École Normale Supérieure de Lyon

Research internship report

Topology of complex affine varieties
From integrals to homology and cohomology

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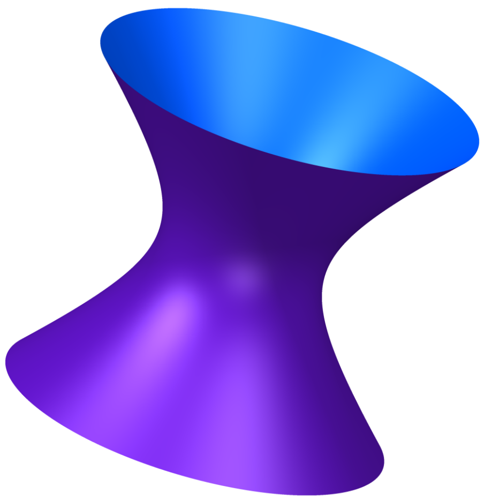
Abstract

This report is motivated by Picard and Simart's work on integrals on algebraic curves and surfaces in the late nineteenth century. After having presented some of the ideas of Picard and Simart, we shall develop the modern tools required to understand and prove two theorems about the topology of affine varieties that are already stated in their work. The first one is a study of algebraic de Rham cohomology, an abstract cohomology theory that can be defined for any algebra over a field. The second one is a computation of the homology groups of regular fibres of holomorphic functions by considering loops around singular values, using ideas from Picard-Lefschetz Theory.



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Introduction

This report has been written after a two-month stay in Instituto de Matemática Pura e Aplicada, in Rio de Janeiro. The work undertaken there aimed to study complex algebraic geometry by coming back to Picard's work, and most notably his book [PS71]. A first prerequisite was to learn the foundations of singular homology and, later, of complex geometry. After that, some time was spent studying the book of Picard directly, trying to understand some of the ideas despite the extraordinary shift in mathematical style that can be felt more than one hundred years after the time of writing. The aim was then to write fully modern and rigorous proofs of two related theorems by Picard, using [Mov19] as a compass and learning new mathematics along the way. We now present a sample of this work, with the hope that it will make the mathematics of yesterday meet that of today.

Terminology and notations

Throughout this text, we will work over an algebraically closed field k, being mostly interested in the case where $k = \mathbb{C}$. Only in Section 3 will we need to work over the field of real numbers, and in this case we shall write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

If I is an ideal of the ring $k[x_1, \dots, x_n]$, we will denote by $V(I) = \{x \in k^n,\ \forall f \in I,\ f(x) = 0\}$ the **vanishing locus** of I. If S is a subset of the affine space k^n, we will denote by $\mathcal{I}(S) = \{f \in k[x_1, \dots, x_n],\ \forall x \in S,\ f(x) = 0\}$ the ideal of S. The maps $I \mapsto V(I)$ and $S \mapsto \mathcal{I}(S)$ are nonincreasing (for the order induced by inclusion). We will call **affine variety** any subset of the affine space k^n that is the vanishing locus of some ideal I of $k[x_1, \dots, x_n]$. We will make frequent use of the following two (equivalent) versions of the Nullstellensatz.

Theorem 0.1 (Nullstellenstaz)**.** (i) *The maximal ideals of $k[x_1, \dots, x_n]$ are the ideals of the form $(x_1 - a_1, \dots, x_n - a_n)$ for $(a_1, \dots, a_n) \in k^n$.*

(ii) *If I is an ideal of $k[x_1, \dots, x_n]$, then $\mathcal{I}(V(I)) = \sqrt{I}$.*

Given a twice differentiable map $f : X \to Y$ between two manifolds, a point $x \in X$ will be called a **regular point** if $\mathrm{d}f(x)$ is onto, and a **singular point** otherwise. A point $y \in Y$ will be called a **regular value** if all points of $f^{-1}(\{y\})$ are regular, and a **singular value** otherwise. A singular point $x \in X$ will be called **degenerate** (respectively **nondegenerate**) if the quadratic form $\mathrm{d}^2 f(x)$ is degenerate (respectively nondegenerate).

Given a map $f : X \to Y$ and a subset $S \subseteq Y$, we shall write $L_S = f^{-1}(S)$ if it is clear from the context that the fibres are considered with respect to f.

Contents

1 Historical perspective: from integrals to algebraic topology

1.1 Elliptic integrals

The story of this report starts with the problem of computing integrals on affine varieties. Perhaps the first historical instance of this problem was the interest in *elliptic integrals* in the eighteenth century, i.e. integrals of the form

$$\int_a^b \frac{p(x)}{\sqrt{q(x)}}\,\mathrm{d}x,$$

where p is a polynomial and q is a polynomial of degree 3 or 4. The above elliptic integral can be rewritten as

$$\int_a^b \frac{p(x)}{y}\,\mathrm{d}x,$$

where integration is done on the so-called *elliptic surface* $S = \{y^2 = q(x)\}$. In this formulation, the integrand has become a rational function of x and y; however, new issues arise and in particular the following one: is the above integral well-defined or does it depend on the path chosen on the surface S between the endpoints a and b? As we are going to see, this problem will lead us to profound questions regarding the topology of elliptic surfaces and of affine varieties in general.

1.2 Integrals on curves and surfaces

In the late nineteenth century, Émile Picard together with Georges Simart undertook an extensive study of integrals on affine varieties in [PS71]. Their aim was to count the number of linearly independent integrals on a complex algebraic curve or surface X, and to classify them as integrals of the first, second or third kind. This classification very much reflects the issue of well-definedness of integrals; we give its definition in the 1-dimensional case (the general case being similar).

Consider a complex algebraic curve $X = \{(x,y) \in \mathbb{C}^2,\ f(x,y) = 0\}$, where f is a polynomial in x and y, and let $\int \omega$ be an integral on X satisfying the *integrability condition* (in a more modern language, ω is a rational 1-form on X such that $\mathrm{d}\omega = 0$).

(i) We say that $\int \omega$ is of the *first kind* if for every choice of holomorphic map $\psi : U \to X$ from an open neighbourhood U of 0 in $\mathbb{C}$, the meromorphic function $z \mapsto \omega(\psi(z)) \cdot \psi'(z)$ is holomorphic at 0.

Equivalently, for every path γ in X with endpoints a and b, $\int_\gamma \omega$ has a finite value that only depends on the homotopy class of γ (with fixed endpoints).

(ii) We say that $\int \omega$ is of the *second kind* if for every choice of holomorphic map $\psi : U \to X$ from an open neighbourhood U of 0 in $\mathbb{C}$, the meromorphic function $z \mapsto \omega(\psi(z)) \cdot \psi'(z)$ has no residue at 0.

Equivalently, for every contractible loop γ on X, $\int_\gamma \omega = 0$.

(iii) Otherwise, we say that $\int \omega$ is of the *third kind*.

1.3 De Rham cohomology

When Picard and Simart explored integrals, they did not have a notion of differential forms; those were introduced later by Élie Cartan and led to the modern concept of de Rham cohomology. However, it can be argued that Picard and Simart were already computing the (rational) de Rham cohomology of surfaces. For them, the main object of study was the integral but they were also considering the integrand under the name of "*différentielle totale*"; moreover, they always assumed that the *integrability condition* was satisfied — in other words they only considered closed forms. Finally, they considered two integrals to be distinct only when their difference was not a rational function of x, y, z — this amounts to identifying with zero any exact differential form.

The following theorem from [PS71, Vol. I, p.113] is therefore a result about the de Rham cohomology of complex algebraic surfaces.

Theorem 1.1 (Picard-Simart)**.** *"Une surface n'a pas, en général, d'intégrale de différentielle totale de première espèce."*

For Picard and Simart, a surface is a variety $X = \left\{(x, y, z) \in \mathbb{C}^3,\ f(x, y, z) = 0\right\}$, where f is a polynomial. The theorem says that, on a 'general surface', all integrals of the first kind are trivial.

Considering integrals of the first kind amounts to computing algebraic rather than rational de Rham cohomology. We shall prove the above theorem in Section 2 after having developed the modern algebraic point of view formally defined by Grothendieck, Atiyah and Hodge, and we shall make more precise what could have been meant by a 'general surface'.

Some of the computations we will do would not have been anachronic when Picard and Simart published their book even though the language would. Indeed, the study of cohomology arguably originated in algebra, in works like that of Picard and Simart, even though it was first formally defined in the context of differential topology following the work of de Rham. Hence, in some sense, algebraic de Rham cohomology goes back to the source of modern cohomology theories.

1.4 *Analysis situs*

Picard and Simart's work on integrals on affine varieties was done at the same time or shortly after Poincaré's foundational work on algebraic topology, which was then known as *analysis situs* or *géométrie de situation*. At the time, the main objects of interest in algebraic topology were the Betti numbers of topological spaces, which were more commonly referred to as *orders of connection*, and which correspond to the modern-day ranks of homology groups. Picard and Simart were interested in the link between the Betti numbers of affine varieties and the integrals on these varieties, and their book provides various results relating the Betti number to the number of independent integrals. One theorem of [PS71, Vol. I, p.85] will be of particular interest to us; it says that 'most algebraic surfaces' have trivial homology of order 1. In Picard's language, the theorem is stated as follows (note that Picard's p_1 is shifted by 1 as compared to the modern first-order Betti number).

Theorem 1.2 (Picard-Simart)**.** *"[Le nombre p_1] est, en général, égal à l'unité ; c'est seulement pour certaines surfaces particulières que p_1 est supérieur à 1."*

Proving a formalised and generalised version of this theorem will be the goal of Section 4. The techniques we shall use are those of Picard-Lefschetz Theory, which originated in Solomon Lefschetz's study of the Betti numbers of affine varieties in the first half of the twentieth century; the key idea will be to consider loops around singular values of holomorphic functions.

2 Algebraic de Rham cohomology

De Rham cohomology is a very powerful tool to study the topology of differentiable manifolds; its rather concrete definition makes computations feasible, yet it provides deep information about the topology of manifolds. Algebraic de Rham cohomology is the analogue in algebraic geometry, where differential forms will be defined using polynomials instead of smooth functions. After having constructed algebraic de Rham cohomology, our goal will be to prove that, with suitable hypotheses, all the cohomology groups of affine varieties of dimension n are trivial up to the order $n-1$.

2.1 Module of Kähler differentials and algebraic de Rham complex

For a k-algebra R, we shall construct the module of differential forms of R over k, our motivation being the case where $R = k\left[x_1, \dots, x_n\right]/I$ for some ideal I of $k\left[x_1, \dots, x_n\right]$. The idea of the construction will be to view R as the ring of functions of $V(I)$ and define differential forms, the exterior differential and the de Rham cohomology in such a way that computations work in the same way as in the differential case.

The **module of Kähler differentials** of R over k is the following R-module:

$$\Omega_R = \left(\bigoplus_{f \in R} R \,\mathrm{d}f\right)/N,$$

where N is the submodule of $\bigoplus_{f \in R} R\,\mathrm{d}f$ generated by $\left\{\mathrm{d}(a_1a_2) - a_1\,\mathrm{d}a_2 - a_2\,\mathrm{d}a_1,\ (a_1,a_2) \in R^2\right\}$ and $\left\{\mathrm{d}(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1\,\mathrm{d}a_1 - \lambda_2\,\mathrm{d}a_2,\ (\lambda_1,\lambda_2) \in k^2,\ (a_1,a_2) \in R^2\right\}$. The module Ω_R is endowed with a k-linear map $\mathrm{d} : R \to \Omega_R$ which is a derivation, i.e. we have the equality

$$\mathrm{d}(a_1a_2) = a_1\,\mathrm{d}a_2 + a_2\,\mathrm{d}a_1,$$

for all $a_1, a_2 \in R$. The elements of Ω_R are called **differential 1-forms** on R. In general, for $m \in \mathbb{N}$, the set of **differential m-forms** on R is defined by the following alternating product:

$$\Omega_R^m = \wedge^m\left(\Omega_R\right).$$

In particular, we have $\Omega_R^0 = R$ and $\Omega_R^1 = \Omega_R$.

For $m \in \mathbb{N}$, we now extend $\mathrm{d} : \Omega_R^0 \to \Omega_R^1$ to a k-linear map $\mathrm{d}^m : \Omega_R^m \to \Omega_R^{m+1}$ called the **exterior differential**, defined in such a way that the following equality holds for all $b, b_1, \dots, b_m \in R$:

$$\mathrm{d}^m\left(b\,\mathrm{d}b_1 \wedge \dots \wedge \mathrm{d}b_m\right) = \mathrm{d}b \wedge \mathrm{d}b_1 \wedge \dots \wedge \mathrm{d}b_m.$$

The family of maps $\left(\mathrm{d}^m\right)_{m\in\mathbb{N}}$ has the property that $\mathrm{d}^{m+1} \circ \mathrm{d}^m = 0$ for all $m \in \mathbb{N}$. Most of the time, we shall omit the superscript from the notation and write $\mathrm{d} : \Omega_R^m \to \Omega_R^{m+1}$.

We can now define the **de Rham complex** of R over k as the following complex of k-vector spaces:

$$0 \to R \xrightarrow{\mathrm{d}^0} \Omega_R \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{m-1}} \Omega_R^m \xrightarrow{\mathrm{d}^m} \Omega_R^{m+1} \xrightarrow{\mathrm{d}^{m+1}} \cdots.$$

The cohomology of this complex will be denoted by $H^\bullet_{\mathrm{dR}}(R)$ and called the **de Rham cohomology** of R over k. In other words, for $m \in \mathbb{N}$, we set $H^m_{\mathrm{dR}}(R) = \operatorname{Ker}\mathrm{d}^m / \operatorname{Im}\mathrm{d}^{m-1}$, where d^{-1} is understood as the zero map $0 \to R$.

Note that a map of k-algebras $\varphi : R_1 \to R_2$ induces maps $\varphi_* : \Omega_{R_1}^m \to \Omega_{R_2}^m$ which are compatible with the wedge product and which make the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_{R_1}^{m-1} \arrow[r, "\mathrm{d}"] \arrow[d, "\varphi_*"'] & \Omega_{R_1}^{m} \arrow[r, "\mathrm{d}"] \arrow[d, "\varphi_*"'] & \Omega_{R_1}^{m+1} \arrow[r, "\mathrm{d}"] \arrow[d, "\varphi_*"'] & \Omega_{R_1}^{m+2} \arrow[r] \arrow[d, "\varphi_*"'] & \cdots \\
\cdots \arrow[r] & \Omega_{R_2}^{m-1} \arrow[r, "\mathrm{d}"] & \Omega_{R_2}^{m} \arrow[r, "\mathrm{d}"] & \Omega_{R_2}^{m+1} \arrow[r, "\mathrm{d}"] & \Omega_{R_2}^{m+2} \arrow[r] & \cdots
\end{tikzcd}$$

As a consequence, φ also induces maps $\varphi_* : H_{\mathrm{dR}}^m(R_1) \to H_{\mathrm{dR}}^m(R_2)$. Hence, algebraic de Rham cohomology defines a functor from the category of k-algebras to the category of graded-commutative k-algebras.

From now on, we will turn our attention to the central case, i.e. the case where $R = k[x_1, \ldots, x_n]/I$ for some ideal I of $k[x_1, \ldots, x_n]$. In this case, in agreement with our geometric intuition, we shall allow ourselves to write Ω_V^m and $H_{\mathrm{dR}}^m(V)$ instead of $\Omega_{k[x_1,\ldots,x_n]/I}^m$ and $H_{\mathrm{dR}}^m(k[x_1, \ldots, x_n]/I)$, where $V = V(I)$ is the vanishing locus of I.

The following proposition describes differential forms on affine varieties.

Proposition 2.1. *Let I be an ideal of $R = k[x_1, \ldots, x_n]$. For $m \in \mathbb{N}$, we have:*

$$\Omega_{V(I)}^m = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} (R/I)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m}.$$

Moreover, in the case where $I = (0)$, the above sum is direct.

We can now compute the de Rham cohomology of the affine n-space.

Theorem 2.2 (Algebraic Poincaré Lemma)**.** *For $m \in \mathbb{N}$, we have:*

$$H_{\mathrm{dR}}^m(k^n) \simeq \begin{cases} k & \text{if } m = 0 \\ 0 & \text{if } m \geqslant 1 \end{cases}.$$

Proof (adapted from [Har75]*). First case*: $m = 0$. We need to show that $\operatorname{Ker} \mathrm{d} \subseteq k$. We note that, for $1 \leqslant i \leqslant n$, the k-linear derivation $\frac{\partial}{\partial x_i} : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ induces a $k[x_1, \ldots, x_n]$-linear map $\psi_i : \Omega_{k^n} \to k[x_1, \ldots, x_n]$ such that $\frac{\partial}{\partial x_i} = \psi_i \circ \mathrm{d}$ (this is actually the universal property of the Kähler differential). Therefore

$$\operatorname{Ker} \mathrm{d} \subseteq \bigcap_{1 \leqslant i \leqslant n} \operatorname{Ker}(\psi_i \circ \mathrm{d}) = \bigcap_{1 \leqslant i \leqslant n} \operatorname{Ker}\left(\frac{\partial}{\partial x_i}\right) = k.$$

Second case: $m \geqslant 1$. It suffices to show that $\operatorname{Ker} \mathrm{d}^m \subseteq \operatorname{Im} \mathrm{d}^{m-1}$ for $m \geqslant 1$. We shall argue by induction on n, the idea being to try to integrate a closed differential form coordinate by coordinate. The result is obvious when $n = 0$ (because in that case $\Omega_{k^n}^m = 0$ for $m \geqslant 1$). Assume the result has been proved for $n - 1$. Let $\omega \in \operatorname{Ker} \mathrm{d}^m$. We can write

$$\omega = \omega_1 + \mathrm{d}x_n \wedge \omega',$$

where $\omega_1 \in \Omega_{k^n}^m$ and $\omega' \in \Omega_{k^n}^{m-1}$ are differential forms not involving $\mathrm{d}x_n$ in the decomposition given by Proposition 2.1. Thus

$$\omega' = \sum_{1 \leqslant i_1 < \cdots < i_{m-1} \leqslant n-1} f_{i_1,\ldots,i_{m-1}}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{m-1}},$$

for some $f_{i_1,\ldots,i_{m-1}} \in k[x_1, \ldots, x_n]$. Consider

$$\eta = \sum_{1 \leqslant i_1 < \cdots < i_{m-1} \leqslant n-1} \left(\int f_{i_1,\ldots,i_{m-1}}\, \mathrm{d}x_n\right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{m-1}},$$

where the symbol $\int \mathrm{d}x_n$ refers to formal integration of polynomials with respect to the variable x_n. Hence

$$\mathrm{d}^{m-1}\eta = \omega_2 + \mathrm{d}x_n \wedge \omega',$$

where $\omega_2 \in \Omega^m_{k^n}$ is a differential form not involving $\mathrm{d}x_n$. As a consequence, we see that

$$\omega - \mathrm{d}^{m-1}\eta = \omega_1 - \omega_2 = \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n-1} g_{i_1,\dots,i_m}\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m},$$

for some $g_{i_1,\dots,i_m} \in k[x_1,\dots,x_n]$. But we know that $\mathrm{d}^m\left(\omega - \mathrm{d}^{m-1}\eta\right) = 0$; if we look at the terms involving $\mathrm{d}x_n$ in this equality, we obtain $\frac{\partial}{\partial x_n} g_{i_1,\dots,i_m} = 0$, which means that $g_{i_1,\dots,i_m} \in k[x_1,\dots,x_{n-1}]$. This proves that

$$\omega - \mathrm{d}^{m-1}\eta \in \Omega^m_{k^{n-1}}.$$

By the induction hypothesis, since $\mathrm{d}^m\left(\omega - \mathrm{d}^{m-1}\eta\right) = 0$, we conclude that there exists $\gamma \in \Omega^{m-1}_{k^{n-1}}$ such that $\omega - \mathrm{d}^{m-1}\eta = \mathrm{d}^{m-1}\gamma$, so $\omega = \mathrm{d}^{m-1}(\eta + \gamma) \in \operatorname{Im} \mathrm{d}^{m-1}$. □

We finish this section by giving an example to illustrate how the algebraic de Rham cohomology of an affine variety can give insight into its topology.

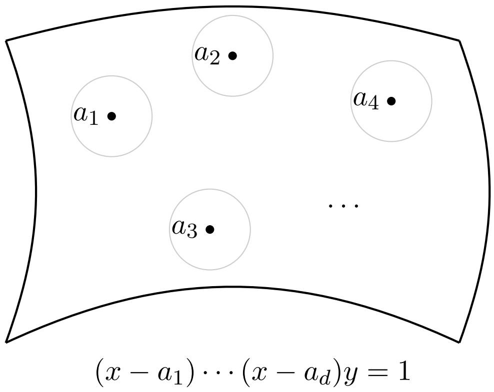


Figure 1: The complex curve $\{(x-a_1)\cdots(x-a_d)y = 1\}$ is isomorphic to the complex plane punctured at d points and has the homotopy type of a bouquet of d circles.

Example. *Consider a polynomial $f(x) = (x-a_1)\cdots(x-a_d) \in \mathbb{C}[x]$ with simple roots and let $V = V(f(x)y - 1) \subseteq \mathbb{C}^2$. Then $H^0_{\mathrm{dR}}(V) = \mathbb{C}$ and the map $\varphi : g \in \mathbb{C}[x] \longmapsto g(x)y\,\mathrm{d}x \in H^1_{\mathrm{dR}}(V)$ induces an isomorphism $H^1_{\mathrm{dR}}(V) \simeq \mathbb{C}[x]/(f) \simeq \mathbb{C}^d$. This agrees with the fact that V is a Riemann surface isomorphic to $\mathbb{C}\setminus\{a_1,\dots,a_d\}$, which has the homotopy type of a bouquet of d circles.*

2.2 Regularity hypotheses for polynomials

In order to study the de Rham cohomology of the affine variety $V(f) = \{x \in k^n,\ f(x) = 0\}$ for some $f \in k[x_1,\dots,x_n]$, we shall need some hypotheses to ensure that f is regular enough. We shall define these hypotheses and give some examples of polynomials satisfying them.

Let $\nu_1,\dots,\nu_n \in \mathbb{N}$. We say that a polynomial $f \in k[x_1,\dots,x_n]$ is **homogeneous of degree d with respect to the weights $\nu_1,\dots,\nu_n$** if it is of the form $f(x_1,\dots,x_n) = \sum_{\lambda\in\Lambda} \alpha_\lambda x_1^{\lambda_1}\cdots x_n^{\lambda_n}$, where $\Lambda \subseteq \mathbb{N}^n$ is such that $\sum_{i=1}^n \nu_i\lambda_i = d$ for all $\lambda \in \Lambda$. This defines a grading on $k[x_1,\dots,x_n]$: given weights $\nu_1,\dots,\nu_n$, every polynomial in $k[x_1,\dots,x_n]$ has a unique decomposition into homogeneous polynomials. This notion will allow us to weaken a little our hypotheses, by imposing conditions not directly on polynomials but on their homogeneous components with respect to arbitrarily chosen weights. We will also need to extend this grading to differential forms by defining the degree of $b_0\,\mathrm{d}b_1\wedge\cdots\wedge\mathrm{d}b_m$ to be $\sum_{i=0}^m\deg b_i$ for all $b_0,\dots,b_m\in k[x_1,\dots,x_n]$.

Proposition 2.3. *Let $g\in k[x_1,\dots,x_n]$ be a homogeneous polynomial with respect to the weights $\nu_1,\dots,\nu_n$. We denote by $J_g=\left(\frac{\partial g}{\partial x_1},\dots,\frac{\partial g}{\partial x_n}\right)$ the **jacobian ideal** of g. Then the following assertions are equivalent:*

(i) *The k-vector space $M_g=k[x_1,\dots,x_n]/J_g$ is finite dimensional.*

(ii) *The vanishing locus of J_g is the single point $\{0\}\subseteq k^n$.*

(iii) *The radical of J_g is the ideal $(x_1,\dots,x_n)\subseteq k[x_1,\dots,x_n]$.*

*If these conditions are satisfied, we say that g is **homogeneous tame**.*

Proof. (i) $\Rightarrow$ (ii) Assume that there exists $z=(z_1,\dots,z_n)\in V(J_g)\backslash\{0\}$. We may assume that $z_1\neq 0$. Since J_g is a homogeneous ideal with respect to the weights $\nu_1,\dots,\nu_n$, this implies that the line kz is included in $V(J_g)$. But note that this line is the vanishing locus of the prime ideal $\mathfrak{p}=(z_1x_2-z_2x_1,\dots,z_1x_n-z_nx_1)\subseteq k[x_1,\dots,x_n]$. In other words $V(J_g)\supseteq V(\mathfrak{p})$. By the Nullstellensatz, this implies that

$$J_g\subseteq\sqrt{J_g}\subseteq\sqrt{\mathfrak{p}}=\mathfrak{p}.$$

Therefore, we have a surjection $k[x_1,\dots,x_n]/J_g\twoheadrightarrow k[x_1,\dots,x_n]/\mathfrak{p}\simeq k[x_1]$. As $k[x_1]$ is infinite dimensional as a k-vector space, so is $k[x_1,\dots,x_n]/J_g$.

(ii) $\Rightarrow$ (iii) This is a direct consequence of the Nullstellensatz.

(iii) $\Rightarrow$ (i) Assume that $\sqrt{J_g}=(x_1,\dots,x_n)$. Therefore, for all $i\in\{1,\dots,n\}$, there exists $p_i\in\mathbb{N}$ such that $x_i^{p_i}\in J_g$. Hence

$$J_g\supseteq\left(x_1^{p_1},\dots,x_n^{p_n}\right).$$

Thus, we have a surjection $k[x_1,\dots,x_n]/\left(x_1^{p_1},\dots,x_n^{p_n}\right)\twoheadrightarrow k[x_1,\dots,x_n]/J_g$. Since the k-vector space $k[x_1,\dots,x_n]/\left(x_1^{p_1},\dots,x_n^{p_n}\right)$ is finite dimensional, so is $k[x_1,\dots,x_n]/J_g$. $\square$

We say that a polynomial $f\in k[x_1,\dots,x_n]$ is **tame** if there exist weights $\nu_1,\dots,\nu_n$ such that the homogeneous component of f of highest degree is homogeneous tame. Here are some simple examples.

Example. (i) *The monomial x^d is tame in $k[x]$. Therefore, every polynomial of $k[x]$ is tame.*

(ii) *If $h\in k[x_1,\dots,x_n]$ is tame, then the **hyperelliptic polynomial***

$$f(x_1,\dots,x_{n+1})=x_{n+1}^2+h(x_1,\dots,x_n)\in k[x_1,\dots,x_{n+1}]$$

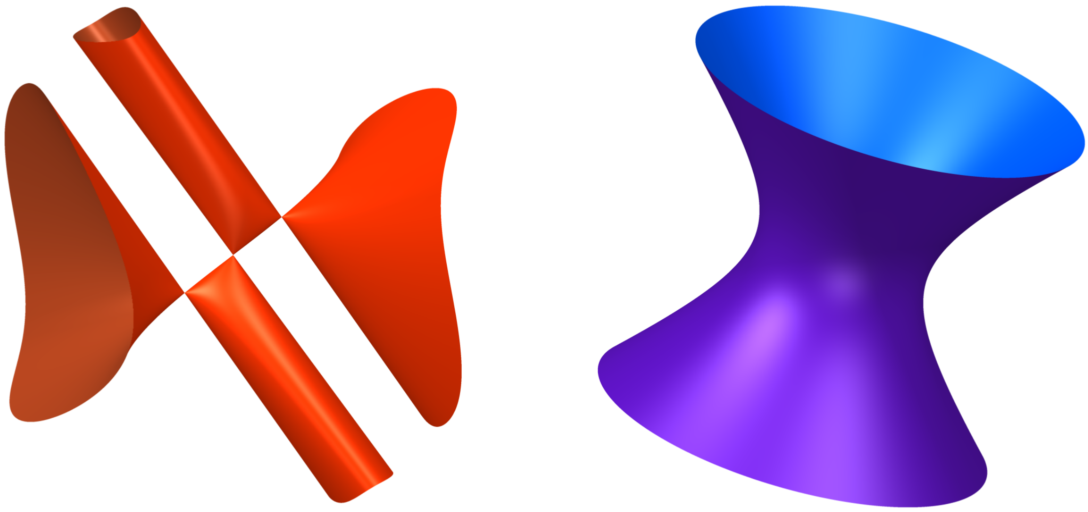
is also tame.

Proof. (i) Note that $\left(\frac{\partial x^d}{\partial x}\right)=\left(x^{d-1}\right)$ and therefore $V(J_{x^d})=\{0\}$.

(ii) Equip $k[x_1,\dots,x_n]$ with weights $\nu_1,\dots,\nu_n$ s.t. h is homogeneous tame. For these weights, denote by h_d the homogeneous polynomial of highest degree in h. We set $\nu_{n+1}=\deg h_d$. With respect to the weights $2\nu_1,\dots,2\nu_n,\nu_{n+1}$, the homogeneous polynomial of highest degree in f is $g(x_1,\dots,x_{n+1})=x_{n+1}^2+h_d(x_1,\dots,x_n)$. Now

$$J_g=\left(\frac{\partial g}{\partial x_1},\dots,\frac{\partial g}{\partial x_{n+1}}\right)=\left(\frac{\partial h_d}{\partial x_1},\dots,\frac{\partial h_d}{\partial x_n},x_{n+1}\right)=J_{h_d}+(x_{n+1}).$$

As h_d is tame, $V(J_{h_d})=\{0\}\subseteq k^n$, and therefore $V_{J_g}=\{0\}\subseteq k^{n+1}$, so g is tame. $\square$



$z^2 = x\,(x^2-1)\,y$ $z^2 = x^2 + y^2 - 1$

Figure 2: Real affine surfaces drawn using the *Surfer* program: on the left, $\{z^2 = x\,(x^2-1)\,y\}$ has three singularities; on the right, $\{z^2 = x^2 + y^2 - 1\}$ is tame and nonsingular.

We will need a second regularity hypothesis.

Proposition 2.4. *Let $f \in k\,[x_1, \ldots, x_n]$. The following assertions are equivalent:*

(i) *The endomorphism of the k-vector space $M_f = k\,[x_1, \ldots, x_n]\,/J_f$ induced by multiplication by f is invertible.*

(ii) *The vanishing locus of $(f) + J_f$ is empty.*

(iii) *There exists a polynomial $\widetilde{f} \in k\,[x_1, \ldots, x_n]$ such that $f\widetilde{f} \equiv 1 \mod J_f$.*

If these conditions are satisfied, we say that f is ***nonsingular****.*

Proof. (iii) $\Rightarrow$ (i) Clear.

(i) $\Rightarrow$ (iii) If the endomorphism of the k-vector space $M_f = k\,[x_1, \ldots, x_n]\,/J_f$ induced by multiplication by f is invertible, then in particular 1 has a preimage, so there exists $\widetilde{f} \in k\,[x_1, \ldots, x_n]$ such that $f\widetilde{f} \equiv 1 \mod J_f$.

(iii) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) If $V\,((f) + J_f) = \varnothing$, then by the Nullstellensatz, the ideal $\sqrt{(f) + J_f}$ contains the constant polynomial 1 and so does the ideal $(f) + J_f$. $\square$

Example. (i) *Every polynomial of $k[x]$ with simple roots is nonsingular.*

(ii) *If $h \in k\,[x_1, \ldots, x_n]$ is nonsingular, then the hyperelliptic polynomial*

$$f\,(x_1, \ldots, x_{n+1}) = x_{n+1}^2 + h\,(x_1, \ldots, x_n) \in k\,[x_1, \ldots, x_{n+1}]$$

is also nonsingular.

Proof. (i) If $f(x) \in k[x]$ has simple roots, then $\gcd\,(f, f') = 1$ and therefore there exist $u(x), v(x) \in k[x]$ such that $f(x)u(x) + f'(x)v(x) = 1$. Hence $f(x)u(x) \equiv 1 \mod J_f$.

(ii) We have $J_f = J_h + (x_{n+1})$. Therefore $M_f = k\,[x_1, \ldots, x_{n+1}]\,/J_f \simeq k\,[x_1, \ldots, x_n]\,/J_h = M_h$ and the endomorphism of M_f induced by multiplication by f corresponds to the endomorphism of M_h induced by multiplication by h. $\square$

2.3 Towards the cohomology of affine varieties: de Rham's Lemma

Our aim is now to show that, if $f \in k[x_1, \ldots, x_{n+1}]$ is sufficiently regular, then the affine variety $V(f)$ has trivial cohomology up to the order $n-1$. We start with the following technical proposition, which describes the link between $\Omega^m_{V(I)} = \Omega^m_{k[x_1,\ldots,x_{n+1}]/I}$ and $\Omega^m_{k^{n+1}} = \Omega^m_{k[x_1,\ldots,x_{n+1}]}$ for any ideal $I \subseteq k[x_1, \ldots, x_{n+1}]$.

Proposition 2.5. *Let I be an ideal of $R = k[x_1, \ldots, x_{n+1}]$. Then the k-linear map $\pi : R \to R/I$ induces maps $\pi_* : \Omega^m_{k^{n+1}} \to \Omega^m_{V(I)}$ and we have*

$$\operatorname{Ker}\left(\pi_* : \Omega^m_{k^{n+1}} \to \Omega^m_{V(I)}\right) = \left\{ f\omega_1 + \mathrm{d}g \wedge \omega_2,\ f, g \in I,\ \omega_1 \in \Omega^m_{k^{n+1}},\ \omega_2 \in \Omega^{m-1}_{k^{n+1}} \right\}.$$

Proof. The inclusion $(\supseteq)$ is clear.

Let us prove $(\subseteq)$. For $m = 0$, the lemma only affirms that the kernel of the projection $\pi : R \to R/I$ is I.

For $m = 1$, note that the map $\pi_* : \Omega^1_{k^{n+1}} \to \Omega^1_{V(I)}$ is induced by the map $\hat{\pi} : \bigoplus_{f\in R} R\,\mathrm{d}f \to \bigoplus_{f\in R/I} (R/I)\,\mathrm{d}f$ given by $\hat{\pi}(g\,\mathrm{d}f) = \pi(g)\,\mathrm{d}(\pi(f))$. We write N_R and $N_{R/I}$ for the respective submodules of $\bigoplus_{f\in R} R\,\mathrm{d}f$ and $\bigoplus_{f\in R/I} (R/I)\,\mathrm{d}f$ defining $\Omega^1_{k^{n+1}}$ and $\Omega^1_{V(I)}$, as in Section 2.1. Determining $\operatorname{Ker}\pi_*$ amounts to finding $\hat{\pi}^{-1}\left(N_{R/I}\right)$. But we have

$$\hat{\pi}^{-1}\left(N_{R/I}\right) = N_R + \left(\bigoplus_{f_I \in I} R\,\mathrm{d}f_I\right) + \left(\bigoplus_{f\in R} I\,\mathrm{d}f\right).$$

The result follows for $m = 1$.

Let $m > 1$. Note that the map $\pi_* : \Omega^m_{k^{n+1}} \to \Omega^m_{V(I)}$ is the map on the alternating product $\Omega^m_{k^{n+1}} = \wedge^m \left(\Omega^1_{k^{n+1}}\right)$ induced by $\pi_* : \Omega^1_{k^{n+1}} \to \Omega^1_{V(I)}$. Going back to the definition of the alternating product as a quotient of the tensor product and reasoning as in the case $m = 1$ yields the result. $\square$

Now that we have this proposition, we see that, for $f \in k[x_1, \ldots, x_{n+1}]$, the de Rham complex of $V(f)$ is similar to the following complex:

$$0 \longrightarrow \Omega^0_{k^{n+1}} \xrightarrow{\mathrm{d}f\wedge\cdot} \Omega^1_{k^{n+1}} \xrightarrow{\mathrm{d}f\wedge\cdot} \cdots \xrightarrow{\mathrm{d}f\wedge\cdot} \Omega^n_{k^{n+1}} \xrightarrow{\mathrm{d}f\wedge\cdot} \Omega^{n+1}_{k^{n+1}}.$$

We are going to prove de Rham's Lemma, which affirms that the above complex is an exact sequence of k-vector spaces provided that f is tame.

The proof of de Rham's Lemma will be by induction, and the tool which will make induction possible is the notion of depth of an ideal. To define it, consider a noetherian ring R. A sequence $(a_i)_{1\leqslant i\leqslant q}$ of elements of R such that a_i is not a divisor of zero in $R/(a_1, \ldots, a_{i-1})$ for all $1 \leqslant i \leqslant q$ is called a **regular sequence**. If $\mathfrak{J}$ is an ideal of R, the **depth** of $\mathfrak{J}$, denoted by $\operatorname{depth}(\mathfrak{J})$, is the maximal length of regular sequences of elements of $\mathfrak{J}$.

We will need to compute the depth of the jacobian ideal of a homogeneous tame polynomial. Towards this aim, we shall show that depth is a geometric property of ideals, i.e. that $\operatorname{depth}(\mathfrak{J})$ only depends on $V(\mathfrak{J})$, or equivalently by the Nullstellensatz, that $\operatorname{depth}(\mathfrak{J}) = \operatorname{depth}\left(\sqrt{\mathfrak{J}}\right)$.

Proposition 2.6. *Let R be a noetherian ring.*

(i) *A sequence $(a_i)_{1\leqslant i\leqslant q}$ is regular in R if and only if a_1 is not a divisor of zero in R and the sequence $(\overline{a_i})_{1<i\leqslant q}$ is regular in $R/(a_1)$.*

(ii) *If $\mathfrak{J}$ is an ideal of R and $a_0 \in \mathfrak{J}$ is the first term of a regular sequence of $\mathfrak{J}$ of maximal length, then the depth of the ideal $\overline{\mathfrak{J}}$ of $R/(a_0)$ is given by* $\operatorname{depth}\left(\overline{\mathfrak{J}}\right) = \operatorname{depth}(\mathfrak{J}) - 1$.

Proof (adapted from [Mur06]*).* (i) This is an obvious consequence of the definition.

(ii) Let $(\overline{a_i})_{1\leqslant i\leqslant q}$ be a regular sequence of $\overline{\mathfrak{J}}$ of maximal length. By (i), we see that $(a_i)_{0\leqslant i\leqslant q}$ is a regular sequence of $\mathfrak{J}$, so that $\operatorname{depth}(\mathfrak{J}) \geqslant q+1 = \operatorname{depth}\left(\overline{\mathfrak{J}}\right)+1$. Conversely, we know by assumption that we can complete a_0 to form a regular sequence $(a_i)_{0\leqslant i\leqslant q}$ of $\mathfrak{J}$ of maximal length. By (i), $(\overline{a_i})_{1\leqslant i\leqslant q}$ is a regular sequence of $\overline{\mathfrak{J}}$, which shows that $\operatorname{depth}\left(\overline{\mathfrak{J}}\right) \geqslant q = \operatorname{depth}(\mathfrak{J})-1$. ☐

Proposition 2.7. *Let R be a noetherian ring.*

(i) *If $(a_i)_{1\leqslant i\leqslant q}$ is a regular sequence of R and $(\xi_i)_{1\leqslant i\leqslant q}$ are elements of R such that $\xi_1 a_1+\cdots+\xi_q a_q=0$, then $\xi_i\in(a_1,\dots,a_q)$ for all $i\in\{1,\dots,q\}$.*

(ii) *If $(a_i)_{1\leqslant i\leqslant q}$ is a regular sequence of R, then the sequence $a_1^t,a_2,\dots,a_q$ is also regular for all $t\in\mathbb{N}^*$.*

(iii) *If $(a_i)_{1\leqslant i\leqslant q}$ is a regular sequence of R, then $(a_i^t)_{1\leqslant i\leqslant q}$ is also regular for all $t\in\mathbb{N}^*$.*

(iv) *For any ideal $\mathfrak{J}$ of R, we have* $\operatorname{depth}(\mathfrak{J})=\operatorname{depth}\left(\sqrt{\mathfrak{J}}\right)$.

Proof (adapted from [Mur06]*).* (i) We use induction on q. If $q=1$, then $\xi_1a_1=0$ implies that $\xi_1=0$ because a_1 is not a divisor of zero. Assume that $q>1$. The fact that $\xi_1a_1+\cdots+\xi_qa_q=0$ implies that $\xi_qa_q\in(a_1,\dots,a_{q-1})$. By regularity, $\xi_q\in(a_1,\dots,a_{q-1})$; write $\xi_q=\lambda_1a_1+\cdots+\lambda_{q-1}a_{q-1}$. Thus

$$(\xi_1+a_q\lambda_1)\,a_1+\cdots+(\xi_{q-1}+a_q\lambda_{q-1})\,a_{q-1}=0.$$

By the induction hypothesis, $\xi_i+a_q\lambda_i\in(a_1,\dots,a_{q-1})$ for all i, and so $\xi_i\in(a_1,\dots,a_q)$.

(ii) We shall use induction on t. The result is obvious for $t=1$. Let $t>1$ and assume that it has been proved that $a_1^{t-1},a_2,\dots,a_q$ is regular. Since a_1^{t-1} is not a divisor of zero in R, neither is a_1^t. We now need to prove that, for $i>1$, a_i is not a divisor of zero in the ring $R/\left(a_1^t,a_2,\dots,a_{i-1}\right)$. Let $b\in R$ such that

$$a_ib\in\left(a_1^t,a_2,\dots,a_{i-1}\right).$$

We write $a_ib=\xi_1a_1^t+\xi_2a_2+\cdots+\xi_{i-1}a_{i-1}$. In particular we have $a_ib\in\left(a_1^{t-1},a_2,\dots,a_{i-1}\right)$ and since $a_1^{t-1},a_2,\dots,a_q$ is regular, we have

$$b\in\left(a_1^{t-1},a_2,\dots,a_{i-1}\right).$$

We write $b=\zeta_1a_1^{t-1}+\zeta_2a_2+\cdots+\zeta_{i-1}a_{i-1}$. Hence, we have

$$(\xi_1a_1-a_i\zeta_1)\,a_1^{t-1}+(\xi_2-a_i\zeta_2)\,a_2+\cdots+(\xi_{i-1}-a_i\zeta_{i-1})\,a_{i-1}=0.$$

By (i), $\xi_1a_1-a_i\zeta_1\in\left(a_1^{t-1},a_2,\dots,a_{i-1}\right)$. Thus, $a_i\zeta_1\in(a_1,\dots,a_{i-1})$. But the sequence $a_1,\dots,a_q$ is regular so $\zeta_1\in(a_1,\dots,a_{i-1})$ and $b=\zeta_1a_1^{t-1}+\zeta_2a_2+\cdots+\zeta_{i-1}a_{i-1}\in\left(a_1^t,a_2,\dots,a_{i-1}\right)$ as required.

(iii) We shall use induction on q. The result is obvious for $q=1$. For $q>1$, we assume that $(a_i)_{1\leqslant i\leqslant q}$ is regular. By (ii), $a_1^t,a_2,\dots,a_q$ is regular, so $(\overline{a_i})_{1<i\leqslant q}$ is regular in the ring $R/\left(a_1^t\right)$ by Proposition 2.6. By the induction hypothesis, $\left(\overline{a_i^t}\right)_{1<i\leqslant q}$ is also regular in $R/\left(a_1^t\right)$ and therefore $(a_i^t)_{1\leqslant i\leqslant q}$ is a regular sequence.

(iv) Since any regular sequence of $\mathfrak{J}$ is also a regular sequence of $\sqrt{\mathfrak{J}}$, we have $\operatorname{depth}(\mathfrak{J}) \leqslant \operatorname{depth}\left(\sqrt{\mathfrak{J}}\right)$. Now, (iii) implies that for any regular sequence of $\sqrt{\mathfrak{J}}$ of length q, there exists a regular sequence of $\mathfrak{J}$ of length q, and therefore $\operatorname{depth}(\mathfrak{J}) = \operatorname{depth}\left(\sqrt{\mathfrak{J}}\right)$. ☐

Corollary 2.8. *Let $g \in k[x_1, \dots, x_{n+1}]$ be a homogeneous tame polynomial. Then the depth of the jacobian ideal $J_g = \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}}\right) \subseteq k[x_1, \dots, x_{n+1}]$ is at least $n+1$.*

Proof. By assumption $\sqrt{J_g} = (x_1, \dots, x_{n+1})$. Now, the sequence $(x_i)_{1 \leqslant i \leqslant n+1}$ is a regular sequence of $\sqrt{J_g}$ and therefore $\operatorname{depth}(J_g) = \operatorname{depth}\left(\sqrt{J_g}\right) \geqslant n+1$. ☐

We are now ready to prove de Rham's Lemma for homogeneous tame polynomials.

Lemma 2.9. *Let R be a noetherian ring that is also a k-algebra such that the R-module Ω_R^1 is free of rank $n+1$, with basis $(\varepsilon_1, \dots, \varepsilon_{n+1})$. Consider $g \in R$, denote by $\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}}$ the coordinates of $\mathrm{d}g$ in the basis ε and set $J_g = \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}}\right) \subseteq R$. Now, consider the following sequence:*

$$0 \longrightarrow \Omega_R^0 \xrightarrow{\mathrm{d}g \wedge \cdot} \Omega_R^1 \xrightarrow{\mathrm{d}g \wedge \cdot} \cdots \xrightarrow{\mathrm{d}g \wedge \cdot} \Omega_R^n \xrightarrow{\mathrm{d}g \wedge \cdot} \Omega_R^{n+1},$$

and write $H^m = \operatorname{Ker}\left(\mathrm{d}g \wedge \cdot : \Omega_R^m \to \Omega_R^{m+1}\right) / \operatorname{Im}\left(\mathrm{d}g \wedge \cdot : \Omega_R^{m-1} \to \Omega_R^m\right)$.

(i) *There exists an integer $\nu \in \mathbb{N}$ such that, for $0 \leqslant m \leqslant n$, we have $J_g^\nu H^m = 0$.*

(ii) *For $0 \leqslant m < \operatorname{depth}(J_g)$, we have $H^m = 0$. In particular, the above sequence is exact if $\operatorname{depth}(J_g) \geqslant n+1$.*

Proof (adapted from [Sai76]*).* (i) Since R is noetherian and Ω_R^1 is finitely generated over R, it suffices to show that for all $i \in \{1, \dots, n+1\}$, for all $m \in \{0, \dots, n\}$ and for all $\omega \in \operatorname{Ker}\left(\mathrm{d}g \wedge \cdot : \Omega_R^m \to \Omega_R^{m+1}\right)$, there exists $\nu \in \mathbb{N}$ such that $\delta^\nu \omega \in \operatorname{Im}\left(\mathrm{d}g \wedge \cdot : \Omega_R^{m-1} \to \Omega_R^m\right)$, with $\delta = \frac{\partial g}{\partial x_i}$.

If δ is nilpotent, there is nothing to prove. Otherwise, we consider the localised ring R_δ. Note that the ideal generated in R_δ by $\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_{n+1}}$ is R_δ itself. We may therefore complete the image of $\mathrm{d}g$ in $\Omega_R^1 \otimes_R R_\delta$ to form a free basis $(\mathrm{d}g, e_1, \dots, e_n)$ of $\Omega_R^1 \otimes_R R_\delta$. Now let $\omega \in \operatorname{Ker}\left(\mathrm{d}g \wedge \cdot : \Omega_R^m \to \Omega_R^{m+1}\right)$. In $\Omega_R^m \otimes_R R_\delta$, we may write

$$\omega = \sum_{1 \leqslant j_1 < \cdots < j_m \leqslant n} a_{j_1, \dots, j_m} e_{j_1} \wedge \cdots \wedge e_{j_m} + \sum_{1 \leqslant j_1 < \cdots < j_{m-1} \leqslant n} b_{j_1, \dots, j_{m-1}} \mathrm{d}g \wedge e_{j_1} \wedge \cdots \wedge e_{j_{m-1}}.$$

The fact that $\mathrm{d}g \wedge \omega = 0$ implies that $a_{j_1, \dots, j_m} = 0$ for all $1 \leqslant j_1 < \cdots < j_m \leqslant n$. Therefore, we can write $\omega = \mathrm{d}g \wedge \eta$ in $\Omega_R^m \otimes_R R_\delta$. This means that there exists $\nu \in \mathbb{N}$ s.t. $\delta^\nu (\omega - \mathrm{d}g \wedge \eta) = 0$. Therefore, $\delta^\nu \omega \in \operatorname{Im}\left(\mathrm{d}g \wedge \cdot : \Omega_R^{m-1} \to \Omega_R^m\right)$.

(ii) We use induction on m. For $m = 0$, the result means that $a = 0$ as soon as $a\,\mathrm{d}g = 0$, which is true because otherwise every element of J_g would be a divisor of zero, in contradiction with $\operatorname{depth}(J_g) > m = 0$. Assume the result has been proved for $m-1$. Let $a \in J_g$ be the first term of a regular sequence of J_g of maximal length (this is possible because $\operatorname{depth}(J_g) > m \geqslant 1$). By (i), there exists $\nu \in \mathbb{N}$ s.t. $a^\nu H^m = 0$. We may actually assume that $\nu = 1$ because a^ν is also the first term of a regular sequence of maximal length by Proposition 2.7. For $\omega \in \Omega_R^m$, we denote by $\overline{\omega}$ the image of ω in $\Omega_R^m \otimes_R R/(a)$. If $\mathrm{d}g \wedge \omega = 0$, since $aH^m = 0$, we can write

$$a\omega = \mathrm{d}g \wedge \eta,$$

for some $\eta \in \Omega_R^{m-1}$. Therefore $\overline{\mathrm{d}g} \wedge \overline{\eta} = \overline{a\omega} = 0$ in $\Omega_R^m \otimes_R R/(a)$. Since $\operatorname{depth}\left(\overline{J_g}\right) = \operatorname{depth}(J_g) - 1 > m-1$ by Proposition 2.6, we may apply the induction hypothesis and conclude that $\overline{\eta} = \overline{\mathrm{d}g} \wedge \overline{\vartheta}$ in $\Omega_R^{m-1} \otimes_R R/(a)$. In other words:

$$\eta = \mathrm{d}g \wedge \vartheta + a\zeta,$$

with $\vartheta \in \Omega_R^{m-2}$ and $\zeta \in \Omega_R^{m-1}$. As a consequence:

$$a\left(\omega - \mathrm{d}g \wedge \zeta\right) = \mathrm{d}g \wedge \eta - a\,\mathrm{d}g \wedge \zeta = \mathrm{d}g \wedge \mathrm{d}g \wedge \vartheta = 0.$$

As a is not a divisor of zero in R, it follows that $\omega \in \operatorname{Im}\left(\mathrm{d}g \wedge \cdot : \Omega_R^{m-1} \to \Omega_R^m\right)$. $\square$

Using Lemma 2.9 and Corollary 2.8, it is clear that de Rham's Lemma is true for homogeneous tame polynomials. We can now prove the general version.

Theorem 2.10 (De Rham's Lemma)**.** *Let $f \in k\left[x_1, \ldots, x_{n+1}\right]$ be a tame polynomial. Then the following sequence is exact:*

$$0 \longrightarrow \Omega_{k^{n+1}}^0 \xrightarrow{\mathrm{d}f\wedge\cdot} \Omega_{k^{n+1}}^1 \xrightarrow{\mathrm{d}f\wedge\cdot} \cdots \xrightarrow{\mathrm{d}f\wedge\cdot} \Omega_{k^{n+1}}^n \xrightarrow{\mathrm{d}f\wedge\cdot} \Omega_{k^{n+1}}^{n+1}.$$

Proof. Let $0 \leqslant m \leqslant n$. We shall prove by induction on $\deg\omega$, for $\omega \in \Omega_{k^{n+1}}^m$, that if $\mathrm{d}f \wedge \omega = 0$ then there exists $\omega_0 \in \Omega_{k^{n+1}}^{m-1}$ such that $\omega = \mathrm{d}f \wedge \omega_0$. If $\omega = 0$, the result is clear. Otherwise, let $\eta \in \Omega_{k^{n+1}}^m$ and $g \in k\left[x_1, \ldots, x_{n+1}\right]$ be the respective homogeneous parts of ω and f of highest degrees. The fact that $\mathrm{d}f \wedge \omega = 0$ implies that

$$\mathrm{d}g \wedge \eta = 0.$$

Using de Rham's Lemma for homogeneous tame polynomials, we obtain the existence of $\eta_0 \in \Omega_{k^{n+1}}^{m-1}$ such that $\eta = \mathrm{d}g \wedge \eta_0$. Therefore, we can write

$$\omega = \mathrm{d}f \wedge \omega_1 + \omega_2,$$

with $\omega_1 \in \Omega_{k^{n+1}}^{m-1}$, $\omega_2 \in \Omega_{k^{n+1}}^m$ and $\deg\omega_2 < \deg\omega$. By the induction hypothesis, since $\mathrm{d}f \wedge \omega_2 = 0$, there exists $\omega_0' \in \Omega_{k^{n+1}}^{m-1}$ such that $\omega_2 = \mathrm{d}f \wedge \omega_0'$, and therefore $\omega = \mathrm{d}f \wedge (\omega_1 + \omega_0')$. $\square$

2.4 De Rham cohomology of nonsingular tame varieties

With de Rham's Lemma in hand, we are ready to compute the algebraic de Rham cohomology of nonsingular tame varieties, i.e. of affine varieties defined as the vanishing locus of a nonsingular tame polynomial. We start with the following lemma. Somewhat unsatisfactorily, it will be necessary to reinforce our regularity hypotheses: in [Mov19], the lemma is proved for nonsingular tame polynomials in the case $m = n$, but we will need it for all $1 \leqslant m \leqslant n$, and therefore we introduce a new condition, saying that a polynomial $f \in k\left[x_1, \ldots, x_{n+1}\right]$ is **strongly nonsingular** if the endomorphism of the k-vector space $\Omega_{k^{n+1}}^{m+1} / \mathrm{d}f \wedge \Omega_{k^{n+1}}^m$ induced by multiplication by f is invertible for all $1 \leqslant m \leqslant n$. Note that strong nonsingularity is indeed stronger than nonsingularity because we have an isomorphism

$$\Omega_{k^{n+1}}^{n+1} / \mathrm{d}f \wedge \Omega_{k^{n+1}}^n \simeq \left(k\left[x_1, \ldots, x_{n+1}\right]/J_f\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n+1},$$

and therefore if f is strongly nonsingular, then the endomorphism of $k\left[x_1, \ldots, x_{n+1}\right]/J_f$ induced by multiplication by f is invertible, i.e. f is nonsingular.

Lemma 2.11. *Let $f \in k\left[x_1, \ldots, x_{n+1}\right]$ be a strongly nonsingular tame polynomial. Let $\omega_1 \in \Omega_{k^{n+1}}^{m+1}$ and $\omega_2 \in \Omega_{k^{n+1}}^m$, with $1 \leqslant m \leqslant n$, such that*

$$f\omega_1 = \mathrm{d}f \wedge \omega_2.$$

Then there exist $\omega_3 \in \Omega_{k^{n+1}}^m$ and $\omega_4 \in \Omega_{k^{n+1}}^{m-1}$ such that

$$\omega_1 = \mathrm{d}f \wedge \omega_3 \qquad \textit{and} \qquad \omega_2 = f\omega_3 - \mathrm{d}f \wedge \omega_4.$$

Proof (adapated from [Mov19]*).* We consider the canonical projection $\pi : \Omega_{k^{n+1}}^{m+1} \to \Omega_{k^{n+1}}^{m+1} / \mathrm{d}f \wedge \Omega_{k^{n+1}}^{m}$. We have

$$f\pi(\omega_1) = \pi(f\omega_1) = \pi(\mathrm{d}f \wedge \omega_2) = 0.$$

But multiplication by f is invertible as an endomorphism of $\Omega_{k^{n+1}}^{m+1} / \mathrm{d}f \wedge \Omega_{k^{n+1}}^{m}$ because f is strongly nonsingular; as a consequence $\omega_1 \in \operatorname{Ker}\pi$. Hence there exists $\omega_3 \in \Omega_{k^{n+1}}^{m}$ s.t. $\omega_1 = \mathrm{d}f \wedge \omega_3$. It follows that:

$$\mathrm{d}f \wedge (f\omega_3 - \omega_2) = f\omega_1 - \mathrm{d}f \wedge \omega_2 = 0.$$

By de Rham's Lemma (Theorem 2.10), there exists $\omega_4 \in \Omega_{k^{n+1}}^{m-1}$ s.t. $f\omega_3 - \omega_2 = \mathrm{d}f \wedge \omega_4$. This concludes the proof. $\square$

We can now prove the main theorem of this section. Note that, for $n = 2$ and $m = 1$, we obtain a more precise statement of Theorem 1.1.

Theorem 2.12. *Let $f \in k[x_1, \dots, x_{n+1}]$ be a strongly nonsingular tame polynomial. Then for all $1 \leqslant m \leqslant n-1$, we have*

$$H_{\mathrm{dR}}^{m}(V(f)) = 0.$$

Proof. We consider the canonical projection $\pi : k[x_1, \dots, x_{n+1}] \to k[x_1, \dots, x_{n+1}]/(f)$, which induces maps $\pi_* : \Omega_{k^{n+1}}^{m} \to \Omega_{V(f)}^{m}$. We want to show that $H_{\mathrm{dR}}^{m}(V(f)) = 0$, in other words

$$\operatorname{Ker}\left(\mathrm{d} : \Omega_{V(f)}^{m} \to \Omega_{V(f)}^{m+1}\right) \subseteq \operatorname{Im}\left(\mathrm{d} : \Omega_{V(f)}^{m-1} \to \Omega_{V(f)}^{m}\right).$$

To do this, we consider $\hat{\omega} \in \operatorname{Ker}\left(\mathrm{d} : \Omega_{V(f)}^{m} \to \Omega_{V(f)}^{m+1}\right)$ and we choose an element $\omega \in \pi_*^{-1}(\omega) \subseteq \Omega_{k^{n+1}}^{m}$. Our aim is to show that $\omega \in \operatorname{Im}\mathrm{d} + \operatorname{Ker}\pi_*$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathrm{d}} & \Omega_{k^{n+1}}^{m-1} & \xrightarrow{\mathrm{d}} & \Omega_{k^{n+1}}^{m} & \xrightarrow{\mathrm{d}} & \Omega_{k^{n+1}}^{m+1} & \xrightarrow{\mathrm{d}} & \cdots \\
 & & \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi_*} & & \Big\downarrow{\scriptstyle \pi_*} & & \\
\cdots & \xrightarrow{\mathrm{d}} & \Omega_{V(f)}^{m-1} & \xrightarrow{\mathrm{d}} & \Omega_{V(f)}^{m} & \xrightarrow{\mathrm{d}} & \Omega_{V(f)}^{m+1} & \xrightarrow{\mathrm{d}} & \cdots
\end{array}$$

Therefore $\mathrm{d}\omega \in \operatorname{Ker}\pi_*$, which means, by Proposition 2.5, that there exist $\omega_1 \in \Omega_{k^{n+1}}^{m+1}$ and $\omega_2 \in \Omega_{k^{n+1}}^{m}$ such that

$$\mathrm{d}\omega = f\omega_1 + \mathrm{d}f \wedge \omega_2 = f(\omega_1 - \mathrm{d}\omega_2) + \mathrm{d}(f\omega_2).$$

Setting $\omega' = \omega - f\omega_2 \in \omega + \operatorname{Ker}\pi_*$ and $\omega_1' = \omega_1 - \mathrm{d}\omega_2 \in \Omega_{k^{n+1}}^{m}$, we have

$$\mathrm{d}\omega' = f\omega_1'.$$

As a consequence

$$0 = \mathrm{d}^2\omega' = \mathrm{d}f \wedge \omega_1' + f\,\mathrm{d}\omega_1'.$$

By Lemma 2.11, there exist $\omega_3 \in \Omega_{k^{n+1}}^{m+1}$ and $\omega_4 \in \Omega_{k^{n+1}}^{m}$ such that

$$\mathrm{d}\omega_1' = -\,\mathrm{d}f \wedge \omega_3 \qquad \text{and} \qquad \omega_1' = f\omega_3 - \mathrm{d}f \wedge \omega_4. \tag{$*$}$$

Thus

$$\mathrm{d}\omega' = f\omega_1' = f^2\omega_3 - f\,\mathrm{d}f \wedge \omega_4 = f^2\left(\omega_3 + \frac{1}{2}\,\mathrm{d}\omega_4\right) - \mathrm{d}\left(\frac{1}{2}f^2\omega_4\right).$$

Setting $\omega'' = \omega' + \frac{1}{2}f^2\omega_4 \in \omega + \operatorname{Ker}\pi_*$ and $\omega_1'' = \omega_3 + \frac{1}{2}\,\mathrm{d}\omega_4 \in \Omega_{k^{n+1}}^{m}$, we obtain

$$\mathrm{d}\omega'' = f^2\omega_1''.$$

Moreover, the equalities $(*)$ tell us that $\deg \omega_3, \deg \omega_4 \leqslant \deg \omega_1' - \deg f \leqslant \deg \omega' - 2\deg f$. Thus, $\deg \omega'' \leqslant \deg \omega'$ and $\deg \omega_1'' < \deg \omega_1'$. Iterating this process, we show that for all $k \geqslant 1$, there exist $\omega^{(k)} \in \omega + \operatorname{Ker} \pi_*$ and $\omega_1^{(k)} \in \Omega^m_{k^{n+1}}$ such that

$$\mathrm{d}\omega^{(k)} = f^k \omega_1^{(k)},$$

and $\deg \omega^{(k)} \leqslant \deg \omega$. With k large enough, we have $\deg \omega^{(k)} < k \deg f$ therefore $\mathrm{d}\omega^{(k)} = 0$. By the Algebraic Poincaré Lemma (Theorem 2.2), $\omega^{(k)} \in \operatorname{Im} \mathrm{d}$ and therefore $\omega \in \operatorname{Im} \mathrm{d} + \operatorname{Ker} \pi_*$. □

3 Interlude: connectedness of real and complex affine varieties

Before going on with the study of the singular homology of complex affine varieties thanks to Picard-Lefschetz Theory, we shall discuss another question related to the topology of affine varieties, namely that of their connectedness properties. Our main aim will be to show that a complex affine variety has only finitely many connected components for the Euclidean topology. This is a result that bears interest in its own right, and from which we hope to draw some consequences which will be useful later.

In most of this section it will not matter whether one is working over the real or complex numbers, but there are some parts where we will need to use sign properties of real numbers; therefore we will state all the results in the real and complex cases, and we will declare it when we need real numbers, noting that a complex affine variety in $\mathbb{C}^n$ is also a real affine variety in $\mathbb{R}^{2n}$, and hence topological results which hold for real affine varieties also hold for complex ones. We will write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

3.1 Finiteness of the set of connected components

The foundation of all the finiteness results we are going to prove is Hilbert's Basis Theorem, according to which $R[x_1, \ldots, x_n]$ is a noetherian ring for any noetherian ring R (which is true in particular if R is a field). Translating this theorem into a more geometric language yields the following.

Proposition 3.1 (Descending Chain Condition)**.** *Let $(V_n)_{n\in\mathbb{N}}$ be a descending sequence of affine varieties in $\mathbb{K}^n$:*

$$V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n \supseteq \cdots.$$

Then the sequence $(V_n)_{n\in\mathbb{N}}$ is eventually constant: there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $V_n = V_{n_0}$.

Proof. For $n \in \mathbb{N}$, write $V_n = V(I_n)$, where $I_n = \mathcal{I}(V_n) \subseteq \mathbb{K}[x_1, \ldots, x_n]$. The fact that $(V(I_n))_{n\in\mathbb{N}}$ is a descending sequence of subsets of $\mathbb{K}^n$ means that $(I_n)_{n\in\mathbb{N}}$ is an ascending sequence of ideals of $\mathbb{K}[x_1, \ldots, x_n]$. But the latter is a noetherian ring and therefore the sequence $(I_n)_{n\in\mathbb{N}}$ is eventually constant, and so is the sequence $(V_n)_{n\in\mathbb{N}}$. □

Using the above principle, the first step will be to show that a zero-dimensional affine variety is finite. To do this, we start by showing how to remove one point from such an affine variety, and then we will apply the Descending Chain Condition.

Lemma 3.2. *Let $V \subseteq \mathbb{K}^n$ be an affine variety and $x_0 \in V$. Assume that there exist $f_1, \ldots, f_n \in \mathcal{I}(V)$ such that the matrix $\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{1\leqslant i,j\leqslant n}$ is invertible. Then $V \setminus \{x_0\}$ is an affine variety.*

Proof (adapted from [Mil68]*).* We may assume that $x_0 = 0$. In this case, since $f_1, \ldots, f_n$ vanish at 0, we may write, for all $1 \leqslant i \leqslant n$ and for all $x \in \mathbb{K}^n$,

$$f_i(x_1, \ldots, x_n) = g_{i1}(x_1, \ldots, x_n)\, x_1 + \cdots + g_{in}(x_1, \ldots, x_n)\, x_n,$$

with $g_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$. Now set $W = \left\{x \in \mathbb{K}^n,\ \det\left(g_{ij}(x)\right)_{1 \leqslant i,j \leqslant n} = 0\right\}$. Note that $0 \notin W$ because $g_{ij}(0) = \frac{\partial f_i}{\partial x_j}(0)$ for all i, j. On the other hand, if $x \in V \backslash \{0\}$, we have $f_i(x) = 0$ for all i, which can be rewritten as

$$\begin{pmatrix} g_{11}(x) & \cdots & g_{1n}(x) \\ \vdots & \ddots & \vdots \\ g_{n1}(x) & \cdots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Since $x \neq 0$, this implies that $\det\left(g_{ij}(x)\right)_{1 \leqslant i,j \leqslant n} = 0$, and therefore $x \in W$. Hence, we see that $V \backslash \{0\} = V \cap W$ is an affine variety. ☐

Proposition 3.3. *Let $V \subseteq \mathbb{K}^n$ be an affine variety whose connected components are points. Then V is a finite set.*

Proof (adapted from [Mil68]*).* Let $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$ such that $V = V(f_1, \ldots, f_k)$. We claim that V contains a point x such that the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n}}$ has rank n. Indeed, if this matrix had rank $r < n$ for all x, then the set of points of V at which the matrix has maximal rank $r_{\max}$ would be a submanifold of $\mathbb{R}^n$ of dimension $n - r_{\max} \geqslant 1$, which contradicts the fact that the connected components of V are points. Therefore, there exists $x_0 \in V$ such that $\mathrm{rk}\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant n}} = n$. Applying Lemma 3.2, we see that $V \backslash \{x_0\}$ is an affine variety.

So far, we have shown that we can remove a point from every nonempty affine variety whose connected components are points and still get an affine variety. Therefore, by iterating this process, we obtain a sequence

$$V \supseteq V_1 \supseteq V_2 \supseteq \cdots,$$

and this sequence is eventually constant by the Descending Chain Condition. This implies that $V_k = \varnothing$ for some k, and therefore V is finite. ☐

Given an affine variety $V \subseteq \mathbb{K}^n$, we write $\Sigma(V)$ to be the set of **singular points** of V. To define it formally, define the **rank** $\mathrm{rk}_x(V)$ of V at a point $x \in V$ by

$$\mathrm{rk}_x(V) = \mathrm{rk}\left(\frac{\partial f}{\partial x_i}(x)\right)_{\substack{f \in \mathcal{I}(V) \\ 1 \leqslant i \leqslant n}}.$$

Then $\Sigma(V)$ is the set of points x of V for which $\mathrm{rk}_x(V)$ is not maximal. Note that $\Sigma(V)$ is an affine variety of $\mathbb{K}^n$ which can be defined using the minors of some jacobian matrices. Moreover, $V \backslash \Sigma(V)$ is a submanifold of $\mathbb{K}^n$ of codimension $\max_{x \in V} \mathrm{rk}_x(V)$.

Now that we have treated the zero-dimensional case, we can prove that an affine variety has only finitely many connected components.

Theorem 3.4. *Let $V \subseteq \mathbb{K}^n$ be an affine variety. Then V has only finitely many connected components for the Euclidean topology of $\mathbb{K}^n$.*

Proof (adapted from [Whi57]*).* We shall work in the real setting, i.e. $\mathbb{K} = \mathbb{R}$, and the theorem will follow in the complex case. Were the theorem false, we could choose a real affine variety V with an infinite number of connected components and such that any proper subvariety of V has a finite number of connected components (otherwise we could produce an infinite strictly decreasing sequence of affine varieties, in contradiction with the Descending Chain Condition). Now V can be written as $V = \Sigma(V) \cup M$, where $M = V \backslash \Sigma(V)$. Since $\Sigma(V)$ is a proper subvariety of V, it has only finitely many connected components, and thus M has infinitely many connected components, and M is in addition a submanifold of $\mathbb{R}^n$ of dimension d. We may moreover assume that $d \geqslant 1$ because the case $d = 0$ is a consequence of Proposition 3.3.

Let Γ be the set of connected components of M and let $N_0 \in \Gamma$. Choose a point $a \in \mathbb{R}^n$ not equidistant from all points of N_0 and consider the following distance function, which is polynomial because we are working over $\mathbb{R}$:

$$\rho_a\left(x_1, \ldots, x_n\right) = \|x-a\|^2 \in \mathbb{R}\left[x_1, \ldots, x_n\right].$$

Given polynomials $f_1, \ldots, f_{n-d} \in \mathbb{R}\left[x_1, \ldots, x_n\right]$ and indices $\lambda_1, \ldots, \lambda_{n-d+1} \in \{1, \ldots, n\}$, consider the jacobian polynomial

$$\Phi_\lambda\left(f_1, \ldots, f_k\right) = \begin{vmatrix} \frac{\partial f_1}{\partial x_{\lambda_1}} & \cdots & \frac{\partial f_{n-d}}{\partial x_{\lambda_1}} & \frac{\partial \rho_a}{\partial x_{\lambda_1}} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_1}{\partial x_{\lambda_{n-d+1}}} & \cdots & \frac{\partial f_{n-d}}{\partial x_{\lambda_{n-d+1}}} & \frac{\partial \rho_a}{\partial x_{\lambda_{n-d+1}}} \end{vmatrix} \in \mathbb{R}\left[x_1, \ldots, x_n\right].$$

Now let V' be the affine variety defined by the ideal generated by $I = \mathcal{I}(V)$ and the set of all polynomials $\Phi_\lambda\left(f_1, \ldots, f_{n-d}\right)$, with $\lambda \in \{1, \ldots, n\}^{n-d+1}$ and $f_1, \ldots, f_{n-d} \in I$. In other words, V' is the set of singular points of the differentiable map $\rho_{a|M} : M \to \mathbb{R}$. Since ρ_a is not constant on the connected manifold N_0, the function ρ_a has regular points on N_0 and so $V' \subsetneq V$. Therefore, V' has only finitely many connected components. But on the other hand, for each connected component $N \in \Gamma$ of M, there exists at least one point y_N of N minimising the distance to a (because N is closed and nonempty); in particular y_N is a singular point of $\rho_{a|M}$ and therefore $y_N \in V'$. This shows that V' is a proper subvariety of V intersecting each connected component of V; therefore V' has infinitely many connected components, a contradiction. □

3.2 Decomposition into submanifolds

The results we are going to prove come from the need to show finiteness results for the set of singularities of real or complex polynomials. This leads us to prove that any real or complex affine variety can be written as the union of finitely many connected submanifolds of $\mathbb{K}^n$. We start by extending Theorem 3.4 to the set of regular points of an affine variety.

Lemma 3.5. *Let $V \subseteq \mathbb{K}^n$ be an affine variety. Then the set $V \backslash \Sigma(V)$ of regular points of V has only finitely many connected components for the Euclidean topology of $\mathbb{K}^n$.*

Proof (adapted from [Mil68]*).* We shall work in the real setting, i.e. $\mathbb{K} = \mathbb{R}$, and the theorem will follow in the complex case. Let $f_1, \ldots, f_k \in \mathbb{R}\left[x_1, \ldots, x_n\right]$ such that $V = V\left(f_1, \ldots, f_k\right)$ and let $g_1, \ldots, g_\ell \in \mathbb{R}\left[x_1, \ldots, x_n\right]$ such that $\Sigma(V) = V\left(g_1, \ldots, g_\ell\right)$. Setting $p = g_1^2 + \cdots + g_\ell^2 \in \mathbb{R}\left[x_1, \ldots, x_n\right]$, we have $\Sigma(V) = V(p)$ (because we are working over $\mathbb{R}$). Now consider the polynomial

$$q\left(x_1, \ldots, x_{n+1}\right) = p\left(x_1, \ldots, x_n\right) x_{n+1} - 1 \in \mathbb{R}\left[x_1, \ldots, x_{n+1}\right],$$

and define

$$W = V\left(f_1, \ldots, f_k, q\right) \subseteq \mathbb{R}^{n+1}.$$

The set W is an affine variety of $\mathbb{R}^{n+1}$, and it is diffeomorphic to $V \backslash \Sigma(V)$ *via* the map $\left(x_1, \ldots, x_n\right) \mapsto \left(x_1, \ldots, x_n, \frac{1}{p\left(x_1, \ldots, x_n\right)}\right)$. But as an affine variety, W has only finitely many connected components for the Euclidean topology (by Theorem 3.4), and so does $V \backslash \Sigma(V)$. □

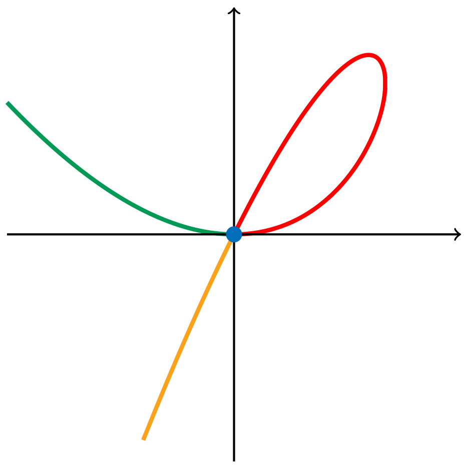
We can now decompose an affine variety into submanifolds of $\mathbb{K}^n$.

Theorem 3.6. *Let $V \subseteq \mathbb{K}^n$ be an affine variety. Then V can be expressed as a finite disjoint union*

$$V = \bigsqcup_{i=1}^{p} M_i,$$

where each M_i is a connected smooth submanifold of $\mathbb{K}^n$.

Proof (adapted from [Mil68]*).* We set $N_1 = V \backslash \Sigma(V)$, $N_2 = \Sigma(V) \backslash \Sigma(\Sigma(V))$, etc. By the Descending Chain Condition (Proposition 3.1), the sequence $V \supseteq \Sigma(V) \supseteq \Sigma(\Sigma(V)) \supseteq \cdots$ is eventually constant, and therefore there exists $q \in \mathbb{N}$ such that $N_i = \varnothing$ for $i > q$. Therefore, $V = \bigsqcup_{i=1}^p N_i$, and each N_i is a smooth submanifold of $\mathbb{K}^n$, with finitely many connected components by Lemma 3.5. Replacing each N_i by the union of its connected components, we obtain the result. □



$x^3 - 2xy + y^2 = 0$

Figure 3: Decomposition of a real affine curve into four connected submanifolds of $\mathbb{R}^2$.

3.3 Applications to finiteness results for singularities

Using what has been done above, we obtain the following theorem, which will not suffice for applications, but which has the advantage of being fully general.

Theorem 3.7. *Every polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ has only finitely many singular values.*

Proof. Let $V = V(f)$ and $\Sigma = \Sigma(V)$. By Theorem 3.6, we can write $\Sigma = \bigsqcup_{i=1}^p M_i$, where M_i is a connected submanifold of $\mathbb{K}^n$. For all $i \in \{1, \dots, p\}$, the map $f_{|M_i} : M_i \to \mathbb{K}$ is smooth and satisfies $\mathrm{d}f_{|M_i} = 0$; as M_i is connected, we conclude that $f_{|M_i}$ is constant, say $f_{|M_i} = \gamma_i$. Therefore, the set of singular values of f is given by

$$f(\Sigma) = \bigcup_{i=1}^p f(M_i) = \{\gamma_1, \dots, \gamma_p\}.$$ □

For later applications, we will actually need to show that the set of *singular points* is finite, which is stronger than the above theorem. This is not true in general, but we can prove it for tame polynomials, which were defined in Section 2.2. We now stop working over the real numbers and we go back to an algebraically closed field k.

Proposition 3.8. *Let $f \in k[x_1, \dots, x_n]$ be a tame polynomial. Then the k-vector space*

$$M_f = k[x_1, \dots, x_n] / J_f$$

is finite-dimensional, where $J_f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ is the jacobian ideal of f.

Proof. Consider weights $\nu_1, \dots, \nu_n$ such that, if $f = f_0 + \cdots + f_d$ with $\deg f_k = k$, then the leading polynomial f_d is homogeneous tame. Therefore, by definition, $M_{f_d} = k[x_1, \dots, x_n] / J_{f_d}$

is finite-dimensional and we may therefore equip M_{f_d} with a k-basis $(b_1, \dots, b_r)$ composed of homogeneous polynomials only.

We claim that $(b_1, \dots, b_r)$ generates $M_f = k[x_1, \dots, x_n]/J_f$ as a k-vector space. To prove it, let $h \in k[x_1, \dots, x_n]$. We want to prove that $h \in kb_1 + \dots + kb_r + J_f$. Since $(b_1, \dots, b_r)$ generates the k-vector space M_{f_d}, there exist $\lambda_1, \dots, \lambda_r \in k$ and $u_1, \dots, u_n \in k[x_1, \dots, x_n]$ such that

$$h = \sum_{j=1}^{r} \lambda_j b_j + \underbrace{\sum_{i=1}^{n} u_i \frac{\partial f_d}{\partial x_i}}_{h'}. \tag{$*$}$$

If $\deg h' > \deg h$, then the homogeneous equation of highest degree induced by $(*)$ does not contain any term of h, and substracting it from $(*)$ allows us to decrease the degree of h'. We may therefore assume that $\deg h' \leqslant \deg h$. Thus

$$h = \sum_{j=1}^{r} \lambda_j b_j + \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i} + \underbrace{\sum_{i=1}^{n} \sum_{k=0}^{d-1} (-u_i) \frac{\partial f_k}{\partial x_i}}_{h_1},$$

and $\deg h_1 < \deg h' \leqslant \deg h$. We have shown that for every $h \in k[x_1, \dots, x_n]$, there exists a polynomial

$$h_1 \in h + kb_1 + \dots + kb_r + J_f,$$

with $\deg h_1 < \deg h$. Applying the same process to h_1 and iterating, we obtain a sequence of polynomials $h_1, h_2, \dots$ with $\deg h > \deg h_1 > \deg h_2 > \cdots$ such that $h_\ell \in h + kb_1 + \dots + kb_r + J_f$ for all ℓ. But by finiteness of the degree, there exists $\ell \geqslant 1$ such that $h_\ell = 0$, so $h \in kb_1 + \dots + kb_r + J_f$. Therefore, $(b_1, \dots, b_r)$ generates the k-vector space M_f. $\square$

We then use the following lemma to translate the above algebraic fact into geometric language.

Lemma 3.9. *Let I be an ideal of $k[x_1, \dots, x_n]$ such that $k[x_1, \dots, x_n]/I$ is a finite-dimensional k-vector space. Then the vanishing locus $V(I) \subseteq k^n$ is finite.*

Proof. We shall write $R = k[x_1, \dots, x_n]$.

We claim that every prime ideal containing I is maximal. Indeed, if $\mathfrak{p} \supseteq I$ is a prime ideal, then we have a surjective map $R/I \twoheadrightarrow R/\mathfrak{p}$. Since R/I is finite-dimensional over k, so is $R/\mathfrak{p}$. Therefore, $R/\mathfrak{p}$ is a finite-dimensional k-algebra that is also an integral domain; as a consequence, $R/\mathfrak{p}$ is a finite field extension of k, so $\mathfrak{p}$ is maximal.

Using the Nullstellensatz, the maximal ideals of R are the ideals $(x_1 - a_1, \dots, x_n - a_n)$ for $a = (a_1, \dots, a_n) \in k^n$, and such a maximal ideal contains I if and only if $a \in V(I)$. Therefore

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \supseteq I}} \mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \mathfrak{m} \supseteq I}} \mathfrak{m} = \bigcap_{a \in V(I)} (x_1 - a_1, \dots, x_n - a_n).$$

We now see that the k-linear map defined by $\varphi : \overline{P} \in R/\sqrt{I} \longmapsto (P(a))_{a \in V(I)} \in k^{V(I)}$ is a well-defined isomorphism. Therefore we have a surjection

$$R/I \twoheadrightarrow R/\sqrt{I} \simeq k^{V(I)}.$$

Since R/I is a finite-dimensional k-vector space, so is $k^{V(I)}$ and hence $V(I)$ is finite. $\square$

Proposition 3.10. *Every tame polynomial $f \in k[x_1, \dots, x_n]$ has only finitely many singular points.*

Proof. The set of singular points of f is $V(J_f)$. But by Proposition 3.8, the k-vector space $k[x_1, \dots, x_n]/J_f$ is finite-dimensional, so by Lemma 3.9, the set $V(J_f)$ is finite. $\square$

4 Singular homology of complex affine varieties

Our goal is now to study the topology of complex affine varieties using entirely different techniques: we shall compute the homology groups of nonsingular varieties using ideas which date back from the works of Picard and Lefschetz. The main idea is that, if we want to study the topology of the regular fibres of a holomorphic function, we should look at what happens near singularities, and try to assemble the information we get from different singularities.

4.1 Homology of fibres near isolated nondegenerate singularities

By the Morse Lemma and the classification of complex quadratic forms, we know that, near a nondegenerate singularity, any holomorphic function can be written up to biholomorphism as $f(z_1, \ldots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2$. Therefore, we will start with the study of this example, which will turn out to be foundational. We recall the notation $L_S = f^{-1}(S)$ if $f : X \to Y$ is a map and $S \subseteq Y$.

Proposition 4.1. *Consider the following map:*

$$f : (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \longmapsto z_1^2 + \cdots + z_{n+1}^2 \in \mathbb{C}.$$

Let $\varepsilon > 0$ *and* $B_\varepsilon = \left\{z \in \mathbb{C}^{n+1},\ \|z\| < \varepsilon\right\}$*. Then for all* $0 < \rho < \varepsilon^2$*, if* $D = \{z \in \mathbb{C},\ |z| = \rho\}$*, we have*

$$H_m(L_\rho) \simeq H_{m+1}(L_D, L_\rho) \simeq \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \mathbb{Z} & \text{if } m = n \end{cases},$$

where the fibres are considered with respect to $f_{|B_\varepsilon}$*.*

Proof (adapted from [Lam81]*).* Note that we have the long exact homology sequence

$$\cdots \longrightarrow H_{m+1}(L_D) \longrightarrow H_{m+1}(L_D, L_\rho) \longrightarrow H_m(L_\rho) \longrightarrow H_m(L_D) \longrightarrow \cdots,$$

which yields the isomorphism $H_m(L_\rho) \simeq H_{m+1}(L_D, L_\rho)$ for $m \geqslant 1$ after using the fact that L_D is contractible. Now, we have

$$L_\rho = \left\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1},\ \sum_{i=1}^{n+1} |z_i|^2 < \varepsilon^2,\ \sum_{i=1}^{n+1} z_i^2 = \rho\right\}.$$

We shall show that L_ρ is diffeomorphic to the disk bundle Q_n of $\mathbb{S}^n$, defined by

$$Q_n = \left\{(x, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1},\ \|x\| = 1,\ \|u\| < 1,\ \langle x, u\rangle = 0\right\},$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the usual norm and scalar product in $\mathbb{R}^{n+1}$. To construct the desired diffeomorphism, set $\Re(z) = (\Re(z_1), \ldots, \Re(z_{n+1}))$ for all $z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$ and likewise for $\Im(z)$. Thus

$$L_\rho = \left\{z \in \mathbb{C}^{n+1},\ \|\Re(z)\|^2 + \|\Im(z)\|^2 < \varepsilon^2,\ \|\Re(z)\|^2 - \|\Im(z)\|^2 = \rho,\ \langle \Re(z), \Im(z)\rangle = 0\right\}.$$

Now, define $\sigma = \sqrt{\frac{1}{2}(\varepsilon^2 - \rho)}$ and set

$$\varphi : z \in L_\rho \longmapsto \left(\frac{\Re(z)}{\|\Re(z)\|}, \frac{\Im(z)}{\sigma}\right) \in Q_n \quad \text{and} \quad \psi : (x, u) \in Q_n \longmapsto \sqrt{\sigma^2 \|u\|^2 + \rho} \cdot x + i\sigma u \in L_\rho.$$

One easily verifies that φ and ψ are inverse diffeomorphisms, which proves that $L_\rho \simeq Q_n$. Now the map $(x, u) \in Q_n \mapsto x \in \mathbb{S}^n$ defines a retraction from Q_n to $\mathbb{S}^n$. Therefore, we obtain

$$H_m(L_\rho) \simeq H_m(Q_n) \simeq H_m(\mathbb{S}^n) \simeq \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \mathbb{Z} & \text{if } m = n \end{cases}. \qquad \square$$

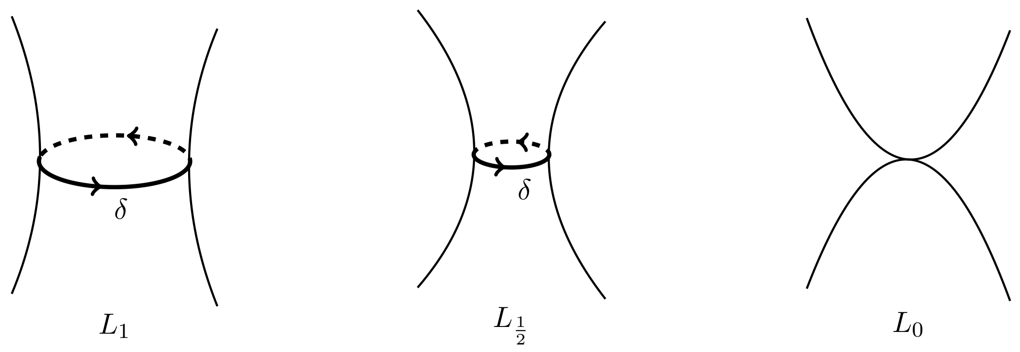


Figure 4: Vanishing of the cycle δ on the fibre L_ρ as $\rho \to 0$ (in complex dimension 1).

Let us try to understand geometrically what happens in complex dimension 1 (i.e. $n = 1$). In this case, the nonsingular fibres are Riemann surfaces. The above proof shows that, for $\rho \neq 0$, the fibre L_ρ is diffeomorphic to the disk bundle Q_1 of $\mathbb{S}^1$, which is actually a cylinder, and the generating cycle δ of $H_1(L_\rho)$ corresponds to a cycle around the cylinder. For $\rho = 0$, L_0 is a copy of two complex lines with identified origins. As ρ goes to 0, the picture is as in Figure 4: the cylinder is progressively pinched and the nontrivial cycle δ vanishes at $\rho = 0$. For this reason, δ is called a **vanishing cycle**.

As mentioned above, the study of the example $z \mapsto z_1^2 + \cdots + z_{n+1}^2$ allows us to understand what happens for any holomorphic function near a nondegenerate singularity, as stated by the following corollary.

Corollary 4.2. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$, with a nondegenerate singularity at 0. If $B \subseteq \mathbb{C}^{n+1}$ is a small enough (open) ball around 0 and $D \subseteq \mathbb{C}$ is a small enough (closed) disk around 0 with $\rho \in \partial D$, we have*

$$H_m(L_\rho) \simeq H_{m+1}(L_D, L_\rho) \simeq \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \mathbb{Z} & \text{if } m = n \end{cases},$$

where the fibres are considered with respect to $f_{|B}$.

Proof. By the Morse Lemma, there exist an open ball $B \subseteq \mathbb{C}^{n+1}$ around 0 and a diffeomorphism $\psi : V \to B$ from an open subset of $\mathbb{C}^{n+1}$ such that, for $z = (z_1, \ldots, z_{n+1}) \in V$,

$$f \circ \psi(z) = z_1^2 + \cdots + z_{n+1}^2.$$

Therefore, we have an isomorphism $H_m(L_\rho) \simeq H_m\left((f \circ \psi)^{-1}(\rho)\right)$ and in the same manner $H_{m+1}(L_D, L_\rho) \simeq H_{m+1}\left((f \circ \psi)^{-1}(D), (f \circ \psi)^{-1}(\rho)\right)$. The result follows from Proposition 4.1. □

4.2 Lifting retractions

We now need a few technical results that will help us lift retractions of subsets of $\mathbb{C}$ to retractions of the fibres over these subsets. The following theorem has a great importance in the study of nonsingular fibres; in particular, it implies that all nonsingular fibres of a proper smooth function are diffeomorphic.

Theorem 4.3 (Ehresmann's Fibration Theorem)**.** *Let $\phi : E \to B$ be a smooth map between two manifolds. Assume that ϕ is a submersion and that ϕ is proper. Then $\phi : E \to B$ is a smooth fibre bundle.*

Proof. See [Ehr52]. □

We will also use the following theorem, which will allow us to lift homotopies in fibre bundles.

Theorem 4.4 (Covering Homotopy Theorem)**.** *Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be two fibre bundles with the same fibre and group. We assume that the space B_1 is normal, locally compact and such that any open covering of B_1 is reducible to a countable covering. Consider a bundle map $(E_1, B_1) \to (E_2, B_2)$, i.e. a pair of maps $h_0 : E_1 \to E_2$, $\overline{h}_0 : B_1 \to B_2$ such that the following diagram commutes:*

$$\begin{tikzcd} E_1 \arrow[r, "h_0"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\overline{h}_0"] & B_2 \end{tikzcd}$$

If $\overline{H} : [0,1] \times B_1 \to B_2$ is a homotopy with $\overline{H}(0, \cdot) = \overline{h}_0$, then there exists a homotopy $H : [0,1] \times E_1 \to E_2$ with $H(0, \cdot) = h_0$ whose induced homotopy is $\overline{H}$ and such that H is stationary with $\overline{H}$: for each $x_1 \in E_1$ and for each interval $[t_1, t_2] \subseteq [0,1]$ such that $\overline{H}(p(x_1), t)$ is constant for $t \in [t_1, t_2]$, then $H(x_1, t)$ is constant for $t \in [t_1, t_2]$.

Proof. See [Ste51]. □

We define the notion of retraction. If $A \subseteq R \subseteq S$ are topological spaces, a **retraction** from S to R over A is a continuous map $r : [0,1] \times S \to S$ such that:

(i) $r(0, \cdot) = \mathrm{id}_S$,

(ii) $r(1, x) \in R$ for all $x \in S$ and $r(1, x) = x$ for all $x \in R$,

(iii) $r(\cdot, x) = x$ for all $x \in A$.

A retraction from S to R is a retraction from S to R over R.

We are now ready to lift retractions using Ehresmann's Fibration Theorem and the Homotopy Covering Theorem.

Proposition 4.5. *Let $f : Y \to B$ be a proper smooth map between manifolds. Let C be the set of singular values of f in B. Consider $A \subseteq R \subseteq S \subseteq B$ such that $S \cap C$ is included in the interior of A in S. Then every retraction from S to R over A can be lifted to a retraction from L_S to L_R over L_A.*

Proof (adapted from [Mov19]*).* By Ehresmann's Fibration Theorem, $f : L_{S \backslash C} \to S \backslash C$ is a smooth fibre bundle. Now, consider a retraction $\overline{r} : [0,1] \times S\backslash C \to S\backslash C$ from $S\backslash C$ to $R\backslash C$ over $A\backslash C$. As $\overline{r}(0, \cdot) = \mathrm{id}_S$, we may apply the Covering Homotopy Theorem with $E_1 = E_2 = L_{S\backslash C}$, $B_1 = B_2 = S\backslash C$ and $\overline{H} = \overline{r}$, to obtain a homotopy $r : [0,1] \times L_{S\backslash C} \to L_{S\backslash C}$. This homotopy will then be a retraction from $L_{S\backslash C}$ to $L_{R\backslash C}$ over $L_{A\backslash C}$. Since $S \cap C$ is included in the interior of A in B, we can extend r to a retraction from L_S to L_R over L_A by setting $r(\cdot, a) = a$ for all $a \in L_A$. □

4.3 Assembling singularities

In Section 4.1, we computed the homology of the regular fibres of a holomorphic function with only one nondegenerate singular point. We can now understand what happens when we have several nondegenerate singular points, thanks to the following lemma, which will also come useful later, after we have treated degeneracy.

Lemma 4.6. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with isolated singular values. We denote by C the set of singular values of f. Let $\beta \in \mathbb{C}\backslash C$. Then for every singular value $c \in C$, we can choose a small (closed) disk D_c around c and an element $b_c \in \partial D_c$ such that, for all $m \geqslant 1$,*

$$H_m\left(L_\beta\right) \simeq H_{m+1}\left(\mathbb{C}^{n+1}, L_\beta\right) \simeq \bigoplus_{c\in C} H_{m+1}\left(L_{D_c}, L_{b_c}\right).$$

Proof (adapted from [Mov19]*).* For $c \in C$, consider a small (closed) disk D_c around c and a path λ_c from β to c intersecting ∂D_c at b_c and set $K_c = D_c \cup \lambda_c$ (see Figure 5). Set

$$K = \bigcup_{c\in C} K_c.$$

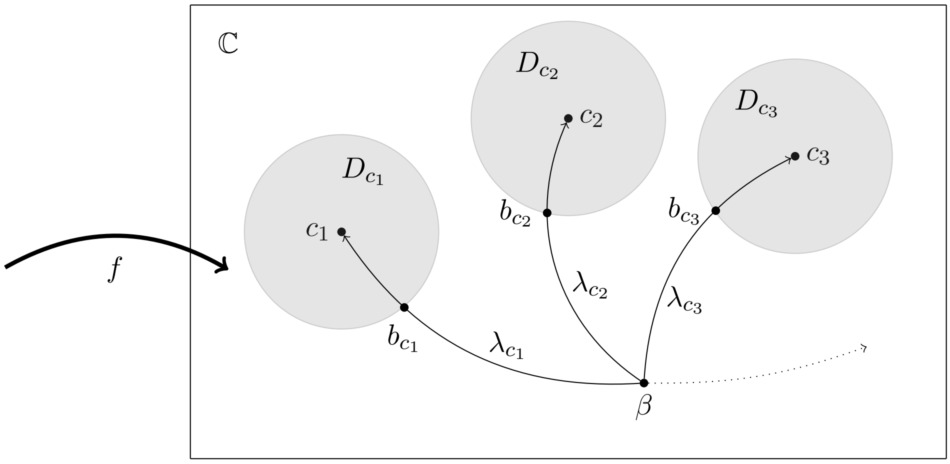


Figure 5: Decomposing the homology of $(\mathbb{C}^{n+1}, L_\beta)$.

There is a retraction from $\mathbb{C}$ to K so Proposition 4.5 implies that

$$H_m\left(\mathbb{C}^{n+1}, L_\beta\right) = H_m\left(L_{\mathbb{C}}, L_\beta\right) \simeq H_m\left(L_K, L_\beta\right).$$

If λ'_c is the path consisting of λ_c started at β but stopped at b_c, we have a retraction from $\lambda' = \bigcup_{c\in C} \lambda'_c$ to $\{\beta\}$ (and this retraction extends to a retraction from K to itself). Applying Proposition 4.5 again, we obtain

$$H_m\left(\mathbb{C}^{n+1}, L_\beta\right) \simeq H_m\left(L_K, L_\beta\right) \simeq H_m\left(L_K, L_{\lambda'}\right).$$

We then use the Excision Property to remove all the paths λ'_c stopped a bit before b_c, and after another retraction to b_c, we obtain

$$H_m\left(\mathbb{C}^{n+1}, L_\beta\right) \simeq H_m\left(L_K, L_{\lambda'}\right) \simeq H_m\left(\bigsqcup_{c\in C} L_{D_c}, \bigsqcup_{c\in C} L_{b_c}\right) \simeq \bigoplus_{c\in C} H_m\left(L_{D_c}, L_{b_c}\right).$$

Using the fact that $\mathbb{C}^{n+1}$ is contractible and writing the long exact homology sequence of the pair $(\mathbb{C}^{n+1}, L_\beta)$, we have

$$H_m\left(L_\beta\right) \simeq H_{m+1}\left(\mathbb{C}^{n+1}, L_\beta\right) \simeq \bigoplus_{c\in C} H_{m+1}\left(L_{D_c}, L_{b_c}\right). \qquad \square$$

The above lemma, together with Corollary 4.2, gives us full understanding of the homology of regular fibres of holomorphic **Morse functions**, i.e. holomorphic functions $f : \mathbb{C}^{n+1} \to \mathbb{C}$ satisfying the following conditions:

(i) Every singular point of f is nondegenerate,

(ii) The restriction of f to the set of singular points is injective.

Corollary 4.7. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a Morse function with isolated singularities. If C is the set of singular values of f and $\beta \in \mathbb{C} \backslash C$, then*

$$H_m\left(L_\beta\right) \simeq H_{m+1}\left(\mathbb{C}^{n+1}, L_\beta\right) \simeq \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \bigoplus_{c \in C} \mathbb{Z} & \text{if } m = n \end{cases}.$$

Proof. Applying Lemma 4.6 gives $H_m\left(L_\beta\right) = \bigoplus_{c \in C} H_{m+1}\left(L_{D_c}, L_{b_c}\right)$. Now, for $c \in C$, we may apply Corollary 4.2 after having chosen a small ball around the only singular point of f in the fibre $f^{-1}(c)$. The result follows. □

4.4 Treatment of degenerate singularities and proof of the main theorem

To motivate the ideas of this section, we recall the following theorem from one-variable complex analysis.

Theorem 4.8. *Let $f : U \to \mathbb{C}$ be a nonconstant holomorphic function defined on an open neighbourhood U of 0 in $\mathbb{C}$ and such that $f(0) = 0$. Consider*

$$k = \min\left\{\ell \geqslant 1,\ f^{(\ell)}(0) \neq 0\right\}.$$

Then f is k-to-one around 0: there exists an open neighbourhood $V \subseteq U$ of 0 such that $f(V)$ is an open neighbourhood of 0 and for all $w \in f(V) \backslash \{0\}$, the set $f^{-1}(\{w\}) \cap V$ has cardinal k.

The above theorem shows that, for a holomorphic function f in one variable with an isolated singularity at the origin, moving away from the singularity turns the fibres into discrete sets of points which are regular with respect to f. For multivariate functions, the principle will be similar: whenever we have an isolated degenerate singularity, we can move away slightly from this singularity and the fibres will split into several nondegenerate singularities.

Lemma 4.9. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$, with a possibly degenerate singularity at 0. If $B \subseteq \mathbb{C}^{n+1}$ is a small enough (open) ball around 0 and $D \subseteq \mathbb{C}$ is a small enough (closed) disk around 0 with $\rho \in \partial D$, we have*

$$H_m\left(L_\rho\right) \simeq H_{m+1}\left(L_D, L_\rho\right) \simeq \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \bigoplus_{\lambda \in \Lambda} \mathbb{Z} & \text{if } m = n \end{cases},$$

for some set Λ, where the fibres are considered with respect to $f_{|B}$.

Proof (adapted from [AGZV88]*).* The idea is to perturbate f with a small linear form to obtain a Morse function and apply Corollary 4.7.

Step 1: We claim that there exist vectors $u \in \mathbb{C}^{n+1}$ which are arbitrarily small and such that the function $f_u : z \mapsto f(z) - \langle u, z\rangle$ is Morse. Indeed, the singular points of f_u are the points $z \in B$ such that $\nabla f(z) = u$, and they are degenerate if and only if z is a singular point of ∇f (and in this case, u is a singular value of ∇f). It follows that, if u is a regular value of ∇f, then f_u has only nondegenerate singular points. But by Sard's Theorem, the set of singular values of ∇f has zero measure (for the Lebesgue measure), in addition to being open. Therefore, after having chosen a certain regular value u of ∇f (which we may choose arbitrarily close to 0), we may perturbate u by an arbitrarily small vector in such a way that f_u is injective on its set of singular points, and therefore is a Morse function.

Step 2: We choose $u_0 \in \mathbb{C}^{n+1}$ such that f_{u_0} is Morse, and we claim that $f_{u_0}^{-1}(\rho)$ remains diffeomorphic to $f^{-1}(\rho)$ for ρ sufficiently small. We may assume that $u_0 \neq \nabla f(0)$, i.e. that 0 is a regular point of f_{u_0} (and therefore 0 is a regular value of f_{u_0} after possibly shrinking B). We now want to apply Ehresmann's Fibration Theorem to the map

$$F : (z, u) \in B \times \mathbb{C}^{n+1} \longmapsto (f_u(z), u) \in \mathbb{C} \times \mathbb{C}^{n+1}.$$

We note that

$$\mathrm{d}F(z, u) \cdot (h, w) = (\langle \nabla f(z) - u, h \rangle - \langle z, w \rangle, w),$$

therefore, (z, u) is a regular point of F if and only if $\nabla f(z) \neq u$ i.e. if and only if z is a regular point of f_u. We have assumed that 0 is a regular value of f_{u_0}, so we can choose a sufficiently small ρ that is a regular value of f_{u_0}. Then ρ will also be a regular value of $f = f_0$ by assumption on f. Hence, (ρ, u_0) and $(\rho, 0)$ are two regular values of F. By Ehresmann's Fibration Theorem (Theorem 4.3), $F^{-1}(\rho, u_0)$ is diffeomorphic to $F^{-1}(\rho, 0)$ and therefore $f_{u_0}^{-1}(\rho)$ is diffeomorphic to $f^{-1}(\rho)$. Therefore, we have an isomorphism $H_m(f^{-1}(\rho)) \simeq H_m(f_{u_0}^{-1}(\rho))$ for all m, and we can conclude using Corollary 4.7. □

We finally obtain this section's main theorem, which gives us the homology of regular fibres of holomorphic functions, and which we will wish to apply to the special case of polynomials.

Theorem 4.10. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with isolated singularities. If β is a regular value of f, then*

$$H_m(L_\beta) \simeq \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \bigoplus_{\lambda \in \Lambda} \mathbb{Z} & \text{if } m = n \end{cases},$$

for some set Λ.

Proof. Apply Lemma 4.6 and Lemma 4.9. □

Applying Theorem 4.10 and using the finiteness of the set of singular points for tame polynomials (Proposition 3.10), we obtain the following result, which implies Theorem 1.2 when $n = 2$ and $m = 1$.

Theorem 4.11. *If $f \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ is a nonsingular tame polynomial, then*

$$H_m(V(f)) = \begin{cases} 0 & \text{if } 1 \leqslant m < n \\ \mathbb{Z}^\mu & \text{if } m = n \end{cases},$$

*where $\mu \in \mathbb{N}$ is the **Milnor number** of $V(f)$.*

Proof. The only thing that remains to prove is that $H_n(V(f))$ is finitely generated. For every singular point of f, the proof of Lemma 4.9 perturbates f by a linear function; the resulting Morse function is therefore a polynomial and has a finite number of critical values by Theorem 3.7. The application of Corollary 4.7 shows therefore that the homology near each singular value is finitely generated. Since f has a finite number of singular values, $H_n(V(f))$ is finitely generated. □

5 Conclusion: at the junction of two paths

Two very different paths – the first one was algebraic, the second one was topological – have led us to similar results about the topology of complex affine varieties. The first result is Theorem 2.12, stating that the algebraic de Rham cohomology of tame nonsingular varieties of dimension n is trivial except at the order n, a generalisation of Picard's Theorem 1.1. The second result is Theorem 4.11, which affirms that the singular homology of nonsingular varieties of dimension n is trivial except at the order n, a generalisation of Picard's Theorem 1.2. We would now like to make these two paths meet; we shall show that the two theorems we have proved are equivalent and that they are therefore two faces of a single phenomenon.

The first thing we need is a classical fact relating singular homology and cohomology.

Theorem 5.1 (Universal Coefficient Theorem)**.** *Let X be a topological space and let G be an arbitrary abelian group. Then there exists a split exact sequence*

$$0 \longrightarrow \operatorname{Ext}\left(H_{m-1}(X), G\right) \longrightarrow H^m\left(X; G\right) \longrightarrow \operatorname{Hom}\left(H_m\left(X\right), G\right) \longrightarrow 0.$$

Proof. See [Mas80]. □

Applying the Universal Coefficient Theorem with $G = \mathbb{C}$, the Ext term in the exact sequence is zero because $\mathbb{C}$ is a divisible group; this implies that $H^m\left(X; \mathbb{C}\right) \simeq \operatorname{Hom}\left(H_m(X), \mathbb{C}\right)$. The following step is the crucial one, which creates a link between the world of topology and that of algebra. It is a result of Grothendieck, following works of Atiyah and Hodge; the three of them can be considered to be the pioneers of algebraic de Rham cohomology.

Theorem 5.2 (Grothendieck)**.** *Let X be a complex nonsingular affine variety. Then the complex cohomology $H^\bullet\left(X; \mathbb{C}\right)$ can be calculated as the cohomology of the algebraic de Rham complex.*

Proof. See [Gro66]. □

Corollary 5.3. *Let X be a complex nonsingular affine variety. Then there is an isomorphism*

$$H^m_{\mathrm{dR}}\left(X\right) \simeq \operatorname{Hom}\left(H_m(X), \mathbb{C}\right).$$

This report's two main theorems are therefore dual results. We have proved the same fact twice, using algebraic techniques first, and then ideas from Picard-Lefschetz Theory. This gives an insight into the wide diversity of methods that can be used to study the topology of algebraic varieties, exploiting the fact that the objects of algebraic geometry lie at the intersection of several different worlds.

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