebraic language.

Throughout, $\Lambda$ will be a ring with unit.

**Definition 1.1.** A module $P$ over $\Lambda$ is said to be *projective* if one of the following equivalent conditions holds:

1. For every surjective map $f : M_1 \to M_2$ (of $\Lambda$-modules) and for every map $g_2 : P \to M_2$, there is a map $g_1 : P \to M_1$ making the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g_1"'] \arrow[d, "g_2"] \\ M_1 \arrow[r, two heads, "f"] & M_2 \end{tikzcd}$$

2. Every short exact sequence $0 \to M \to N \to P \to 0$ splits.
3. The module $P$ is a retract of a free module $F$, i.e. there are maps $j : P \hookrightarrow F$ and $s : F \twoheadrightarrow P$ such that $s \circ j = \mathrm{id}_P$.

**Definition 1.2.** A *projective resolution* of a module $M$ over $\Lambda$ is an exact sequence of $\Lambda$-modules

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0,$$

where each $P_i$ is projective.

Now fix a group $G$, and consider the group ring $\Lambda G$.

Fix a $\Lambda G$-module $M$. Take a projective resolution of $\Lambda$ (seen as a $\Lambda G$-module with trivial $G$-action) over $\Lambda G$, and apply $\mathrm{Hom}_{\Lambda G}(-, M)$ to obtain a cochain complex

$$\cdots \leftarrow \mathrm{Hom}_{\Lambda G}(P_{n+1}, M) \xleftarrow{d^{n+1}} \mathrm{Hom}_{\Lambda G}(P_n, M) \xleftarrow{d^n} \mathrm{Hom}_{\Lambda G}(P_n, M) \leftarrow \\ \cdots \leftarrow \mathrm{Hom}_{\Lambda G}(P_0, M).$$

**Definition 1.3.** The *cohomology* over the ring $\Lambda$ of the group $G$ with coefficients in the $\Lambda G$-module $M$ is the cohomology of the above cochain complex:

$$H^n_\Lambda(G; M) = \mathrm{Ker}\, d^{n+1} / \mathrm{Im}\, d^n.$$

This does not depend on the choice of a projective resolution.

**Proposition 1.4.** *If $M$ is a $\Lambda G$-module, then it can also be seen as a $\mathbb{Z}G$-module, and there are isomorphisms*

$$H^*_\Lambda(G; M) \cong H^*_\mathbb{Z}(G; M) \otimes \Lambda.$$

*Proof.* This is essentially because tensoring a projective $\mathbb{Z}G$-module by $\Lambda$ yields a projective $\Lambda G$-module. $\square$

# 2 Topological interpretation of group cohomology

There is a natural way to construct a projective resolution from the classifying space of a group. Let $X$ be a $K(G,1)$ space which is also a CW-complex, and let $\tilde{X}$ be the universal cover of $X$. For each $n \geq 0$, denote by $\mathrm{Cell}_n\left(\tilde{X}\right)$ the set of $n$-cells of $\tilde{X}$, and set

$$P_n = \bigoplus_{c \in \mathrm{Cell}_n\left(\tilde{X}\right)} \Lambda c.$$

The action $G \curvearrowright \mathrm{Cell}_n\left(\tilde{X}\right)$ induces an action $G \curvearrowright P_n$, which turns $P_n$ into a $\Lambda G$-module. Moreover, the boundary maps $\partial_n : \mathrm{Cell}_n\left(\tilde{X}\right) \to \mathrm{Cell}_{n-1}\left(\tilde{X}\right)$

induce $\Lambda G$-linear maps $d_n : P_n \to P_{n-1}$. This yields a projective resolution of $\Lambda$ over $\Lambda G$ (here, $P_0 \to \Lambda$ is the augmentation map defined by $c \mapsto 1$ for each basis element $c$ of $P_0$). Computing the cohomology of $G$ using this projective resolution, one recovers the singular cohomology of $X$ (at least when $\Lambda = \mathbb{Z}$ and $M$ has trivial $G$-action):

**Proposition 2.1.** *If $A$ is an abelian group (seen as a $\mathbb{Z}G$-module with trivial $G$-action), then there are isomorphisms*

$$H^*_{\mathbb{Z}}(G; A) \cong H^*_{\text{sing}}(K(G, 1); A).$$

# 3 Cohomological dimension

The key notion of these notes is the following:

**Definition 3.1.** Let $G$ be a group and let $\Lambda$ be a ring. The *cohomological dimension* of $G$ over $\Lambda$ is defined by

$$\mathrm{cd}_\Lambda(G) = \sup\{n \geq 0 \mid H^n_\Lambda(G; M) \neq 0 \text{ for some } \Lambda G\text{-module } M\} \in \mathbb{N}_{\geq 0} \cup \{\infty\}.$$

The cohomological dimension of a group $G$ is related to the length of the projective resolutions of $\Lambda$ over $\Lambda G$. More precisely:

**Definition 3.2.** The *projective dimension* of a $\Lambda$-module $M$ — denoted by $\mathrm{projdim}_\Lambda(M)$ — is the smallest integer $n$ such that there is a length-$n$ projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$

**Proposition 3.3.** *Given a group $G$ and a ring $\Lambda$, there is an equality*

$$\mathrm{cd}_\Lambda(G) = \mathrm{projdim}_{\Lambda G}(\Lambda).$$

*Proof.* ($\leq$) If $\Lambda$ has a length-$n$ projective resolution over $\Lambda G$, then it is clear that $H^k_\Lambda(G; M) = 0$ for every $\Lambda G$-module $M$ and for every $k > n$.

($\geq$) Assume that $\mathrm{cd}_\Lambda(G) \leq n$; we want to construct a length-$n$ projective resolution of $\Lambda$ over $\Lambda G$. Start with a projective resolution

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \Lambda \to 0$$

over $\Lambda G$. Let $M = \operatorname{Ker} d_n = \operatorname{Im} d_{n+1} \leq P_n$, and consider the cochain $\alpha \in \mathrm{Hom}_{\Lambda G}(P_{n+1}, M)$ given by the map $d_{n+1}$ with target restricted to $M$. Note that $d^{n+2}\alpha = 0$ since $d_{n+1} \circ d_{n+2} = 0$, so $\alpha$ is a cocycle. But $H^{n+1}_\Lambda(G; M) = 0$ by assumption, so $\alpha$ is a coboundary, i.e. there exists $\beta \in \mathrm{Hom}_{\Lambda G}(P_n, M)$ such that $\alpha = d^{n+1}\beta = \beta \circ d_{n+1}$. If $j : M \hookrightarrow P_n$ is the inclusion, then we have

$$\beta \circ j \circ \alpha = \beta \circ d_{n+1} = \alpha,$$

so $\beta \circ j = \mathrm{id}_M$ since $\alpha : P_{n+1} \to M$ is surjective by definition of $M$. Now define $\gamma : P_n \to P_n$ by $x \mapsto x - \beta(x)$. Check that $\gamma \circ \gamma = \gamma$, which implies that $\mathrm{Im}\,\gamma$ is a retract of $P_n$, and therefore a projective module. Moreover, the map

$$d_n : \mathrm{Im}\,\gamma \to \mathrm{Ker}\, d_{n-1}$$

is an isomorphism of $\Lambda G$-modules, so $\mathrm{Ker}\, d_{n-1}$ is also projective. This gives a length-$n$ projective resolution

$$0 \to \mathrm{Ker}\, d_{n-1} \to P_{n-1} \to \cdots \to P_0 \to \Lambda \to 0. \qquad \square$$

**Remark 3.4.** The proof of Proposition 3.3 implies that, if $\mathrm{cd}_\Lambda(G) \leq 1$, then the kernel of the augmentation map $\Lambda G \xrightarrow{\varepsilon} G$ is projective.

If in addition $G$ is finitely generated, then the module $P_1$ in the above proof can be chosen to be finitely generated over $\Lambda G$, and $\mathrm{Ker}\,\varepsilon$ is a finitely generated projective $\Lambda G$-module.

**Proposition 3.5.** *1.* $\mathrm{cd}_\Lambda(G) \leq \mathrm{cd}_\mathbb{Z}(G)$.

*2. If* $H \leq G$, *then* $\mathrm{cd}_\Lambda(H) \leq \mathrm{cd}_\Lambda(G)$.

*Proof.* 1. This follows from Proposition 1.4.

2. A projective $\Lambda G$-module is also a projective $\Lambda H$-module. $\square$

**Example 3.6.** 1. Let $F$ be a (nontrivial) free group. Then there is a $K(F,1)$ space which is a graph. This yields a length-1 projective resolution of $\Lambda$ over $\Lambda F$ as explained in §2, so

$$\mathrm{cd}_\Lambda(F) \leq 1.$$

Since $H^1_\Lambda(F;\Lambda) \cong \Lambda^{\oplus \,\mathrm{rk}\, F}$, this is an equality.

2. Let $G = \mathbb{Z}/k = \langle t \mid t^k = 1 \rangle$ for some integer $k \geq 2$. Then there is a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules given by:

$$\cdots \to \mathbb{Z}G \xrightarrow{\cdot(t-1)} \mathbb{Z}G \xrightarrow{\cdot k} \mathbb{Z}G \xrightarrow{\cdot(t-1)} \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Therefore, one can check that $H^{2i}_\mathbb{Z}(G;\mathbb{Z}) = \mathbb{Z}/n$ for all $i$, and therefore

$$\mathrm{cd}_\mathbb{Z}(G) = \infty.$$

It follows in particular that any group $G$ with $\mathrm{cd}_\mathbb{Z}(G) < \infty$ is torsion-free. However, $\mathrm{cd}_\mathbb{Q}(G) = 0$ if $G$ is finite!

3. Let $M$ be an aspherical connected $n$-manifold. Then a cellular decomposition of $M$ yields a length-$n$ projective resolution of $\Lambda$ over $\Lambda\pi_1 M$, so $\mathrm{cd}_\Lambda(\pi_1 M) \leq n$. If in addition $M$ is orientable, then Poincaré duality implies that

$$H^n_\Lambda(\pi_1 M;\Lambda) \cong \Lambda \otimes H_0^{\mathrm{sing}}(M;\mathbb{Z}) \cong \Lambda,$$

so $\mathrm{cd}_\Lambda(\pi_1 M) = n$.

# 4 Geometric dimension

The above examples suggest that the following invariant might bring useful information on the cohomological dimension:

**Definition 4.1.** The *geometric dimension* $\mathrm{gd}(G)$ of a group $G$ is the minimal dimension of a $K(G,1)$ space (or $\infty$ if there is no finite-dimensional $K(G,1)$).

**Example 4.2.**

1. If $F$ is a nontrivial free group, then $\mathrm{gd}(F) = 1$.

   Conversely, a group of geometric dimension 1 admits a classifying space which is a graph, and must therefore be free.

2. A group has geometric dimension at most 2 if and only if it admits an aspherical presentation.

Since a $n$-dimensional $K(G,1)$ space gives a length-$n$ projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ as explained above, we have the following:

**Proposition 4.3.** *For any group $G$ and any ring $\Lambda$, there are inequalities*

$$\mathrm{cd}_\Lambda(G) \leq \mathrm{cd}_\mathbb{Z}(G) \leq \mathrm{gd}(G).$$

In fact, the inequality $\mathrm{cd}_\mathbb{Z} \leq \mathrm{gd}$ turns out to be an equality in many cases:

**Theorem 4.4** (Eilenberg–Ganea '57 [2])**.** *For any group $G$, there is an equality $\mathrm{cd}_\mathbb{Z}(G) = \mathrm{gd}(G)$, except possibly in one of the following cases:*

1. $\mathrm{cd}_\mathbb{Z}(G) = 1$ *and* $\mathrm{gd}(G) \in \{2, 3\}$,
2. $\mathrm{cd}_\mathbb{Z}(G) = 2$ *and* $\mathrm{gd}(G) = 3$.

Stallings' contribution (i.e. Theorem 0.1) was to prove that any group of cohomological dimension 1 has geometric dimension 1. In other words, Case 1 cannot occur (at least for finitely generated groups — this was then generalised by Swan to all groups):

**Corollary 4.5** (Eilenberg–Ganea [2], Stallings [3], Swan [4])**.** *For any group $G$, there is an equality $\mathrm{cd}_\mathbb{Z}(G) = \mathrm{gd}(G)$, except possibly if $\mathrm{cd}_\mathbb{Z}(G) = 2$ and $\mathrm{gd}(G) = 3$.*

It is still not known whether or not there exists a group $G$ with $\mathrm{cd}_\mathbb{Z}(G) = 2$ and $\mathrm{gd}(G) = 3$.

# 5 Proof of the Stallings–Swan Theorem

We follow Stallings' proof [3, §6] and start with a very general algebraic lemma.

Throughout, $\Lambda$ is a ring with unit. Given a $\Lambda$-module $M$, we will write $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$. Hence, there is a natural map $M \to M^{**}$.

**Lemma 5.1.** *Let $P$ and $Q$ be finitely generated projective $\Lambda$-modules.*

1. *The canonical homomorphism $P \to P^{**}$ is an isomorphism.*

2. *If $\varphi : P \to Q$ is a homomorphism such that $\varphi^* : Q^* \to P^*$ is an isomorphism, then $\varphi$ is an isomorphism.*

*Proof.* 1. The assumption that $P$ is finitely generated projective means that there exists a finitely generated free $\Lambda$-module $F$, and maps $j : P \hookrightarrow F$ and $s : F \twoheadrightarrow P$ such that $s \circ j = \mathrm{id}_P$. Now consider the following commutative diagram:

$$\begin{tikzcd}
P \arrow[rrr] \arrow[dr, "j"] \arrow[ddd, "\mathrm{id}_P"'] & & & P^{**} \arrow[dl, "j^{**}"'] \arrow[ddd, "\mathrm{id}_{P^{**}}"] \\
& F \arrow[r, "\cong"] \arrow[dl, "s"] & F^{**} \arrow[dr, "s^{**}"'] & \\
& & & \\
P \arrow[rrr] & & & P^{**}
\end{tikzcd}$$

One can see that the composition $P^{**} \xrightarrow{j^{**}} F^{**} \xleftarrow{\cong} F \xrightarrow{s} P$ is the inverse of $P \to P^{**}$.

2. The inverse of $\varphi$ can be seen to be the composition

$$Q \xrightarrow{\cong} Q^{**} \xrightarrow{(\varphi^*)^{-1}} P^{**} \xleftarrow{\cong} P. \qquad \square$$

We now fix a group $G$. The connection between $\mathrm{cd}_{\mathbb{Z}}(G)$ and the number of ends of $G$ is uncovered by the following:

**Lemma 5.2.** *Let $G$ be a nontrivial finitely generated group with $\mathrm{cd}_{\mathbb{Z}}(G) \leq 1$. Then $G$ has more than one end.*

*Proof.* We start with the following observations:

- $G$ is infinite (otherwise $\mathrm{cd}_{\mathbb{Z}} G = \infty$ — see Example 3.6.2).

- $H^0_{\mathbb{F}_2}(G; \mathbb{F}_2 G) = 0$. Indeed, let $X$ be a $K(G, 1)$. Then $H^0_{\mathbb{F}_2}(G; \mathbb{F}_2 G)$ can be interpreted as the group of 0-cocycles on the universal cover $\tilde{X}$. A 0-cochain is a map $\alpha : \tilde{X}^{(0)} \to \mathbb{F}_2 G$ that is $G$-equivariant. If $\alpha$ is a cocycle, then it takes the same value on adjacent vertices; but $\tilde{X}$ is connected, so $\alpha$ must be constant. Now, $G$-equivariance implies that the value of $\alpha$ must be 0.

- $H^1_{\mathbb{F}_2}(G; \mathbb{F}_2 G) = 0$ if and only if $G$ has at most one end by Definition 0.3 (see also Remark 5.3 below).

Moreover, we have $\mathrm{cd}_{\mathbb{F}_2}(G) \leq \mathrm{cd}_{\mathbb{Z}}(G) \leq 1$, so Remark 3.4 yields an exact sequence

$$0 \to P \xrightarrow{\partial} \mathbb{F}_2 G \xrightarrow{\varepsilon} \mathbb{F}_2 \to 0, \qquad (1)$$

where $\varepsilon$ is the augmentation map and $P$ is a finitely generated projective $\mathbb{F}_2G$-module. Hence, we can compute $H^*_{\mathbb{F}_2}(G;\mathbb{F}_2G)$ as the cohomology of the cochain complex $0 \leftarrow P^* \xleftarrow{\partial^*} (\mathbb{F}_2G)^*$. In other words, there is an exact sequence

$$0 \to H^0_{\mathbb{F}_2}(G;\mathbb{F}_2G) \to (\mathbb{F}_2G)^* \xrightarrow{\partial^*} P^* \to H^1_{\mathbb{F}_2}(G;\mathbb{F}_2G) \to 0.$$

But $H^0_{\mathbb{F}_2}(G;\mathbb{F}_2G) = 0$. If $H^1_{\mathbb{F}_2}(G;\mathbb{F}_2G) = 0$, then $\partial^*$ would be an isomorphism, and so would $\partial$ by Lemma 5.1.2. But this would contradict the exact sequence (1), so $H^1_{\mathbb{F}_2}(G;\mathbb{F}_2G) \neq 0$ and $G$ has more than one end. $\square$

**Remark 5.3.** The cohomological definition of ends (Definition 0.3) is ambiguous as to what ring the cohomology should be computed over. However, note that there is an isomorphism of $\mathbb{F}_2$-vector spaces

$$H^1_{\mathbb{Z}}(G;\mathbb{F}_2G) \cong H^1_{\mathbb{F}_2}(G;\mathbb{F}_2G),$$

and it doesn't matter whether the cohomology is computed over $\mathbb{Z}$ or $\mathbb{F}_2$.

We can now prove the main theorem:

**Theorem 5.4.** *Let $G$ be a finitely generated group with $\mathrm{cd}_{\mathbb{Z}}(G) \leq 1$. Then $G$ is free.*

*Proof.* We argue by induction on the minimum number of generators of $G$ — which we denote by $\mathrm{rk}\, G$. If $\mathrm{rk}\, G = 0$, then $G$ is trivial.

Otherwise, Lemma 5.2 implies that $G$ has two or infinitely many ends. If $G$ has two ends, then since it is torsion-free (as $\mathrm{cd}(G) < \infty$) it must be isomorphic to $\mathbb{Z}$ (by Theorem 0.2.2), which is free of rank 1.

Now assume that $\mathrm{rk}\, G > 1$ and $G$ has infinitely many ends. By Theorem 0.2.1, $G$ splits over a finite subgroup $H$. Again, $G$ is torsion-free, so $H$ must be trivial, and $G$ does in fact split as a nontrivial free product

$$G = G_1 * G_2.$$

We have $\mathrm{cd}_{\mathbb{Z}}\, G_1, \mathrm{cd}_{\mathbb{Z}}\, G_2 \leq \mathrm{cd}_{\mathbb{Z}}\, G \leq 1$, and by Grushko's Theorem, $\mathrm{rk}\, G_1, \mathrm{rk}\, G_2 < \mathrm{rk}\, G$. Hence, by induction, $G_1$ and $G_2$ are both free, and so is $G$. $\square$

# 6 Two generalisations

Stallings' Theorem was given two major generalisations.

Swan removed the finite generation assumption (he also obtained a similar result over an arbitrary ring assuming that $G$ is torsion-free):

**Theorem 6.1** (Swan '69 [4])**.** *Any group $G$ with $\mathrm{cd}_{\mathbb{Z}}(G) \leq 1$ is free.*

Dunwoody gave a complete characterisation of groups of cohomological dimension 1 over an arbitrary ring:

**Definition 6.2.** A group $G$ is said to have *no $\Lambda$-torsion* if the order of every finite subgroup of $G$ is a unit in $\Lambda$.

**Theorem 6.3** (Dunwoody '79 [1])**.** *Let $G$ be a group and let $\Lambda$ be a ring. The following are equivalent:*

1. $\mathrm{cd}_\Lambda(G) \leq 1$.
2. *$G$ splits as a graph of groups where every vertex group is finite and has no $\Lambda$-torsion.*

# References


[1] M. J. Dunwoody, *Accessibility and groups of cohomological dimension one*, Proc. London Math. Soc. (3) **38** (1979), no. 2, 193–215. MR531159

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[4] Richard G. Swan, *Groups of cohomological dimension one*, J. Algebra **12** (1969), 585–610. MR240177