École Normale Supérieure de Lyon

Research internship report

Geometry of Coxeter groups
and applications to algebraic problems

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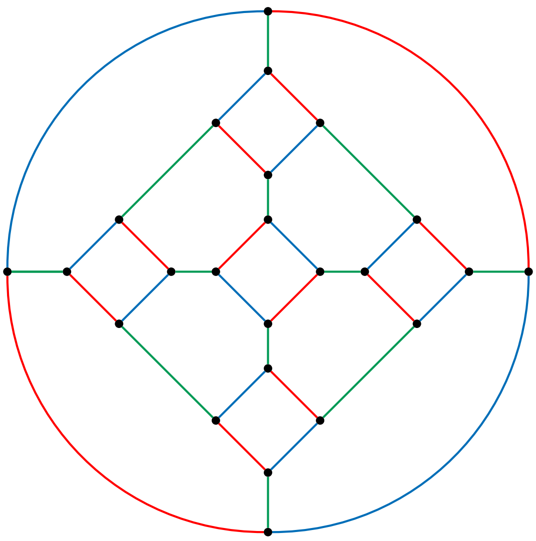
Abstract

Coxeter groups are a class of groups with a combinatorial definition. In this report, we study their geometric properties. After having introduced general concepts from geometric group theory and metric geometry, we show that Coxeter groups are CAT(0). We then show how this geometric result can lead to the solution of algebraic problems, for instance the conjugacy problem.



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Introduction

This report is the result of a six-week long research internship at the Institute for Algebra and Geometry of the Karlsruhe Institute of Technology. The aim of the internship was to learn the foundations of geometric group theory through the example of Coxeter groups. It involved learning the basics of the combinatorial theory of Coxeter groups, geometric group theory, hyperbolic geometry, metric geometry and the theory of buildings. Concretely, the work was mainly bibliographical, the two principal references being [Dav08] and [BH99]; but it also involved attending seminars, for example Gye-Seon Lee's lectures on Coxeter groups and geometry at the University of Heidelberg. We present here an outline of the work that has been done during the internship, focusing on the results leading to Moussong's Theorem, which states that Coxeter groups are CAT(0).

Contents

1 Coxeter groups from an algebraic point of view

We will start by defining Coxeter systems, and then introduce a first way of viewing a Coxeter group as a geometric object, which will give us some insight into the combinatorial structure of Coxeter groups.

1.1 Coxeter systems

In order to define a Coxeter system and a Coxeter group, we will use the vocabulary of presentations: given a set S and a set $\mathcal{R}$ of words on the alphabet $S \cup S^{-1}$, the group defined by the **presentation** $\langle S \mid \mathcal{R} \rangle$ is the quotient of the free group $F(S)$ on S by the normal subgroup $\langle\langle \mathcal{R} \rangle\rangle$ generated by $\mathcal{R}$: $\langle S \mid \mathcal{R} \rangle = F(S) / \langle\langle \mathcal{R} \rangle\rangle$.

A **Coxeter matrix** on a finite set S is a symmetric matrix $(m_{s,t})_{(s,t)\in S^2}$ with entries in $\mathbb{N}^* \cup \{\infty\}$ such that $m_{s,t} = 1$ if and only if $s = t$. A **Coxeter system** is a pair (W, S), where W is a group and S is a finite subset of W, associated to a Coxeter matrix $(m_{s,t})_{(s,t)\in S^2}$ on S, such that a presentation for W is given by:

$$W = \langle S \mid \{(st)^{m_{s,t}},\ s,t \in S,\ m_{s,t} < \infty\}\rangle .$$

If (W, S) is a Coxeter system, W will be called a **Coxeter group**. Associated to any Coxeter system (W, S), there is a **length function**: if $w \in W$, define $\ell_S(w)$ to be the length of a word of minimal length on S representing w.

Given a Coxeter system, many combinatorial questions arise naturally. For example, if s, t are two generators, we know that the order of (st) divides $m_{s,t}$, but is it *equal* to $m_{s,t}$? More generally, given two words a and b on the alphabet S, can we decide whether a and b represent the same element of W? This problem is called the **word problem**. Another problem is to decide whether the elements of W represented by a and b are conjugate; this is called the **conjugacy problem**. Using elegant geometric methods, we shall give answers to these seemingly simple combinatorial problems.

1.2 The canonical representation

The simplest examples of Coxeter groups are given by Euclidean reflection groups, i.e. subgroups of $O(n)$ generated by reflections across hyperplanes. An arbitrary Coxeter system (W, S) may not be representable in such a way, but we shall construct a linear representation of W which has similar properties. We first define the **cosine matrix** $C = (c_{s,t})_{(s,t)\in S^2}$ of (W, S) by:

$$c_{s,t} = -\cos\left(\frac{\pi}{m_{s,t}}\right),$$

where $(m_{s,t})_{(s,t)\in S^2}$ is the Coxeter matrix of (W, S) (with the convention $\cos\left(\frac{\pi}{\infty}\right) = \cos(0) = 1$). We then define B to be the symmetric bilinear form on $V = \mathbb{R}^S$ whose matrix in the canonical basis $(e_s)_{s\in S}$ is C. For $s \in S$, consider $H_s = \operatorname{Ker} B(e_s, \cdot)$ (this is the hyperplane orthogonal to e_s relative to B). As e_s is anisotropic, we have $V = \mathbb{R}e_s \oplus H_s$. Therefore, we can define σ_s to be the reflection through H_s parallel to $\mathbb{R}e_s$ (i.e. $\sigma_s(x) = x - 2B(e_s, x)\, e_s$ for $x \in V$); it is clear that σ_s preserves B. In order to be able to define a group homomorphism $W \to GL(V)$, we need to check firstly that the relations defining W are carried into $GL(V)$ by the map $s \mapsto \sigma_s$:

Lemma 1.1. *Let (W, S) be a Coxeter system. With the above notations, $\sigma_s\sigma_t$ has order $m_{s,t}$ for all $s, t \in S$.*

Proof. Let $V_{s,t} = \mathbb{R}e_s + \mathbb{R}e_t$. Note that $V_{s,t}$ is σ_s-stable and σ_t-stable. For $\lambda, \mu \in \mathbb{R}$, we have:

$$B(\lambda e_s + \mu e_t, \lambda e_s + \mu e_t) = \lambda^2 - 2\lambda\mu\cos\left(\frac{\pi}{m_{s,t}}\right) + \mu^2 = \left(\lambda - \mu\cos\left(\frac{\pi}{m_{s,t}}\right)\right)^2 + \mu^2\sin^2\left(\frac{\pi}{m_{s,t}}\right).$$

Hence, the restriction of B to $V_{s,t}$ is positive semidefinite, and it is nondegenerate precisely when $m_{s,t} < \infty$. We now have two separate cases:

- If $m_{s,t} < \infty$, B is a euclidean scalar product on $V_{s,t}$, and σ_s and σ_t are both orthogonal reflections. Moreover, $B(e_s, e_t) = \cos\left(\pi\left(1 - \frac{1}{m_{s,t}}\right)\right)$, so the angle between e_s and e_t is $\pi\left(1 - \frac{1}{m_{s,t}}\right)$. Thus, the lines H_s and H_t make an angle of $\frac{\pi}{m_{s,t}}$. From this, we deduce that $\sigma_s\sigma_t$ is a rotation through the angle $\frac{2\pi}{m_{s,t}}$ (on the euclidean plane $(V_{s,t}, B)$), so the endomorphism of $V_{s,t}$ induced by $\sigma_s\sigma_t$ is of order $m_{s,t}$. Moreover, as the restriction of B to $V_{s,t}$ is nondegenerate, we have $V = V_{s,t} \oplus (H_s \cap H_t)$. But the restriction of $\sigma_s\sigma_t$ to $H_s \cap H_t$ is the identity, which proves that the order of $\sigma_s\sigma_t$ is $m_{s,t}$.

- If $m_{s,t} = \infty$, we see that $\sigma_s(u) = \sigma_t(u) = u$, where $u = e_s + e_t$. Therefore, $\forall k \in \mathbb{N}^*$, $(\sigma_s\sigma_t)^k (e_s) = 2ku + e_s \neq e_s$, and $\sigma_s\sigma_t$ has infinite order. $\square$

In particular, we have $(\sigma_s\sigma_t)^{m_{s,t}} = \mathrm{id}_V$ for all $s, t \in S$. Therefore, the homomorphism $F(S) \to GL(V)$ given by $s \mapsto \sigma_s$ induces a homomorphism $\sigma : W \to GL(V)$. This homomorphism is called the **canonical representation** of (W, S). We shall see σ as an action of W on V and write $w(x)$ instead of $(\sigma(w))(x)$ for $w \in W$ and $x \in V$.

Define $\Phi = \{w(e_s),\ w \in W,\ s \in S\} \subseteq V$. The set Φ is called a **root system** for (W, S) and its elements are called **roots**. A root $\alpha \in \Phi$ is said to be **positive** (resp. **negative**), which shall be denoted by $\alpha > 0$ (resp. $\alpha < 0$), if all its coefficients in the basis $(e_s)_{s\in S}$ are nonnegative (resp. nonpositive). There is a strong link between the length function and the sign of roots:

Proposition 1.2. *Let (W, S) be a Coxeter system. For $w \in W$ and $s \in S$, we have:*

(i) *If $\ell_S(ws) > \ell_S(w)$, then $w(e_s) > 0$.*

(ii) *If $\ell_S(ws) < \ell_S(w)$, then $w(e_s) < 0$.*

Proof. It is enough to prove the first statement (because if $\ell_S(ws) < \ell_S(w)$, then $\ell_S(w's) > \ell_S(w')$, with $w' = ws$). We shall prove it by induction on $\ell_S(w)$. If $\ell_S(w) = 0$, then $w = 1$ and it is clear that $w(e_s) = e_s > 0$.

Let $w \in W$ with $\ell_S(w) > 0$ and assume that the result is proven for elements of length less than $\ell_S(w)$. If we take $s' \in S$ to be the last letter in a minimal word on S representing W, we have $\ell_S(ws') < \ell_S(w)$. We have $s' \neq s$ because $\ell_S(ws) > \ell_S(w)$. Put $I = \{s, s'\}$ and $W_I = \langle I \rangle \leq W$. W_I is a subgroup of W isomorphic to a dihedral group; and it has a length function ℓ_I relative to its generating set I. Now consider:

$$A = \left\{v \in W,\ v^{-1}w \in W_I,\ \ell_S(v) + \ell_S\left(v^{-1}w\right) = \ell_S(w)\right\}.$$

The set A is nonempty because $w \in A$. Choose $v \in A$ of minimal length and put $v_I = v^{-1}w \in W_I$. Hence $w = vv_I$ and $\ell_S(w) = \ell_S(v) + \ell_S(v_I)$. Note that $ws' \in A$, so $\ell_S(v) \leq \ell_S(ws') < \ell_S(w)$. Therefore, the induction hypothesis applies to v. Moreover, if $\ell_S(vs) < \ell_S(v)$, then we could prove that $vs \in A$, which would contradict the choice of v. Therefore, $\ell_S(vs) > \ell_S(v)$, which gives $v(e_s) > 0$. Likewise, we obtain $v(e_{s'}) > 0$. As $w = vv_I$, it suffices to proves that v_I maps e_s to a linear combination of e_s and $e_{s'}$ with nonnegative coefficients. Observe firstly that $\ell_I(v_I s) \geq \ell_I(v_I)$, so any minimal word on I representing v_I must end in s'. Write $m = m_{s,s'}$ and consider two cases. If $m = \infty$, we prove the desired statement by a direct calculation (noting that $B(e_s, e_{s'}) = 1$). If $m < \infty$, note that $\ell_I(v_I) \leq m$, and any element of length m in W_I has a reduced expression ending in s, so $\ell_I(v_I) < m$. Using the fact that W_I is a reflection group in the euclidean plane, it is easy to show the desired statement. $\square$

Theorem 1.3. *The canonical representation of a Coxeter system is faithful.*

Proof. Let (W, S) be a Coxeter system and $\sigma : W \to GL(V)$ be its canonical representation. If $w \in W\backslash\{1\}$, then $\ell_S(w) \geq 1$. By choosing $s \in S$ to be the last letter of a minimal word on S representing w, we have $\ell_S(ws) < \ell_S(w)$, which implies (thanks to Proposition 1.2), that $w(e_s) < 0$. But $e_s > 0$, so $w(e_s) \neq e_s$. Hence, $\sigma(w) \neq \mathrm{id}_V$. $\square$

1.3 Combinatorial consequences

Proposition 1.4. *Let (W, S) be a Coxeter system with Coxeter matrix $(m_{s,t})_{(s,t)\in S^2}$. For $s, t \in S$, the order of (st) is $m_{s,t}$.*

Proof. Let $\sigma : W \to GL(V)$ be the canonical representation of (W, S). We know that the order of (st) divides $m_{s,t}$. But if there existed $1 \leq k < m_{s,t}$ such that $(st)^k = 1$, we would have $(\sigma(s)\sigma(t))^k = \sigma\left((st)^k\right) = \mathrm{id}_V$, which is impossible according to Lemma 1.1. $\square$

The definition of a Coxeter system which we have chosen here is the one given in [Hum90]. Both [Bou68] and [Dav08] give a different definition, saying that (W, S) is a Coxeter system if W is a group with a generating set S such that $W = \langle S \mid \{(st)^{m_{s,t}},\ s, t \in S,\ m_{s,t} < \infty\}\rangle$, where $m_{s,t}$ is the order of st in W for $s, t \in S$. In this definition, the Coxeter matrix is intrinsic to the Coxeter system, whereas in our definition, there was a slight abuse in writing (W, S) when we really meant $\left(W, S, (m_{s,t})_{(s,t)\in S^2}\right)$. However, Proposition 1.4 proves that the two definitions are equivalent, which was by no means obvious.

Using Proposition 1.2, we shall prove that Coxeter systems have a solvable word problem. To do this, consider the dual $\sigma^* : W \to GL\left(V^*\right)$ of the canonical representation $\sigma : W \to GL\left(V\right)$. To each generator $s \in S$ we associate two half spaces of V^* defined by:

$$A_s^+ = \{f \in V^*,\ \langle f, e_s\rangle > 0\} \qquad \text{and} \qquad A_s^- = \{f \in V^*,\ \langle f, e_s\rangle < 0\}.$$

We also define the **fundamental chamber** of (W, S) by $C = \bigcap_{s\in S} A_s^+$. The following lemma describes the action of W on the fundamental chamber:

Lemma 1.5. *Let (W, S) be a Coxeter system. For $w \in W$ and $s \in S$, we have:*

(i) *If $\ell_S(sw) > \ell_S(w)$, then $w(C) \subseteq A_s^+$.*

(ii) *If $\ell_S(sw) < \ell_S(w)$, then $w(C) \subseteq A_s^-$.*

Proof. Suppose that $\ell_S(sw) > \ell_S(w)$, i.e. $\ell_S\left(w^{-1}s\right) > \ell_S\left(w^{-1}\right)$. Proposition 1.2 implies that $w^{-1}\left(e_s\right) > 0$. Let $f \in C$ and write $f = \sum_{s\in S} \lambda_s e_s^*$, with $(\lambda_s)_{s\in S} \in \left(\mathbb{R}_+^*\right)^S$, where $(e_s^*)_{s\in S}$ is the dual basis of $(e_s)_{s\in S}$. We have:

$$\langle w\left(f\right), e_s\rangle = \left\langle f, w^{-1}\left(e_s\right)\right\rangle = \sum_{s'\in S} \overbrace{\lambda_{s'}}^{>0} \underbrace{\left\langle e_{s'}^*, w^{-1}\left(e_s\right)\right\rangle}_{\geq 0} \geq 0.$$

Moreover, if we had $\left\langle e_{s'}^*, w^{-1}\left(e_s\right)\right\rangle = 0$ for all $s \in S$, we would have $w^{-1}\left(e_s\right) = 0$, so $e_s = 0$, which is false. Hence $\langle w\left(f\right), e_s\rangle > 0$ and $w(f) \in A_s^+$. The proof of the second statement is similar. □

Theorem 1.6. *Coxeter systems have a solvable word problem.*

Proof. Let (W, S) be a Coxeter system. Consider the canonical representation $\sigma : W \to GL(V)$ and its dual representation $\sigma^* : W \to GL\left(V^*\right)$. Write $(e_s^*)_{s\in S}$ for the dual basis of $(e_s)_{s\in S}$. Consider $f = \sum_{s\in S} e_s^* \in C$. We are going to show that the stabiliser $\mathrm{Stab}(f)$ of f in W is trivial. Therefore, in order to determine whether a word $s_1 \cdots s_k \in F(S)$ represents the identity in W, one only needs to compute $s_1 \cdots s_k(f)$: the result is f if and only if $s_1 \cdots s_k \overset{W}{=} 1$.

It remains to prove that $\mathrm{Stab}(f) = \{1\}$. Let $w \in \mathrm{Stab}(f)$. Note that, because of Lemma 1.5, we have $w(C) \subseteq \bigcap_{s\in S} A_s^{\varepsilon_s}$, where $(\varepsilon_s)_{s\in S} \in \{\pm 1\}^S$. But since $f = w(f) \in C \cap w(C)$, it follows that $\varepsilon_s = +1$ for all $s \in S$. Applying Lemma 1.5 again, we see that $\ell_S(sw) > \ell_S(w)$ for all $s \in S$, which implies that $w = 1$. □

2 Elements of geometric group theory

Geometric group theory consists in studying groups by making them operate on interesting topological or metric spaces. We will endow a group with a geometric structure, and we shall study the relations between this structure and spaces on which the group operates. This will lead us to first examples of algebraic properties originating from geometric methods.

2.1 Cayley graphs and word metrics

Let Γ be a group with a generating set S. We define the **Cayley graph** $\operatorname{Cay}(\Gamma, S)$ as the oriented graph whose vertices are elements of Γ and where there is an edge labeled by $s \in S$ going from $g \in \Gamma$ to $h \in \Gamma$ precisely when $h = gs$. If s is of order 2, note that there is an s-edge from g to h if and only if there is an s-edge from h to g; in this case, we shall only draw one undirected edge between g and h rather than two directed edges, as in Figure 1 (this is always the case for Coxeter systems). The Cayley graph is endowed with the structure of a **metric graph** by defining the length of each edge to be 1; this metric is locally well-defined, and the distance between two arbitrary points is taken to be the length of a shortest path joining these two points. The following remark will be fundamental for working with Cayley graphs:

Remark. *A path between two vertices in the Cayley graph* $\operatorname{Cay}(\Gamma, S)$ *corresponds to a word on* $S \cup S^{-1}$*. Moreover, this correspondence preserves lengths.*

As S generates Γ, the preceding remark shows that $\operatorname{Cay}(\Gamma, S)$ is connected. Moreover, for any element $g \in \Gamma$, the distance between 1 and g is equal to the length of a minimal word $(s_1, \ldots, s_k)$ on $S \cup S^{-1}$ such that $g = s_1 \cdots s_k$. This is the **length** of g relative to S, denoted by $\ell(g)$ or $\ell_S(g)$ (note that this notion of length is the same as for Coxeter systems). Moreover, the restriction to Γ of the metric on $\operatorname{Cay}(\Gamma, S)$ will be called the **word metric** on Γ (relative to S).

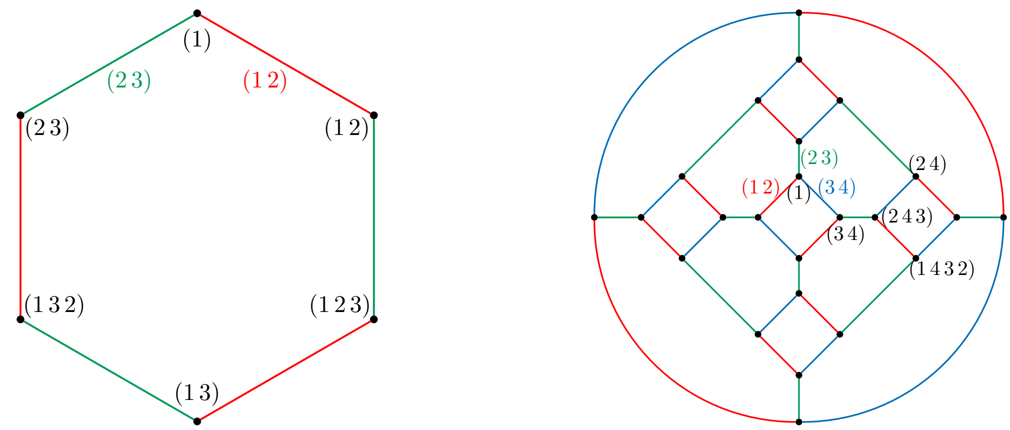

Figure 1: The Cayley graphs of $(\mathfrak{S}_3, \{(1\,2), (2\,3)\})$ and $(\mathfrak{S}_4, \{(1\,2), (2\,3), (3\,4)\})$.

There is a natural geometric action of Γ on $\operatorname{Cay}(\Gamma, S)$. Here is some vocabulary which shall be used to describe the properties of this action: if a group Γ acts by homeomorphisms on a topological space X, we shall say that the action is **proper** if for any compact set $K \subseteq X$, the set $\{\gamma \in \Gamma,\ \gamma K \cap K \neq \varnothing\}$ is finite; and we shall say that the action is **cocompact** if there exists a compact set $K \subseteq X$ such that $X = \Gamma \cdot K$.

Proposition 2.1. *Let* Γ *be a group with a generating set* S*.*

(i) *The left action of* Γ *on* Γ *by translation induces an action of* Γ *on* $\operatorname{Cay}(\Gamma, S)$ *by isometries.*

(ii) *If* S *is finite, the action of* Γ *on* $\operatorname{Cay}(\Gamma, S)$ *is proper and cocompact.*

Proof. (i) It is enough to prove that the action of Γ preserves adjacency in the Cayley graph. To do this, consider $\gamma \in \Gamma$ and take two vertices $g, h \in \Gamma$ of $\operatorname{Cay}(\Gamma, S)$. Suppose there is an s-edge from g to h, i.e. $h = gs$. Then $\gamma h = (\gamma g)\, s$, so there is indeed an s-edge from γg to γh.

(ii) To see that the action of Γ on $\operatorname{Cay}(\Gamma, S)$ is proper, take a compact set $K \subseteq X$. Choose $r > 0$ so that $K \subseteq B(1, r)$, where $B(1, r)$ is the open ball with center 1 and radius r. If

$\gamma \in \Gamma$ is such that $\gamma K \cap K \neq \varnothing$, then $\gamma B(1,r) \cap B(1,r) \neq \varnothing$, so $d(1,\gamma) < 2r$. This proves that:

$$\{\gamma \in \Gamma,\ \gamma K \cap K \neq \varnothing\} \subseteq \{\gamma \in \Gamma,\ d(1,\gamma) < 2r\} = \{\gamma \in \Gamma,\ \ell(\gamma) < 2r\}.$$

As S is finite, these sets are finite, so the action of Γ on $\mathrm{Cay}(\Gamma, S)$ is proper.

Now note that the action of Γ on itself by translation is transitive. Therefore, for all $s \in S$, the action of Γ on the set of s-edges of $\mathrm{Cay}(\Gamma, S)$ is transitive. Hence, if K_s is an s-edge for $s \in S$, we have $\mathrm{Cay}(\Gamma, S) = \Gamma \cdot \left(\bigcup_{s\in S} K_s\right)$, so the action is cocompact. □

2.2 Geodesic spaces and length spaces

The central idea of what follows will be to compare paths in the Cayley graph of (Γ, S) (which correspond to words on $S \cup S^{-1}$) to paths in a metric space X on which Γ acts geometrically. Therefore, it will be necessary to add an additional hypothesis on X, which will ensure that paths in X are well-behaved.

If X is a metric space and $c : [a,b] \to X$ is a path in X, the **length** of c is defined by:

$$\ell(c) = \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} d\left(c\left(t_j\right), c\left(t_{j+1}\right)\right) \geq d\left(c(a), c(b)\right).$$

The space X is said to be a **length space** (resp. a **geodesic space**) if for any two points $x, y \in X$, the distance between x and y is the *infimum* (resp. the *minimum*) of the lengths of paths from x to y (as a consequence, X is path-connected). A path c is said to be a **geodesic path** if $d\left(c(s), c(t)\right) = |s-t|$ for all $s, t \in [a,b]$. It is said to be a **linearly reparametrised geodesic** if there exists $\lambda > 0$ such that $t \mapsto c(\lambda t)$ is a geodesic path. The space X is geodesic if and only if any two points of X are joined by a geodesic path. If this geodesic path is always unique, X is said to be **uniquely geodesic**.

Remark. *All metric graphs are geodesic spaces; in particular, Cayley graphs are geodesic spaces.*

2.3 Quasi-isometry

The problem we now face is that, given a group Γ, there may be several different generating sets S leading to different Cayley graphs and different word metrics. For example, $\mathrm{Cay}\left(\mathbb{Z}, \{1\}\right)$ is not isometric to $\mathrm{Cay}\left(\mathbb{Z}, \{2,3\}\right)$. To get round this problem, we define the notion of quasi-isometry. If X and Y are metric spaces, a map $f : X \to Y$ is called a **quasi-isometric embedding** if there exist $\lambda \geq 1$ and $\varepsilon \geq 0$ such that the following inequalities hold for all $x, x' \in X$:

$$\frac{1}{\lambda} d\left(x, x'\right) - \varepsilon \leq d\left(f(x), f\left(x'\right)\right) \leq \lambda d\left(x, x'\right) + \varepsilon.$$

The map f is called a **quasi-isometry** if it is a quasi-isometric embedding and has a **quasi-inverse**, i.e. a map $g : Y \to X$ such that $d\left(g \circ f, \mathrm{id}_X\right) < \infty$ and $d\left(f \circ g, \mathrm{id}_Y\right) < \infty$. In this case, g is also a quasi-isometric embedding, and the spaces X and Y are said to be **quasi-isometric**. Therefore, the quasi-isometry of metric spaces is an equivalence relation.

Proposition 2.2. *Let Γ be a* finitely generated *group. Consider S and S' two* finite *generating sets for G, with associated word metrics d_S and $d_{S'}$. Then the metric spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are quasi-isometric. Therefore, Γ has a unique natural metric structure up to quasi-isometry.*

Proof. We will show that $\mathrm{id}_\Gamma : (\Gamma, d_S) \to (\Gamma, d_{S'})$ is a quasi-isometric embedding, which will imply that it is a quasi-isometry because it has an inverse map, namely $\mathrm{id}_\Gamma : (\Gamma, d_{S'}) \to (\Gamma, d_S)$. With $\lambda = \max\left\{\max_{s\in S} \ell_{S'}(s), \max_{s'\in S'} \ell_S\left(s'\right)\right\}$, we have $\frac{1}{\lambda}\ell_S(g) \leq \ell_{S'}(g) \leq \lambda \ell_S(g)$ for $g \in \Gamma$. □

2.4 The fundamental observation of geometric group theory

The following lemma is very general, and tells us how to obtain a generating set for a group given a geometric action of this group on a topological space.

Lemma 2.3. *Let Γ be a group. Suppose Γ acts by homeomorphisms on a connected topological space X and consider an open subset $\mathcal{U} \subseteq X$ such that $X = \Gamma \cdot \mathcal{U}$. Then $S = \{\gamma \in \Gamma,\ \gamma\mathcal{U} \cap \mathcal{U} \neq \varnothing\}$ is a generating set for Γ.*

Proof. Consider $V = \langle S \rangle \cdot \mathcal{U}$ and $V' = (\Gamma \backslash \langle S \rangle) \cdot \mathcal{U}$. Due to the definition of S, we have $V \cap V' = \varnothing$. Therefore, V and V' are disjoint open subsets of X, and $X = V \sqcup V'$. Moreover, $V \supseteq \mathcal{U} \supsetneq \varnothing$, and X is connected, so $V = X$ and $V' = \varnothing$, which implies that $\Gamma \backslash \langle S \rangle = \varnothing$. □

The following theorem is sometimes called *the fundamental observation of geometric group theory*; it helps us understand the relation between a group and a metric space on which it acts. We call a metric space **proper** if its closed balls are compact.

Theorem 2.4 (Švarc-Milnor)**.** *Let Γ be a group. Suppose Γ acts properly cocompactly by isometries on a proper length space X. Then:*

(i) *Γ is finitely generated.*

(ii) *For all $x_0 \in X$, the map $g \in \Gamma \longmapsto gx_0 \in X$ is a quasi-isometry between Γ and X.*

Proof. (i) As the action of Γ on X is cocompact, let $x_0 \in X$ and $r > 0$ such that $X = \Gamma \cdot B(x_0, r)$. Define $S = \{\gamma \in \Gamma,\ \gamma B(x_0, 3r) \cap B(x_0, 3r) \neq \varnothing\}$. The set S is finite because the operation of Γ on X is proper, and X is proper. Since $X = \Gamma \cdot B(x_0, 3r)$, Lemma 2.3 guarantees that S generates Γ.

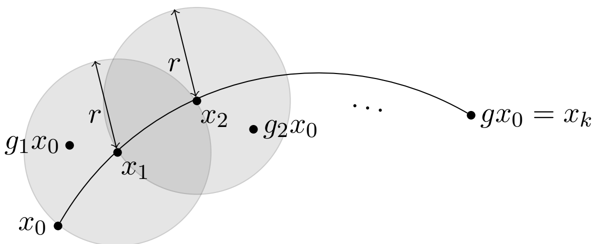

Figure 2: Proof of the Švarc-Milnor Theorem.

(ii) Consider $f : g \in \Gamma \longmapsto g \cdot x_0 \in X$. We shall prove that f is a quasi-isometry. Let $g \in \Gamma$. Let $k \in \mathbb{N}^*$ such that $d_X(x_0, gx_0) = (k-1)r + r'$, with $0 \leq r' < r$. As X is a length space, choose a path c in X from x_0 to gx_0 of length $\ell(c) = d_X(x_0, gx_0) + \varepsilon$, with $0 \leq \varepsilon \leq (r - r')$. Now separate c into $(k-1)$ subpaths of length r and one extra subpath of length $r' + \varepsilon$. Write $x_0, x_1, \ldots, x_{k-1}, x_k = gx_0$ for the corresponding points in X. For $j \in \{1, \ldots, k-1\}$, there exists $g_j \in \Gamma$ such that $x_j \in g_j B(x_0, r)$. Set $g_0 = 1$ and $g_k = g$. For $j \in \{0, \ldots, k-1\}$, we have $g_{j+1}x_0 \in g_{j+1}B(x_0, 3r) \cap g_j B(x_0, 3r)$ so $s_j = g_j^{-1}g_{j+1} \in S$. As $g = s_0 \cdots s_{k-1}$, the length of g relative to S is at most k:

$$\ell(g) \leq k = \frac{(k-1)r}{r} + 1 \leq \frac{1}{r} d_X(x_0, gx_0) + 1.$$

Now, consider $g = s'_0 \cdots s'_{m-1}$ a reduced expression for g relative to S (i.e. $m = \ell(g)$). For $j \in \{0, \ldots, m-1\}$, we know that $s'_j B(x_0, 3r) \cap B(x_0, 3r) \neq \varnothing$ (because $s'_j \in S$), which implies that $d_X\left(s'_j x_0, x_0\right) < 6r$. Thus:

$$d_X(x_0, gx_0) \leq \sum_{j=0}^{m-1} d_X\left(s'_0 \cdots s'_{j-1}x_0, s'_0 \cdots s'_j x_0\right) = \sum_{j=0}^{m-1} d_X\left(x_0, s'_j x_0\right) < 6mr = 6r\ell(g).$$

To summarise, we have proved that $r\ell(g) - r \leq d_X(x_0, gx_0) \leq 6r\ell(g)$ for all $g \in \Gamma$. As Γ acts by isometries, we obtain $rd_\Gamma(g,h) - r \leq d_X(f(g), f(h)) \leq 6rd_\Gamma(g,h)$ for all $g, h \in \Gamma$, where d_Γ is the word metric on Γ relative to S. This proves that $f : \Gamma \to X$ is a quasi-isometric embedding.

Moreover, for $x \in X$, if $g(x) \in \Gamma$ is such that $d_X(x, g(x)x_0) < r$, then the map $g : X \to \Gamma$ is a quasi-inverse for f, so f is a quasi-isometry. □

Corollary 2.5. *If Γ is a group with a finite generating set S, Γ is quasi-isometric to* $\mathrm{Cay}(\Gamma, S)$.

To illustrate the depth of the Švarc-Milnor Theorem, here is a purely algebraic property, proved using geometric ideas:

Proposition 2.6. *If a group Γ is finitely generated, then every subgroup $H \leq \Gamma$ of finite index is also finitely generated.*

Proof. Take S a finite set of generators for Γ. We know that Γ acts properly cocompactly by isometries on $\mathrm{Cay}(\Gamma, S)$, which is a proper length space. The induced action of H on $\mathrm{Cay}(\Gamma, S)$ is proper and by isometries. It is also cocompact because H has finite index in Γ. Therefore, the Švarc-Milnor Theorem guarantees that H is finitely generated. □

3 Nonpositive curvature

We are going to define a notion of curvature for a metric space X. To do this, the idea will be to compare triangles in X with triangles in a well-known space in which we already have a notion of curvature (given by differential geometry, for instance). We will see later that information about the curvature of a space on which a group acts translates into algebraic properties of the group itself.

3.1 $\mathrm{CAT}(0)$ and $\mathrm{CAT}(1)$ spaces

Informally, $\mathrm{CAT}(0)$ spaces are metric spaces in which triangles are thinner than in the **Euclidean plane** $\mathbb{E}^2$, equipped with the standard scalar product and the induced distance $d_{\mathbb{E}^2}$. Likewise, $\mathrm{CAT}(1)$ spaces are spaces in which triangles are thinner than in the **2-sphere** $\mathbb{S}^2$, equipped with the distance $d_{\mathbb{S}^2}$ defined by $d_{\mathbb{S}^2}(A,B) = \arccos\left(\langle A, B\rangle_{\mathbb{R}^3}\right)$ for $A, B \in \mathbb{S}^2$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ is the Euclidean scalar product on $\mathbb{R}^3$. These spaces have the following properties:

Proposition 3.1. (i) $\mathbb{E}^2$ *is a uniquely geodesic space and for all $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_+)^3$ satisfying the triangle inequality, there exists a triangle in $\mathbb{E}^2$ whose edges have lengths ℓ_1, ℓ_2, ℓ_3.*

(ii) $\mathbb{S}^2$ *is a geodesic space and any pair of points a distance less than π apart can be joined by a unique geodesic path. For all $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_+)^3$ satisfying the triangle inequality and such that $\ell_1 + \ell_2 + \ell_3 < 2\pi$, there exists a triangle in $\mathbb{S}^2$ whose edges have lengths ℓ_1, ℓ_2, ℓ_3.*

A **geodesic triangle** Δ in a metric space X consists of three points x_1, x_2, x_3 together with three geodesic paths $[x_1, x_2], [x_2, x_3], [x_3, x_1]$. According to Proposition 3.1, there always exists a geodesic triangle $\overline{\Delta}$ with vertices $\overline{x}_1, \overline{x}_2, \overline{x}_3$ in $\mathbb{E}^2$ such that $d_X(x_i, x_j) = d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j)$ for all i, j. Once we have the triangle $\overline{\Delta}$, we can construct comparison points for any point on edges of Δ by choosing the comparison point for $p \in [x_i, x_j]$ to be the unique point $\overline{p} \in [\overline{x}_i, \overline{x}_j]$ such that $d_X(p, x_i) = d_{\mathbb{E}^2}(\overline{p}, \overline{x}_i)$. We shall say that Δ satisfies the $\mathrm{CAT}(0)$ **inequality** if $d_X(p,q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q})$ for all $p, q \in \Delta$. Likewise, if the perimeter of Δ is less than 2π, there exists a comparison triangle $\overline{\Delta}$ in $\mathbb{S}^2$ and one can define the $\mathrm{CAT}(1)$ inequality in the same manner.

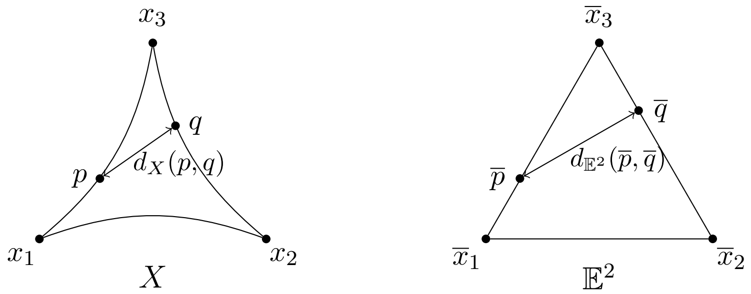

Figure 3: The CAT(0) inequality.

The space X is said to be a CAT(0) **space** if it is a geodesic space such that any geodesic triangle satisfies the CAT(0) inequality; X is said to be a CAT(1) space if any two points a distance less than π apart can be joined by a geodesic path and if any triangle of perimeter less than 2π satisfies the CAT(1) inequality. Moreover, X is said to be **nonpositively curved** if it is locally a CAT(0) space (i.e. for $x \in X$, there exists $r > 0$ such that $B(x, r)$ is CAT(0)).

Remark. *The Euclidean plane $\mathbb{E}^2$ is a* CAT(1) *space. Therefore, any* CAT(0) *space is* CAT(1)*.*

3.2 Properties of CAT(0) spaces

Proposition 3.2. *Let X be a* CAT(0) *space.*

(i) *X is uniquely geodesic.*

(ii) *The distance on X is convex, i.e. for any pair of linearly reparametrised geodesics $c, c' : [0,1] \to X$, the following inequality holds for all $t \in [0,1]$:*

$$d\left(c(t), c'(t)\right) \leq (1-t) \cdot d\left(c(0), c'(0)\right) + t \cdot d\left(c(1), c'(1)\right).$$

(iii) *X is contractible.*

Proof. (i) Let $x, y \in X$ and consider two geodesic paths $[x, y]$ and $[x, y]'$ from x to y. Let $p \in [x, y]$. Write $[x, p]$ and $[p, y]$ for the subpaths of $[x, y]$ respectively starting and ending at p. Then $[x, p], [p, y], [x, y]'$ is a geodesic triangle in X and the comparison triangle in $\mathbb{E}^2$ is degenerate (i.e. its vertices are aligned). Therefore, thanks to the CAT(0) inequality, the triangle in X is also degenerate, which proves that $p \in [x, y]'$. Hence, $[x, y] = [x, y]'$.

(ii) In the case where $c(0) = c'(0)$, the desired inequality can be obtained by taking a comparison triangle for the triangle $\Delta\left(c(0), c(1), c'(1)\right)$. In the general case, we introduce a linearly reparametrised geodesic $c'' : [0,1] \to X$ from $c(0)$ to $c'(1)$. We then apply the previous special case, firstly to c and c'', and then to c'' and c' with reverse orientation.

(iii) Fix a point $x_0 \in X$. For any $x \in X$, let $\gamma_x : [0,1] \to X$ be the unique linearly reparametrised geodesic from x_0 to x. Define a map $H : [0,1] \times X \to X$ by $H(t, x) = \gamma_x(t)$. We have $H(0, \cdot) = x_0$ and $H(1, \cdot) = \mathrm{id}_X$; it remains to prove that H is continuous. Thanks to the convexity of d, we have, for all $t, t' \in [0,1]$ and $x, x' \in X$:

$$\begin{aligned} d\left(H(t,x), H\left(t', x'\right)\right) &= d\left(\gamma_x(t), \gamma_{x'}\left(t'\right)\right) \\ &\leq d\left(\gamma_x(t), \gamma_{x'}(t)\right) + d\left(\gamma_{x'}(t), \gamma_{x'}\left(t'\right)\right) \\ &\leq td\left(\gamma_x(1), \gamma_{x'}(1)\right) + (t-t')d\left(\gamma_{x'}(0), \gamma_{x'}(1)\right) \\ &= td(x, x') + (t-t')d\left(x_0, x'\right) \\ &\leq d(x, x') + (t-t')d\left(x_0, x'\right). \end{aligned}$$

Therefore, H is a homotopy from the constant map at x_0 to id_X. $\square$

3.3 The exponential map and the Cartan-Hadamard Theorem

Our aim will be to show that, under suitable hypotheses, if a space X is nonpositively curved (i.e. if it is locally CAT(0)), then it is (globally) CAT(0). In differential or Riemannian geometry, one of the important tools for obtaining global properties thanks to local one is the exponential map; this is why we are going to define an analogue of the exponential map for metric spaces.

Let X be a metric space. A **local geodesic path** in X is a path $c : I \to X$, where $I \subseteq \mathbb{R}$ is an interval, such that for every $t_0 \in I$, there exists an $\varepsilon > 0$ such that $d\left(c(t), c\left(t'\right)\right) = |t - t'|$ for all $t, t' \in I$ with $|t - t_0| + |t' - t_0| \leq \varepsilon$. A **linearly reparametrised local geodesic** is a path $c : I \to X$ such that there exists a constant $\lambda > 0$ such that $t \mapsto c\left(\lambda t\right)$ is a local geodesic.

Fix a point $x_0 \in X$. We define the **tangent space** $\widetilde{X}_{x_0}$ of X at x_0 to be the set of all linearly reparametrised local geodesics $c : [0, 1] \to X$ such that $c(0) = x_0$, together with the constant path $\widetilde{x}_0$ at x_0. We define a metric on $\widetilde{X}_{x_0}$ by $d\left(c, c'\right) = \sup_{t\in[0,1]} d\left(c(t), c'(t)\right)$. Finally, we define the **exponential map** by $\exp : c \in \widetilde{X}_{x_0} \longmapsto c(1) \in X$. The following proposition highlights the analogy with the differential case:

Proposition 3.3. *Suppose that X is a complete metric space, with a locally convex metric, and fix $x_0 \in X$.*

(i) *$\widetilde{X}_{x_0}$ is a complete contractible metric space.*

(ii) *$\exp : \widetilde{X}_{x_0} \to X$ is a local isometry.*

(iii) *There is a unique local geodesic path joining $\widetilde{x}_0$ to each point of $\widetilde{X}_{x_0}$.*

(iv) *If X is connected, then $\exp : \widetilde{X}_{x_0} \to X$ is a universal covering map.*

(v) *If X is connected, then there is a unique local geodesic path between each pair of points of $\widetilde{X}_{x_0}$, and these local geodesics vary continuously with their endpoints.*

Proof. See [BH99], Chapter II.4. □

Therefore, the exponential map provides an explicit construction of the universal covering of X. Using this construction, one can prove the following theorem:

Theorem 3.4 (Cartan-Hadamard)**.** *Let X be a complete connected metric space.*

(i) *If the metric on X is locally convex, then the induced length metric on the universal covering $\widetilde{X}$ is convex.*

(ii) *If X is nonpositively curved, then $\widetilde{X}$ (with the induced length metric) is* CAT(0).

Proof. See [BH99], Chapter II.4. □

Corollary 3.5. *Let X be a complete simply-connected geodesic space. If X is nonpositively curved, then X is* CAT(0).

3.4 Berestovskii's Theorem

For the last part of this report, it will be necessary to understand the curvature of cell complexes. Berestovskii's Theorem, which supplies information about the curvature of the **cone** Cone(Y) over a metric space Y, will be our main tool in order to achieve this aim. To define Cone(Y), consider the quotient of $[0, \infty[\times Y$ by the equivalence relation which consists in identifying the points whose first coordinate is 0, and equip it with the metric defined by $d_{\mathrm{Cone}(Y)}\left(\left(t, y\right), \left(t', y'\right)\right)^2 = t^2 + t'^2 - 2tt' \cos\left(\min\left\{\pi, d_Y\left(y, y'\right)\right\}\right)$.

Theorem 3.6 (Berestovskii)**.** *A metric space Y is* CAT(1) *if and only if* Cone(Y) *is* CAT(0).

Proof. See [BH99], Theorem II.3.14. □

If L is a Euclidean cell complex and v is a vertex of L, we define the **link** $\mathrm{Lk}(v, L)$ of v in L to be the set of unit vectors at v that point into L. The link $\mathrm{Lk}(v, L)$ has the structure of a spherical cell complex. If L has only *finitely many isometry types of cells*, then there exists an open ball centred at v in L which is isometric to an open ball centred at v in $\mathrm{Cone}\,(\mathrm{Lk}\,(v, L))$. This leads to the following result:

Corollary 3.7. *A Euclidean cell complex L with only finitely many isometry types of cells has nonpositive curvature if and only if the link of every vertex of L is a* $\mathrm{CAT}(1)$ *space.*

4 Coxeter groups from a geometric point of view

The aim of this report's last part will be, given a Coxeter group W, to construct a Euclidean $\mathrm{CAT}(0)$ cell complex on which W acts properly cocompactly by isometries. Remarkably, this fact has many algebraic applications. In particular, knowing that Coxeter groups act on $\mathrm{CAT}(0)$ spaces will enable us to solve their conjugacy problem, i.e. to decide whether or not two words on the set of generators represent conjugate elements of W.

4.1 The Davis complex

Let (W, S) be a Coxeter system. A subset $T \subseteq S$ is said to be **spherical** if it generates a finite subgroup of W. In this case, the subgroup $\langle T \rangle$ is called a **spherical subgroup** and its (left) cosets are called **spherical cosets**. Write $\mathscr{S}$ for the set of spherical subsets of S and define the **nerve** $\mathscr{N}$ of (W, S) to be the poset $\mathscr{S} \backslash \{\varnothing\}$, ordered by inclusion; the nerve will be considered as an abstract simplicial complex.

Consider the poset $W\mathscr{S}$ of all spherical cosets of (W, S), ordered by inclusion. The **Davis complex** Σ is defined as the **flag complex** of the poset $W\mathscr{S}$, i.e. the abstract simplicial complex whose set of vertices is $W\mathscr{S}$ and where a subset $\Delta \subseteq W\mathscr{S}$ spans a simplex if and only if it is totally ordered. Here is a first observation aiming to understand what the flag complex of a poset is:

Remark. *Let Λ be a convex cell complex (i.e. a complex whose cells are convex polytopes). If $\mathcal{P}$ is the poset of cells of Λ, then the flag complex of $\mathcal{P}$ is the barycentric subdivision of Λ.*

We are now going to relate Σ to a geometric object. Suppose for the moment that W is finite. Recall the definition of the **fundamental chamber** associated to (W, S) from Section 1.3. Define a **Coxeter polytope** of (W, S) to be the convex hull of an orbit Wx, where x is any point of the fundamental chamber. As shown by the following proposition, Coxeter polytopes are closely related to the Davis complex:

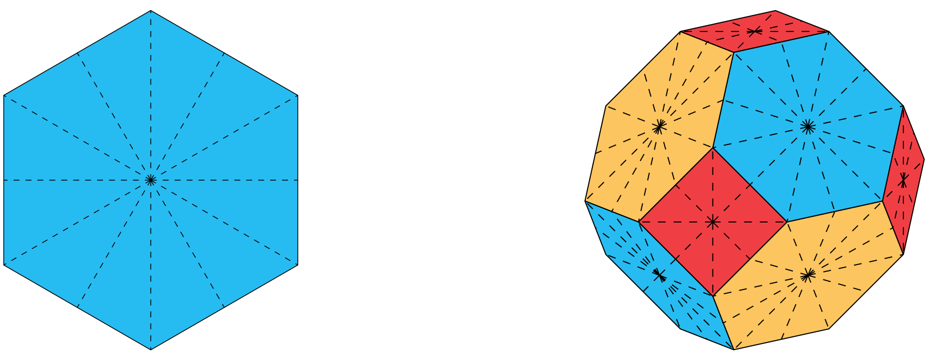

Figure 4: Coxeter polytopes of $(\mathfrak{S}_3, \{(1\,2), (2\,3)\})$ and $(\mathfrak{S}_4, \{(1\,2), (2\,3), (3\,4)\})$, with their barycentric subdivision.

Proposition 4.1. *Let (W, S) be a Coxeter system, with W finite. Write $\mathcal{C}_x$ for the Coxeter polytope of (W, S) associated to a point x lying in the fundamental chamber. Then the map $w \mapsto wx$ induces an isomorphism of posets between $W\mathscr{S}$ and the poset of faces of $\mathcal{C}_x$.*

Proof. See [Dav08], Lemma 7.3.3. □

Corollary 4.2. *Let (W, S) be a Coxeter system, with W finite. Then the Davis complex Σ of (W, S) is the barycentric subdivision of any Coxeter polytope of (W, S).*

We do not suppose anymore that W is finite. If $T \subseteq S$ is a spherical subset, note that $(\langle T\rangle, T)$ is a Coxeter system and the poset $(W\mathscr{S})_{\leq w\langle T\rangle}$ is isomorphic to $\langle T\rangle \mathscr{S}_{\leq T}$ for all $w \in W$. According to Corollary 4.2, the flag complex of $(W\mathscr{S})_{\leq w\langle T\rangle}$ is the barycentric subdivision of a Coxeter polytope associated to $\langle T\rangle$. Therefore, we can equip Σ with a new cell structure, its **natural cell structure**, where each cell is a Coxeter polytope. With this new structure, the vertex set of Σ is W and its 1-skeleton is $\mathrm{Cay}(W, S)$. We observe that the 2-skeleton of Σ is the complex formed by attaching a 2-cell to each loop in $\mathrm{Cay}(W, S)$ corresponding to a relation given by the Coxeter matrix. This leads to the following proposition:

Lemma 4.3. *Let (W, S) be a Coxeter system and write $\mathcal{R} = \{(st)^{m_{s,t}},\ s, t \in S,\ m_{s,t} < \infty\} \subseteq F(S)$ for the set of relations given by the Coxeter matrix. Then $\pi_1(\mathrm{Cay}(W, S)) \simeq \langle\!\langle \mathcal{R}\rangle\!\rangle \trianglelefteq F(S)$.*

Proposition 4.4. *The Davis complex Σ of a Coxeter system (W, S) is simply connected.*

Thanks to the natural cell structure on Σ, it is possible to define a metric on Σ by specifying a metric for each Coxeter polytope. Coxeter polytopes are subsets of a real finite-dimensional vector space, so they naturally come with the Euclidean metric; but the question remains of the choice of the Coxeter polytope, because the Coxeter polytopes associated to different points of the fundamental chamber are not isometric. It is not difficult to see that, if (W, S) is a Coxeter system with W finite, given a choice of positive real numbers $(\ell_s)_{s\in S}$ indexed by S, one can find a point x in the fundamental chamber (by solving a linear system) such that, in the Coxeter polytope associated to x, each s-edge has length ℓ_s. For instance there exists a Coxeter polytope, unique up to isometry, such that each edge has length 1; this defines a **canonical Coxeter polytope** for any Coxeter system (W, S), with W finite. If (W, S) is any Coxeter system (where W may be infinite), we now equip its Davis complex Σ with the length metric defined so as to extend the metric structure defined on Coxeter polytopes. The following proposition will be useful for studying the curvature of the Davis complex:

Proposition 4.5. *Let (W, S) be a Coxeter system. The link of each vertex of the Davis complex Σ is isomorphic (as an abstract simplicial complex) to the nerve $\mathscr{N}$.*

4.2 Gromov's Lemma

Thanks to Corollaries 3.5 and 3.7, we see that, under certain hypotheses, in order to prove that a Euclidean complex is $\mathrm{CAT}(0)$, one only needs to prove that its link, a spherical complex, is $\mathrm{CAT}(1)$. Our main tool to do this will be Gromov's Lemma, which gives a combinatorial condition for spherical complexes to be $\mathrm{CAT}(1)$. Before proving it, we need a few lemmata. To state them, we use the following vocabulary: a spherical complex is said to be **all-right** if each edge has length $\frac{\pi}{2}$.

Lemma 4.6. *Let K be an all-right spherical complex. Suppose that ℓ is a geodesic circle and $v \in \mathrm{Vert}(K)$. Then each connected component of $\ell \cap B\left(v, \frac{\pi}{2}\right)$ must have length π.*

Proof. Let ℓ' be a connected component of $\ell \cap B\left(v, \frac{\pi}{2}\right)$. Consider the endpoints a and b of ℓ', i.e. the points of ℓ' which are a distance $\frac{\pi}{2}$ from v. Choose a point m in the relative interior of ℓ'. Now consider a comparison triangle $\Delta(\overline{v}, \overline{a}, \overline{m}) \subseteq \mathbb{S}^2$ for $\Delta(v, a, m)$. If $\overline{v}$ is chosen to lie on the North pole, $\overline{a}$ is on the equator and $\overline{m}$ is in the open Northern hemisphere. Now choose $\overline{b}$ to be a comparison point for b, lying on the side of $[\overline{v}, \overline{m}]$ opposite to $\overline{a}$ (i.e. $\overline{b}$ is determined by its distances from $\overline{v}$ and $\overline{m}$). As $\overline{b}$ is at a distance $\frac{\pi}{2}$ from $\overline{v}$, it also lies on the equator. Moreover, since ℓ' is a geodesic circle, $[\overline{a}, \overline{m}] \cup \left[\overline{m}, \overline{b}\right]$ is a local geodesic, i.e. $\overline{b}$ lies on the same great circle as $\overline{a}$ and $\overline{m}$. Hence $\overline{a}$ and $\overline{b}$ are both in the intersection of two great circles, namely the equator and the one passing through $\overline{m}$. These two great circles are distinct since $\overline{m}$ lies in the open Northern hemisphere. Therefore, $\overline{a}$ and $\overline{b}$ are opposite poles, so their distance is exactly π. ☐

Lemma 4.7. *Let L be an all-right spherical complex. If two distinct vertices of L are a distance less than π apart, then they are adjacent.*

Proof. This is due to the fact that, in an all-right spherical simplex, the distance between a vertex and any point of the opposite hyperface is exactly $\frac{\pi}{2}$. ☐

We also need the following proposition. A complex is said to satisfy the **link condition** if the link of each vertex is CAT(1).

Proposition 4.8. *Let L be a spherical complex with only finitely many isometry types of cells. Then L is* CAT(1) *if and only if L satisfies the link condition and contains no isometrically embedded circle of length less than 2π.*

Proof. See [BH99], Theorem II.5.4. ☐

We are now ready to prove Gromov's Lemma. A simplicial complex is said to be a **flag complex** if any finite set of pairwise adjacent vertices spans a simplex.

Lemma 4.9 (Gromov)**.** *Let L be a finite dimensional all-right spherical complex. If L is a flag complex, then it is* CAT(1)*.*

Proof. We proceed by induction on the dimension of L. The statement is clear if $\dim L = 0$. Suppose the statement is true for any all-right spherical complex of dimension less than $\dim L$. In particular, for all $v \in L$, the statement is true for $\mathrm{Lk}\,(v, L)$. But since $\mathrm{Lk}\,(v, L)$ is an all-right spherical flag complex, it is CAT(1). This proves that L satisfies the link condition. In order to prove that L is CAT(1), it remains to prove that L contains no isometrically embedded circle of length less than 2π (because of Proposition 4.8).

Let ℓ be an isometrically embedded circle of L and suppose for contradiction that ℓ has length $< 2\pi$. Let $V = \left\{v \in \mathrm{Vert}(L),\ \ell \cap B\left(v, \frac{\pi}{2}\right) \neq \varnothing\right\}$. For $v \in V$, the length of $\ell \cap B\left(v, \frac{\pi}{2}\right)$ is at least π according to Lemma 4.6. Therefore, if $v, v' \in V$, $B\left(v, \frac{\pi}{2}\right)$ and $B\left(v', \frac{\pi}{2}\right)$ cannot be disjoint (for otherwise ℓ would have length at least 2π). This proves that $d\left(v, v'\right) < \pi$ for all $v, v' \in V$. Due to Lemma 4.7, the vertices of V are pairwise adjacent so V spans a simplex Δ in L, because L is a flag complex and V is finite. Now $\ell \subseteq \Delta$, which is a contradiction because a spherical simplex cannot contain an isometrically embedded circle of length less than 2π.

Therefore, L satisfies the link condition and does not contain any isometrically embedded circle of length less than 2π, so L is CAT(1) according to Proposition 4.8. ☐

4.3 Coxeter groups are CAT(0)

We now reach this report's main theorem. A group is said to be CAT(0) if it acts properly cocompactly by isometries on a proper CAT(0) space (according to Theorem 2.4, this implies that it is finitely generated and quasi-isometric to a CAT(0) space). Firstly, we make the following observation:

Proposition 4.10. *Let (W, S) be a Coxeter system and let Σ be its Davis complex. Then W acts on Σ properly cocompactly by isometries.*

We can prove that a special class of Coxeter groups are CAT(0):

Theorem 4.11. *Let (W, S) be a right-angled Coxeter system, i.e. a Coxeter system such that the Coxeter matrix has entries in $\{1, 2, \infty\}$. Then W is* CAT(0).

Proof. It suffices to prove that the Davis complex Σ, a proper simply connected geodesic space, is CAT(0). According to Corollary 3.5, Σ is CAT(0) if and only if it is nonpositively curved; according to Corollary 3.7, Σ is nonpositively curved if and only if the link of each vertex is CAT(1). But the link of each vertex is isomorphic to the nerve $\mathscr{N}$ of (W, S), so in the light of Lemma 4.9, it is enough to prove that $\mathscr{N}$ is a flag complex. To do this, let $T \subseteq \operatorname{Vert}(\mathscr{N}) = S$ be a finite set of pairwise adjacent vertices. Since (W, S) is right-angled, we have $m_{s,t} = 2$ for all $s, t \in T$ with $s \neq t$. Therefore, we see that $\langle T \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{|T|}$, which is finite, so T spans a simplex in $\mathscr{N}$, and $\mathscr{N}$ is a flag complex. □

In his PhD thesis, Moussong generalised Theorem 4.11 to any Coxeter system. The method used consists in generalising Gromov's Lemma to be able to apply it to the nerve of any Coxeter system.

Theorem 4.12 (Moussong)**.** *The Davis complex of any Coxeter group is* CAT(0). *In particular, Coxeter groups are* CAT(0).

Proof. See [Mou88]. □

4.4 Solvability of the conjugacy problem for Coxeter groups

Using Theorem 4.12, a geometric result, we are going to give an answer to an algebraic question which we asked at the beginning of this report, namely the **conjugacy problem**. In Theorem 1.6, we have already seen that a Coxeter system (W, S) has a solvable **word problem**, i.e. one can decide whether two words on S represent the same element of W.

A finitely generated group Γ is said to have the **quasi-monotone conjugacy property** if for any finite generating set S, there is a constant $K \geq 1$ such that if two words $u, v \in F(S)$ are conjugate in Γ, then there exists a word $w = s_1 \cdots s_n$ with $s_i \in S \cup S^{-1}$ such that $w^{-1}uw \overset{\Gamma}{=} v$ and $\ell_S\left((s_1 \cdots s_i)^{-1} u (s_1 \cdots s_i)\right) \leq K \max\{\ell_S(u), \ell_S(v)\}$ for all $1 \leq i \leq n$. Due to Proposition 2.2, it is enough to check that there exists one finite generating set S satisfying the above condition.

Proposition 4.13. *If Γ is a* CAT(0) *group, then Γ has the quasi-monotone conjugacy property.*

Proof. Let X be a proper CAT(0) space on which Γ acts properly cocompactly by isometries. Fix $x_0 \in X$ and $r > 0$ s.t. $X = \Gamma \cdot B\left(x_0, \frac{r}{3}\right)$. Set $S = \{\gamma \in \Gamma,\ \gamma B(x_0, r+1) \cap B(x_0, r+1) \neq \varnothing\}$. According to Lemma 2.3, S is a finite generating set for Γ.

For $\gamma \in \Gamma$, let $c_\gamma : [0, \ell] \to X$ be a geodesic segment from x_0 to γx_0. For $k \in \mathbb{N}^*$ with $k < \ell$, set $w_k \in \Gamma$ such that $d(c_\gamma(k), w_k x_0) < \frac{r}{3}$, and define $w_0 = 1$ and $w_k = \gamma$ if $k \geq \ell$. Hence, for all $k \in \mathbb{N}^*$, $s_k = w_{k-1}^{-1} w_k \in S$. Define $\sigma_\gamma \overset{F(S)}{=} s_1 \cdots s_{\lceil \ell \rceil} \overset{\Gamma}{=} \gamma$.

By the Švarc-Milnor Theorem (Theorem 2.4), there are constants $\lambda \geq 1$, $\varepsilon \geq 0$ such that $\gamma \mapsto \gamma x_0$ is a (λ, ε)-quasi-isometric-embedding, as in Section 2.3.

Let $u, v \in F(S)$ such that there exists $\gamma \in \Gamma$ with $v \overset{\Gamma}{=} \gamma^{-1} u \gamma$. We consider the geodesic paths $c_\gamma : [0, \ell] \to X$ and $c'_\gamma = u c_\gamma$ joining respectively x_0 to γx_0 and $u x_0$ to $u \gamma x_0 = \gamma v x_0$. Set $w \overset{F(S)}{=} \sigma_\gamma \overset{F(S)}{=} s_1 \cdots s_m$, with $m = \lceil \ell \rceil$. We have $v \overset{\Gamma}{=} \gamma^{-1} u \gamma \overset{\Gamma}{=} w^{-1} u w$. Now let $k \in \{1, \ldots, m\}$

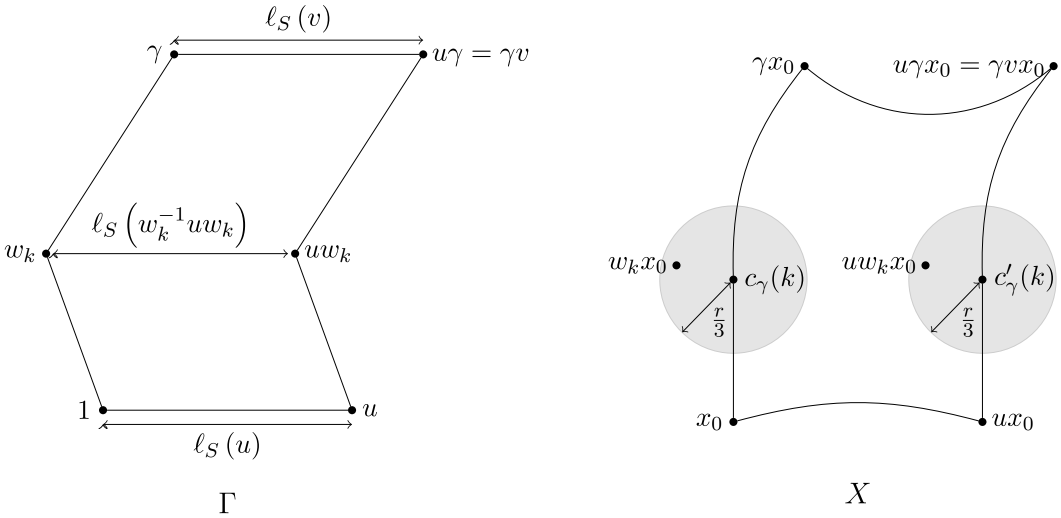

Figure 5: The quasi-monotone conjugacy property in a CAT(0) group.

and consider $w_k = s_1 \cdots s_k$. We have:

$$\begin{aligned}
\ell_S\left(w_k^{-1}uw_k\right) &= d\left(w_k, uw_k\right) \\
&\leq \lambda d\left(w_k x_0, uw_k x_0\right) + \lambda\varepsilon \\
&\leq \lambda d\left(c_\gamma(k), c'_\gamma(k)\right) + \lambda\varepsilon + \frac{2}{3}\lambda r \\
&\leq \lambda \max\left\{d\left(c_\gamma(0), c'_\gamma(0)\right), d\left(c_\gamma(\ell), c'_\gamma(\ell)\right)\right\} + \lambda\varepsilon + \frac{2}{3}\lambda r \\
&\qquad\qquad\text{because } X \text{ is CAT(0), so } d \text{ is convex} \\
&\leq \lambda^2 \max\left\{d\left(1, u\right), d\left(\gamma, u\gamma\right)\right\} + 2\lambda\varepsilon + \frac{2}{3}\lambda r \\
&= \lambda^2 \max\left\{d\left(1, u\right), d\left(\gamma, \gamma v\right)\right\} + 2\lambda\varepsilon + \frac{2}{3}\lambda r \\
&= \lambda^2 \max\left\{\ell_S\left(u\right), \ell_S\left(v\right)\right\} + 2\lambda\varepsilon + \frac{2}{3}\lambda r.
\end{aligned}$$

Now choose $K = \lambda^2 + 2\lambda\varepsilon + \frac{2}{3}\lambda r > 0$ (which is independent of u and v). We may suppose that $u \neq 1$ and $v \neq 1$. Hence:

$$\begin{aligned}
\ell_S\left(w_k^{-1}uw_k\right) &\leq \lambda^2 \max\left\{\ell_S\left(u\right), \ell_S\left(v\right)\right\} + \left(2\lambda\varepsilon + \frac{2}{3}\lambda r\right) \max\left\{\ell_S\left(u\right), \ell_S\left(v\right)\right\} \\
&= K \max\left\{\ell_S\left(u\right), \ell_S\left(v\right)\right\}.
\end{aligned}$$

□

Using the preceding lemma, we are ready to prove that Coxeter groups have a solvable conjugacy problem:

Proposition 4.14. *Let Γ be a group with a finite generating set S. If Γ has a quasi-monotone conjugacy property with a computable constant K for the set S and if Γ has a solvable word problem, then Γ has a solvable conjugacy problem.*

Proof. For $n \in \mathbb{N}$, let $V(n) = \{w \in F(S), \ \ell_S(w) \leq n\}$. We define a relation $\mathfrak{C}$ on $V(n)$ by $v_1 \mathfrak{C} v_2$ if and only if there exists $a \in S \cup S^{-1}$ such that $a^{-1}v_1 a = v_2$. The group Γ has a solvable word problem and S is finite; therefore, given $v_1, v_2 \in V(n)$, one can decide whether $v_1 \mathfrak{C} v_2$. Thus, we can algorithmically construct a finite graph $\mathcal{G}(n)$ with vertex set $V(n)$ that has an edge joining v_1 and v_2 if and only if $v_1 \mathfrak{C} v_2$.

Now fix $u, v \in F(S)$. According to the quasi-monotone conjugacy property, u and v are conjugate in Γ if and only if there exists a path from u to v in the graph $\mathcal{G}(n)$, with $n \geq K \max\{\ell_S(u), \ell_S(v)\}$. Therefore, we can decide whether u and v are conjugate in Γ. $\square$

Theorem 4.15. *Coxeter systems have a solvable conjugacy problem.*

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