ÉCOLE NORMALE SUPÉRIEURE DE LYON

Research internship report

Geometry of Coxeter groups
and applications to algebraic problems

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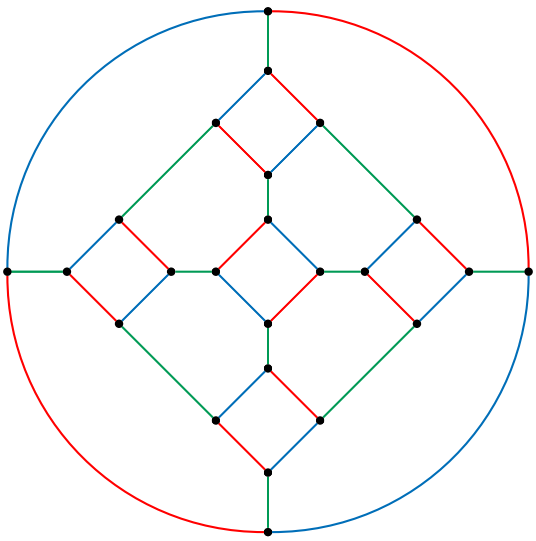
Abstract

Coxeter groups are a class of groups with a combinatorial definition. In this report, we study their geometric properties. After having introduced general concepts from geometric group theory and metric geometry, we show that Coxeter groups are CAT(0). We then show how this geometric result can lead to the solution of algebraic problems, for instance the conjugacy problem.



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Introduction

This report is the result of a six-week long research internship at the Institute for Algebra and Geometry of the Karlsruhe Institute of Technology. The aim of the internship was to learn the foundations of geometric group theory through the example of Coxeter groups. It involved learning the basics of the combinatorial theory of Coxeter groups, geometric group theory, hyperbolic geometry, metric geometry and the theory of buildings. Concretely, the work was mainly bibliographical, the two principal references being [Dav08] and [BH99]; but it also involved attending seminars, for example Gye-Seon Lee's lectures on Coxeter groups and geometry at the University of Heidelberg. We present here an outline of the work that has been done during the internship, focusing on the results leading to Moussong's Theorem, which states that Coxeter groups are CAT(0).

Contents

1 Coxeter groups from an algebraic point of view

We will start by defining Coxeter systems, and then introduce a first way of viewing a Coxeter group as a geometric object, which will give us some insight into the combinatorial structure of Coxeter groups.

1.1 Coxeter systems

In order to define a Coxeter system and a Coxeter group, we will use the vocabulary of presentations: given a set S and a set $\mathcal{R}$ of words on the alphabet $S \cup S^{-1}$, the group defined by the **presentation** $\langle S \mid \mathcal{R} \rangle$ is the quotient of the free group $F(S)$ on S by the normal subgroup $\langle\langle \mathcal{R} \rangle\rangle$ generated by $\mathcal{R}$: $\langle S \mid \mathcal{R} \rangle = F(S) / \langle\langle \mathcal{R} \rangle\rangle$.

A **Coxeter matrix** on a finite set S is a symmetric matrix $(m_{s,t})_{(s,t)\in S^2}$ with entries in $\mathbb{N}^* \cup \{\infty\}$ such that $m_{s,t} = 1$ if and only if $s = t$. A **Coxeter system** is a pair (W, S), where W is a group and S is a finite subset of W, associated to a Coxeter matrix $(m_{s,t})_{(s,t)\in S^2}$ on S, such that a presentation for W is given by:

$$W = \langle S \mid \{(st)^{m_{s,t}},\ s,t \in S,\ m_{s,t} < \infty\} \rangle .$$

If (W, S) is a Coxeter system, W will be called a **Coxeter group**. Associated to any Coxeter system (W, S), there is a **length function**: if $w \in W$, define $\ell_S(w)$ to be the length of a word of minimal length on S representing w.

Given a Coxeter system, many combinatorial questions arise naturally. For example, if s, t are two generators, we know that the order of (st) divides $m_{s,t}$, but is it *equal* to $m_{s,t}$? More generally, given two words a and b on the alphabet S, can we decide whether a and b represent the same element of W? This problem is called the **word problem**. Another problem is to decide whether the elements of W represented by a and b are conjugate; this is called the **conjugacy problem**. Using elegant geometric methods, we shall give answers to these seemingly simple combinatorial problems.

1.2 The canonical representation

The simplest examples of Coxeter groups are given by Euclidean reflection groups, i.e. subgroups of $O(n)$ generated by reflections across hyperplanes. An arbitrary Coxeter system (W, S) may not be representable in such a way, but we shall construct a linear representation of W which has similar properties. We first define the **cosine matrix** $C = (c_{s,t})_{(s,t)\in S^2}$ of (W, S) by:

$$c_{s,t} = -\cos\left(\frac{\pi}{m_{s,t}}\right),$$

where $(m_{s,t})_{(s,t)\in S^2}$ is the Coxeter matrix of (W, S) (with the convention $\cos\left(\frac{\pi}{\infty}\right) = \cos(0) = 1$). We then define B to be the symmetric bilinear form on $V = \mathbb{R}^S$ whose matrix in the canonical basis $(e_s)_{s\in S}$ is C. For $s \in S$, consider $H_s = \operatorname{Ker} B(e_s, \cdot)$ (this is the hyperplane orthogonal to e_s relative to B). As e_s is anisotropic, we have $V = \mathbb{R}e_s \oplus H_s$. Therefore, we can define σ_s to be the reflection through H_s parallel to $\mathbb{R}e_s$ (i.e. $\sigma_s(x) = x - 2B(e_s, x)\, e_s$ for $x \in V$); it is clear that σ_s preserves B. In order to be able to define a group homomorphism $W \to GL(V)$, we need to check firstly that the relations defining W are carried into $GL(V)$ by the map $s \mapsto \sigma_s$:

Lemma 1.1. *Let (W, S) be a Coxeter system. With the above notations, $\sigma_s\sigma_t$ has order $m_{s,t}$ for all $s, t \in S$.*

Proof. See [Hum90], Section 5.3. □

In particular, we have $(\sigma_s\sigma_t)^{m_{s,t}} = \mathrm{id}_V$ for all $s, t \in S$. Therefore, the homomorphism $F(S) \to GL(V)$ given by $s \mapsto \sigma_s$ induces a homomorphism $\sigma : W \to GL(V)$. This homomorphism is called the **canonical representation** of (W, S). We shall see σ as an action of W on V and write $w(x)$ instead of $(\sigma(w))(x)$ for $w \in W$ and $x \in V$.

Define $\Phi = \{w(e_s),\ w \in W,\ s \in S\} \subseteq V$. The set Φ is called a **root system** for (W, S) and its elements are called **roots**. A root $\alpha \in \Phi$ is said to be **positive** (resp. **negative**), which shall be denoted by $\alpha > 0$ (resp. $\alpha < 0$), if all its coefficients in the basis $(e_s)_{s\in S}$ are nonnegative (resp. nonpositive). There is a strong link between the length function and the sign of roots:

Proposition 1.2. *Let (W, S) be a Coxeter system. For $w \in W$ and $s \in S$, we have:*

(i) *If* $\ell_S(ws) > \ell_S(w)$*, then* $w(e_s) > 0$.

(ii) *If* $\ell_S(ws) < \ell_S(w)$*, then* $w(e_s) < 0$.

Proof. See [Hum90], Section 5.4. □

Theorem 1.3. *The canonical representation of a Coxeter system is faithful.*

Proof. Let (W, S) be a Coxeter system and $\sigma : W \to GL(V)$ be its canonical representation. If $w \in W \backslash \{1\}$, then $\ell_S(w) \geq 1$. By choosing $s \in S$ to be the last letter of a minimal word on S representing w, we have $\ell_S(ws) < \ell_S(w)$, which implies (thanks to Proposition 1.2), that $w(e_s) < 0$. But $e_s > 0$, so $w(e_s) \neq e_s$. Hence, $\sigma(w) \neq \mathrm{id}_V$. □

1.3 Combinatorial consequences

Proposition 1.4. *Let* (W, S) *be a Coxeter system with Coxeter matrix* $(m_{s,t})_{(s,t)\in S^2}$*. For* $s, t \in S$*, the order of* (st) *is* $m_{s,t}$*.*

Proof. Let $\sigma : W \to GL(V)$ be the canonical representation of (W, S). We know that the order of (st) divides $m_{s,t}$. But if there existed $1 \leq k < m_{s,t}$ such that $(st)^k = 1$, we would have $(\sigma(s)\sigma(t))^k = \sigma\left((st)^k\right) = \mathrm{id}_V$, which is impossible according to Lemma 1.1. □

Using Proposition 1.2, we shall prove that Coxeter systems have a solvable word problem. To do this, consider the dual $\sigma^* : W \to GL(V^*)$ of the canonical representation $\sigma : W \to GL(V)$. To each generator $s \in S$ we associate two half spaces of V^* defined by:

$$A_s^+ = \{f \in V^*,\ \langle f, e_s\rangle > 0\} \qquad \text{and} \qquad A_s^- = \{f \in V^*,\ \langle f, e_s\rangle < 0\}.$$

We also define the **fundamental chamber** of (W, S) by $C = \bigcap_{s\in S} A_s^+$. The following lemma describes the action of W on the fundamental chamber:

Lemma 1.5. *Let* (W, S) *be a Coxeter system. For* $w \in W$ *and* $s \in S$*, we have:*

(i) *If* $\ell_S(sw) > \ell_S(w)$*, then* $w(C) \subseteq A_s^+$.

(ii) *If* $\ell_S(sw) < \ell_S(w)$*, then* $w(C) \subseteq A_s^-$.

Proof. This is merely a rewriting of Proposition 1.2 (note that ws has been replaced by sw because we are now working with the dual of σ). □

Theorem 1.6. *Coxeter systems have a solvable word problem.*

Proof. Let (W, S) be a Coxeter system. Consider the canonical representation $\sigma : W \to GL(V)$ and its dual representation $\sigma^* : W \to GL(V^*)$. Write $(e_s^*)_{s\in S}$ for the dual basis of $(e_s)_{s\in S}$. Consider $f = \sum_{s\in S} e_s^* \in C$. We are going to show that the stabiliser $\mathrm{Stab}(f)$ of f in W is trivial. Therefore, in order to determine whether a word $s_1 \cdots s_k \in F(S)$ represents the identity in W, one only needs to compute $s_1 \cdots s_k(f)$: the result is f if and only if $s_1 \cdots s_k \overset{W}{=} 1$.

It remains to prove that $\mathrm{Stab}(f) = \{1\}$. Let $w \in \mathrm{Stab}(f)$. Note that, because of Lemma 1.5, we have $w(C) \subseteq \bigcap_{s\in S} A_s^{\varepsilon_s}$, where $(\varepsilon_s)_{s\in S} \in \{\pm 1\}^S$. But since $f = w(f) \in C \cap w(C)$, it follows that $\varepsilon_s = +1$ for all $s \in S$. Applying Lemma 1.5 again, we see that $\ell_S(sw) > \ell_S(w)$ for all $s \in S$, which implies that $w = 1$. □

2 Elements of geometric group theory

Geometric group theory consists in studying groups by making them operate on interesting topological or metric spaces. We will endow a group with a geometric structure, and we shall study the relations between this structure and spaces on which the group operates. This will lead us to first examples of algebraic properties originating from geometric methods.

2.1 Cayley graphs and word metrics

Let Γ be a group with a generating set S. We define the **Cayley graph** $\mathrm{Cay}(\Gamma, S)$ as the oriented graph whose vertices are elements of Γ and where there is an edge labeled by $s \in S$ going from $g \in \Gamma$ to $h \in \Gamma$ precisely when $h = gs$. If s is of order 2, note that there is an s-edge from g to h if and only if there is an s-edge from h to g; in this case, we shall only draw one undirected edge between g and h rather than two directed edges, as in Figure 1 (this is always the case for Coxeter systems). The Cayley graph is endowed with the structure of a **metric graph** by defining the length of each edge to be 1; this metric is locally well-defined, and the distance between two arbitrary points is taken to be the length of a shortest path joining these two points. The following remark will be fundamental for working with Cayley graphs:

Remark. *A path between two vertices in the Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *corresponds to a word on* $S \cup S^{-1}$*. Moreover, this correspondence preserves lengths.*

As S generates Γ, the preceding remark shows that $\mathrm{Cay}(\Gamma, S)$ is connected. Moreover, for any element $g \in \Gamma$, the distance between 1 and g is equal to the length of a minimal word $(s_1, \ldots, s_k)$ on $S \cup S^{-1}$ such that $g = s_1 \cdots s_k$. This is the **length** of g relative to S, denoted by $\ell(g)$ or $\ell_S(g)$ (note that this notion of length is the same as for Coxeter systems). Moreover, the restriction to Γ of the metric on $\mathrm{Cay}(\Gamma, S)$ will be called the **word metric** on Γ (relative to S).

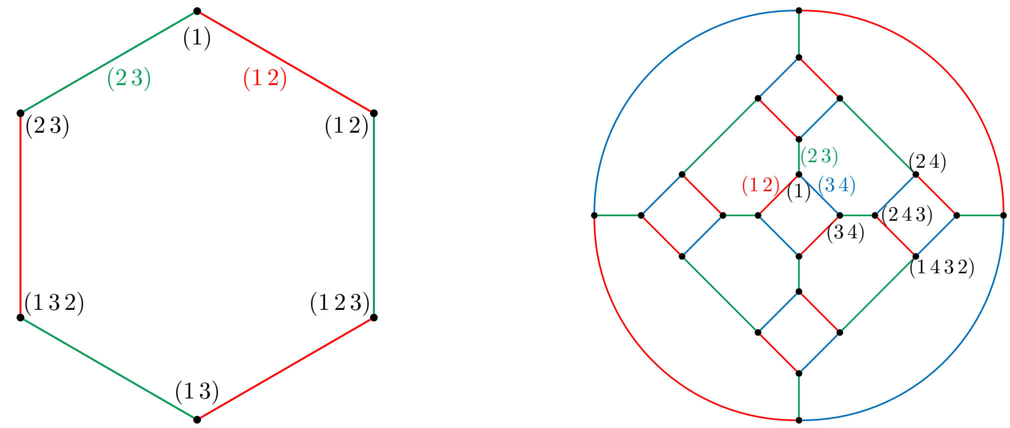

Figure 1: The Cayley graphs of $(\mathfrak{S}_3, \{(1\,2), (2\,3)\})$ and $(\mathfrak{S}_4, \{(1\,2), (2\,3), (3\,4)\})$.

There is a natural geometric action of Γ on $\mathrm{Cay}(\Gamma, S)$. Here is some vocabulary which shall be used to describe the properties of this action: if a group Γ acts by homeomorphisms on a topological space X, we shall say that the action is **proper** if for any compact set $K \subseteq X$, the set $\{\gamma \in \Gamma,\ \gamma K \cap K \neq \varnothing\}$ is finite; and we shall say that the action is **cocompact** if there exists a compact set $K \subseteq X$ such that $X = \Gamma \cdot K$.

Proposition 2.1. *Let* Γ *be a group with a generating set* S.

(i) *The left action of* Γ *on* Γ *by translation induces an action of* Γ *on* $\mathrm{Cay}(\Gamma, S)$ *by isometries.*

(ii) *If* S *is finite, the action of* Γ *on* $\mathrm{Cay}(\Gamma, S)$ *is proper and cocompact.*

2.2 Geodesic spaces and length spaces

The central idea of what follows will be to compare paths in the Cayley graph of (Γ, S) (which correspond to words on $S \cup S^{-1}$) to paths in a metric space X on which Γ acts geometrically. Therefore, it will be necessary to add an additional hypothesis on X, which will ensure that paths in X are well-behaved.

If X is a metric space and $c : [a, b] \to X$ is a path in X, the **length** of c is defined by:

$$\ell(c) = \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} d\left(c\left(t_j\right), c\left(t_{j+1}\right)\right) \geq d\left(c(a), c(b)\right).$$

The space X is said to be a **length space** (resp. a **geodesic space**) if for any two points $x, y \in X$, the distance between x and y is the *infimum* (resp. the *minimum*) of the lengths of paths from x to y (as a consequence, X is path-connected). A path c is said to be a **geodesic path** if $d(c(s), c(t)) = |s - t|$ for all $s, t \in [a, b]$. It is said to be a **linearly reparametrised geodesic** if there exists $\lambda > 0$ such that $t \mapsto c(\lambda t)$ is a geodesic path. The space X is geodesic if and only if any two points of X are joined by a geodesic path. If this geodesic path is always unique, X is said to be **uniquely geodesic**.

Remark. *All metric graphs are geodesic spaces; in particular, Cayley graphs are geodesic spaces.*

2.3 Quasi-isometry

The problem we now face is that, given a group Γ, there may be several different generating sets S leading to different Cayley graphs and different word metrics. For example, $\mathrm{Cay}(\mathbb{Z}, \{1\})$ is not isometric to $\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$. To get round this problem, we define the notion of quasi-isometry. If X and Y are metric spaces, a map $f : X \to Y$ is called a **quasi-isometric embedding** if there exist $\lambda \geq 1$ and $\varepsilon \geq 0$ such that the following inequalities hold for all $x, x' \in X$:

$$\frac{1}{\lambda} d(x, x') - \varepsilon \leq d(f(x), f(x')) \leq \lambda d(x, x') + \varepsilon.$$

The map f is called a **quasi-isometry** if it is a quasi-isometric embedding and has a **quasi-inverse**, i.e. a map $g : Y \to X$ such that $d(g \circ f, \mathrm{id}_X) < \infty$ and $d(f \circ g, \mathrm{id}_Y) < \infty$. In this case, g is also a quasi-isometric embedding, and the spaces X and Y are said to be **quasi-isometric**. Therefore, the quasi-isometry of metric spaces is an equivalence relation.

Proposition 2.2. *Let* Γ *be a* finitely generated *group. Consider* S *and* S' *two* finite *generating sets for* G*, with associated word metrics* d_S *and* $d_{S'}$*. Then the metric spaces* (Γ, d_S) *and* $(\Gamma, d_{S'})$ *are quasi-isometric. Therefore,* Γ *has a unique natural metric structure up to quasi-isometry.*

Proof. We will show that $\mathrm{id}_\Gamma : (\Gamma, d_S) \to (\Gamma, d_{S'})$ is a quasi-isometric embedding, which will imply that it is a quasi-isometry because it has an inverse map, namely $\mathrm{id}_\Gamma : (\Gamma, d_{S'}) \to (\Gamma, d_S)$. With $\lambda = \max\left\{\max_{s\in S} \ell_{S'}(s), \max_{s'\in S'} \ell_S(s')\right\}$, we have $\frac{1}{\lambda}\ell_S(g) \leq \ell_{S'}(g) \leq \lambda \ell_S(g)$ for $g \in \Gamma$. □

2.4 The fundamental observation of geometric group theory

The following lemma is very general, and tells us how to obtain a generating set for a group given a geometric action of this group on a topological space.

Lemma 2.3. *Let* Γ *be a group. Suppose* Γ *acts by homeomorphisms on a connected topological space* X *and consider an open subset* $\mathcal{U} \subseteq X$ *such that* $X = \Gamma\cdot\mathcal{U}$*. Then* $S = \{\gamma \in \Gamma,\ \gamma\mathcal{U} \cap \mathcal{U} \neq \varnothing\}$ *is a generating set for* Γ*.*

Proof. Consider $V = \langle S\rangle\cdot\mathcal{U}$ and $V' = (\Gamma \setminus \langle S\rangle)\cdot\mathcal{U}$. Due to the definition of S, we have $V \cap V' = \varnothing$. Therefore, V and V' are disjoint open subsets of X, and $X = V \sqcup V'$. Moreover, $V \supseteq \mathcal{U} \supsetneq \varnothing$, and X is connected, so $V = X$ and $V' = \varnothing$, which implies that $\Gamma \setminus \langle S\rangle = \varnothing$. □

The following theorem is sometimes called *the fundamental observation of geometric group theory*; it helps us understand the relation between a group and a metric space on which it acts. We call a metric space **proper** if its closed balls are compact.

Theorem 2.4 (Švarc-Milnor)**.** *Let Γ be a group. Suppose Γ acts properly cocompactly by isometries on a proper length space X. Then:*

(i) *Γ is finitely generated.*

(ii) *For all $x_0 \in X$, the map $g \in \Gamma \longmapsto gx_0 \in X$ is a quasi-isometry between Γ and X.*

Proof. As the action of Γ on X is cocompact, let $x_0 \in X$ and $r > 0$ such that $X = \Gamma \cdot B(x_0, r)$. Define $S = \{\gamma \in \Gamma,\ \gamma B(x_0, 3r) \cap B(x_0, 3r) \neq \varnothing\}$. Lemma 2.3 guarantees that S is a (finite) generating set for Γ. We then equip Γ with its length metric relative to S and we prove that the map $f : g \in \Gamma \longmapsto gx_0 \in X$ is a quasi-isometry. Constructing a quasi-inverse is easy using the fact that $X = \Gamma \cdot B(x_0, r)$. To prove that it is a quasi-isometric embedding, one uses the facts that (1) a word on S representing an element $g \in \Gamma$ gives an upper-bound on the distance $d_X(x_0, gx_0)$ and (2) given a path c from x_0 to gx_0 in X, one can construct a word on S representing g by choosing elements $g_i \in \Gamma$ such that $g_i x_0$ is not too far from c, as in Figure 2. In both cases, lengths of paths in X and lengths of words on S are closely related. ☐

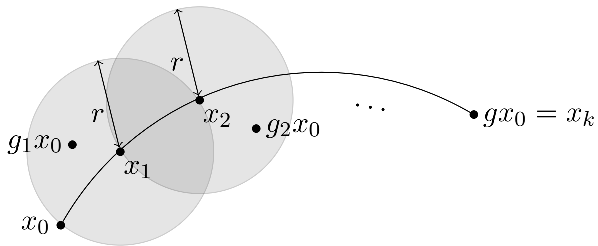

Figure 2: Proof of the Švarc-Milnor Theorem.

Corollary 2.5. *If Γ is a group with a finite generating set S, Γ is quasi-isometric to* $\mathrm{Cay}(\Gamma, S)$.

To illustrate the depth of the Švarc-Milnor Theorem, here is a purely algebraic property, proved using geometric ideas:

Proposition 2.6. *If a group Γ is finitely generated, then every subgroup $H \leq \Gamma$ of finite index is also finitely generated.*

Proof. Take S a finite set of generators for Γ. We know that Γ acts properly cocompactly by isometries on $\mathrm{Cay}(\Gamma, S)$, which is a proper length space. The induced action of H on $\mathrm{Cay}(\Gamma, S)$ is proper and by isometries. It is also cocompact because H has finite index in Γ. Therefore, the Švarc-Milnor Theorem guarantees that H is finitely generated. ☐

3 Nonpositive curvature

We are going to define a notion of curvature for a metric space X. To do this, the idea will be to compare triangles in X with triangles in a well-known space in which we already have a notion of curvature (given by differential geometry, for instance). We will see later that information about the curvature of a space on which a group acts translates into algebraic properties of the group itself.

3.1 CAT(0) and CAT(1) spaces

Informally, CAT(0) spaces are metric spaces in which triangles are thinner than in the **Euclidean plane** $\mathbb{E}^2$, equipped with the standard scalar product and the induced distance $d_{\mathbb{E}^2}$. Likewise, CAT(1) spaces are spaces in which triangles are thinner than in the **2-sphere** $\mathbb{S}^2$, equipped with the distance $d_{\mathbb{S}^2}$ defined by $d_{\mathbb{S}^2}(A,B) = \arccos\left(\langle A,B\rangle_{\mathbb{R}^3}\right)$ for $A,B\in\mathbb{S}^2$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ is the Euclidean scalar product on $\mathbb{R}^3$. These spaces have the following properties:

Proposition 3.1. (i) *$\mathbb{E}^2$ is a uniquely geodesic space and for all $(\ell_1,\ell_2,\ell_3)\in(\mathbb{R}_+)^3$ satisfying the triangle inequality, there exists a triangle in $\mathbb{E}^2$ whose edges have lengths ℓ_1,ℓ_2,ℓ_3.*

(ii) *$\mathbb{S}^2$ is a geodesic space and any pair of points a distance less than π apart can be joined by a unique geodesic path. For all $(\ell_1,\ell_2,\ell_3)\in(\mathbb{R}_+)^3$ satisfying the triangle inequality and such that $\ell_1+\ell_2+\ell_3<2\pi$, there exists a triangle in $\mathbb{S}^2$ whose edges have lengths ℓ_1,ℓ_2,ℓ_3.*

A **geodesic triangle** Δ in a metric space X consists of three points x_1,x_2,x_3 together with three geodesic paths $[x_1,x_2],[x_2,x_3],[x_3,x_1]$. According to Proposition 3.1, there always exists a geodesic triangle $\overline{\Delta}$ with vertices $\overline{x}_1,\overline{x}_2,\overline{x}_3$ in $\mathbb{E}^2$ such that $d_X(x_i,x_j) = d_{\mathbb{E}^2}(\overline{x}_i,\overline{x}_j)$ for all i,j. Once we have the triangle $\overline{\Delta}$, we can construct comparison points for any point on edges of Δ by choosing the comparison point for $p\in[x_i,x_j]$ to be the unique point $\overline{p}\in[\overline{x}_i,\overline{x}_j]$ such that $d_X(p,x_i) = d_{\mathbb{E}^2}(\overline{p},\overline{x}_i)$. We shall say that Δ satisfies the CAT(0) **inequality** if $d_X(p,q)\le d_{\mathbb{E}^2}(\overline{p},\overline{q})$ for all $p,q\in\Delta$. Likewise, if the perimeter of Δ is less than 2π, there exists a comparison triangle $\overline{\Delta}$ in $\mathbb{S}^2$ and one can define the CAT(1) inequality in the same manner.

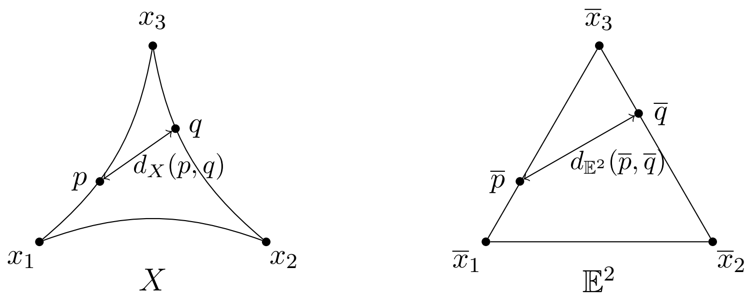

Figure 3: The CAT(0) inequality.

The space X is said to be a CAT(0) **space** if it is a geodesic space such that any geodesic triangle satisfies the CAT(0) inequality; X is said to be a CAT(1) space if any two points a distance less than π apart can be joined by a geodesic path and if any triangle of perimeter less than 2π satisfies the CAT(1) inequality. Moreover, X is said to be **nonpositively curved** if it is locally a CAT(0) space (i.e. for $x\in X$, there exists $r>0$ such that $B(x,r)$ is CAT(0)).

Remark. *The Euclidean plane $\mathbb{E}^2$ is a* CAT(1) *space. Therefore, any* CAT(0) *space is* CAT(1).

3.2 Properties of CAT(0) spaces

Proposition 3.2. *Let X be a* CAT(0) *space.*

(i) *X is uniquely geodesic.*

(ii) *The distance on X is convex, i.e. for any pair of linearly reparametrised geodesics $c,c':[0,1]\to X$, the following inequality holds for all $t\in[0,1]$:*

$$d\left(c(t),c'(t)\right)\le(1-t)\cdot d\left(c(0),c'(0)\right)+t\cdot d\left(c(1),c'(1)\right).$$

(iii) *X is contractible.*

3.3 The Cartan-Hadamard Theorem

In what follows, we shall need to be able to deduce that a space is CAT(0) using the fact that it is **nonpositively curved**, i.e. locally CAT(0). Our tool for going from local to global will be the metric version of the Cartan-Hadamard Theorem:

Theorem 3.3 (Cartan-Hadamard)**.** *Let X be a complete connected metric space.*

(i) *If the metric on X is locally convex, then the induced length metric on the universal covering $\widetilde{X}$ is convex.*

(ii) *If X is nonpositively curved, then $\widetilde{X}$ (with the induced length metric) is* CAT(0)*.*

Proof. See [BH99], Chapter II.4. □

Corollary 3.4. *Let X be a complete simply-connected geodesic space. If X is nonpositively curved, then X is* CAT(0)*.*

3.4 Berestovskii's Theorem

For the last part of this report, it will be necessary to understand the curvature of cell complexes. Berestovskii's Theorem, which supplies information about the curvature of the **cone** $\mathrm{Cone}(Y)$ over a metric space Y, will be our main tool in order to achieve this aim. To define $\mathrm{Cone}(Y)$, consider the quotient of $[0,\infty[\times Y$ by the equivalence relation which consists in identifying the points whose first coordinate is 0, and equip it with the metric defined by $d_{\mathrm{Cone}(Y)}\left((t,y),(t',y')\right)^2 = t^2 + {t'}^2 - 2tt'\cos\left(\min\left\{\pi, d_Y\left(y,y'\right)\right\}\right)$.

Theorem 3.5 (Berestovskii)**.** *A metric space Y is* CAT(1) *if and only if* $\mathrm{Cone}(Y)$ *is* CAT(0)*.*

Proof. See [BH99], Theorem II.3.14. □

If L is a Euclidean cell complex and v is a vertex of L, we define the **link** $\mathrm{Lk}(v,L)$ of v in L to be the set of unit vectors at v that point into L. The link $\mathrm{Lk}(v,L)$ has the structure of a spherical cell complex. If L has only *finitely many isometry types of cells*, then there exists an open ball centred at v in L which is isometric to an open ball centred at v in $\mathrm{Cone}\left(\mathrm{Lk}\left(v,L\right)\right)$. This leads to the following result:

Corollary 3.6. *A Euclidean cell complex L with only finitely many isometry types of cells has nonpositive curvature if and only if the link of every vertex of L is a* CAT(1) *space.*

4 Coxeter groups from a geometric point of view

The aim of this report's last part will be, given a Coxeter group W, to construct a Euclidean CAT(0) cell complex on which W acts properly cocompactly by isometries. Remarkably, this fact has many algebraic applications. In particular, knowing that Coxeter groups act on CAT(0) spaces will enable us to solve their conjugacy problem, i.e. to decide whether or not two words on the set of generators represent conjugate elements of W.

4.1 The Davis complex

Let (W,S) be a Coxeter system. A subset $T \subseteq S$ is said to be **spherical** if it generates a finite subgroup of W. In this case, the subgroup $\langle T\rangle$ is called a **spherical subgroup** and its (left) cosets are called **spherical cosets**. Write $\mathscr{S}$ for the set of spherical subsets of S and define the **nerve** $\mathscr{N}$ of (W,S) to be the poset $\mathscr{S}\backslash\{\varnothing\}$, ordered by inclusion; the nerve will be considered as an abstract simplicial complex.

Consider the poset $W\mathscr{S}$ of all spherical cosets of (W, S), ordered by inclusion. The **Davis complex** Σ is defined as the **flag complex** of the poset $W\mathscr{S}$, i.e. the abstract simplicial complex whose set of vertices is $W\mathscr{S}$ and where a subset $\Delta \subseteq W\mathscr{S}$ spans a simplex if and only if it is totally ordered. Here is a first observation aiming to understand what the flag complex of a poset is:

Remark. *Let Λ be a convex cell complex (i.e. a complex whose cells are convex polytopes). If $\mathcal{P}$ is the poset of cells of Λ, then the flag complex of $\mathcal{P}$ is the barycentric subdivision of Λ.*

We are now going to relate Σ to a geometric object. Suppose for the moment that W is finite. Recall the definition of the **fundamental chamber** associated to (W, S) from Section 1.3. Define a **Coxeter polytope** of (W, S) to be the convex hull of an orbit Wx, where x is any point of the fundamental chamber. As shown by the following proposition, Coxeter polytopes are closely related to the Davis complex:

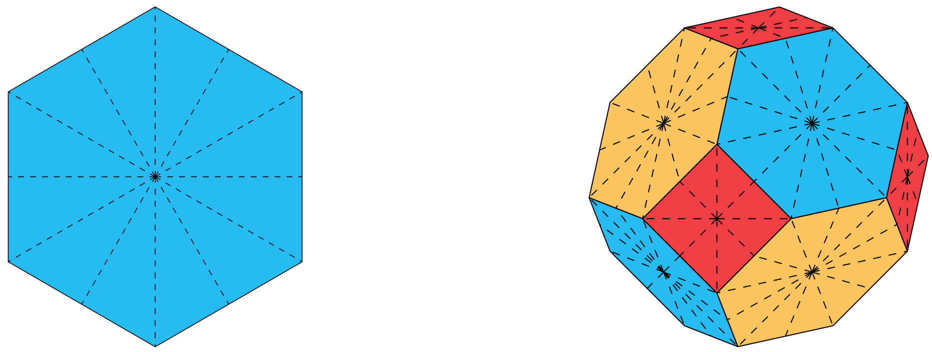

Figure 4: Coxeter polytopes of $(\mathfrak{S}_3, \{(1\,2), (2\,3)\})$ and $(\mathfrak{S}_4, \{(1\,2), (2\,3), (3\,4)\})$, with their barycentric subdivision.

Proposition 4.1. *Let (W, S) be a Coxeter system, with W finite. Write $\mathcal{C}_x$ for the Coxeter polytope of (W, S) associated to a point x lying in the fundamental chamber. Then the map $w \mapsto wx$ induces an isomorphism of posets between $W\mathscr{S}$ and the poset of faces of $\mathcal{C}_x$.*

Proof. See [Dav08], Lemma 7.3.3. □

Corollary 4.2. *Let (W, S) be a Coxeter system, with W finite. Then the Davis complex Σ of (W, S) is the barycentric subdivision of any Coxeter polytope of (W, S).*

We do not suppose anymore that W is finite. If $T \subseteq S$ is a spherical subset, note that $(\langle T \rangle, T)$ is a Coxeter system and the poset $(W\mathscr{S})_{\leq w\langle T \rangle}$ is isomorphic to $\langle T \rangle \mathscr{S}_{\leq T}$ for all $w \in W$. According to Corollary 4.2, the flag complex of $(W\mathscr{S})_{\leq w\langle T \rangle}$ is the barycentric subdivision of a Coxeter polytope associated to $\langle T \rangle$. Therefore, we can equip Σ with a new cell structure, its **natural cell structure**, where each cell is a Coxeter polytope. With this new structure, the vertex set of Σ is W and its 1-skeleton is $\mathrm{Cay}(W, S)$. We observe that the 2-skeleton of Σ is the complex formed by attaching a 2-cell to each loop in $\mathrm{Cay}(W, S)$ corresponding to a relation given by the Coxeter matrix. This leads to the following proposition:

Lemma 4.3. *Let (W, S) be a Coxeter system and write $\mathcal{R} = \{(st)^{m_{s,t}},\ s, t \in S,\ m_{s,t} < \infty\} \subseteq F(S)$ for the set of relations given by the Coxeter matrix. Then $\pi_1(\mathrm{Cay}(W, S)) \simeq \langle\langle \mathcal{R} \rangle\rangle \trianglelefteq F(S)$.*

Proposition 4.4. *The Davis complex Σ of a Coxeter system (W, S) is simply connected.*

Thanks to the natural cell structure on Σ, it is possible to define a metric on Σ by specifying a metric for each Coxeter polytope. Coxeter polytopes are subsets of a real finite-dimensional vector space, so they naturally come with the Euclidean metric; but the question remains of the choice of the Coxeter polytope, because the Coxeter polytopes associated to different points of the fundamental chamber are not isometric. It is not difficult to see that, if (W,S) is a Coxeter system with W finite, given a choice of positive real numbers $(\ell_s)_{s\in S}$ indexed by S, one can find a point x in the fundamental chamber (by solving a linear system) such that, in the Coxeter polytope associated to x, each s-edge has length ℓ_s. For instance there exists a Coxeter polytope, unique up to isometry, such that each edge has length 1; this defines a **canonical Coxeter polytope** for any Coxeter system (W,S), with W finite. If (W,S) is any Coxeter system (where W may be infinite), we now equip its Davis complex Σ with the length metric defined so as to extend the metric structure defined on Coxeter polytopes. The following proposition will be useful for studying the curvature of the Davis complex:

Proposition 4.5. *Let (W,S) be a Coxeter system. The link of each vertex of the Davis complex Σ is isomorphic (as an abstract simplicial complex) to the nerve $\mathscr{N}$.*

4.2 Gromov's Lemma

Thanks to Corollaries 3.4 and 3.6, we see that, under certain hypotheses, in order to prove that a Euclidean complex is CAT(0), one only needs to prove that its link, a spherical complex, is CAT(1). Our main tool to do this will be Gromov's Lemma, which gives a combinatorial condition for spherical complexes to be CAT(1). A simplicial complex is said to be a **flag complex** if any finite set of pairwise adjacent vertices spans a simplex. A spherical complex is said to be **all-right** if each edge has length $\frac{\pi}{2}$.

Lemma 4.6 (Gromov)**.** *Let L be a finite dimensional all-right spherical complex. If L is a flag complex, then it is* CAT(1)*.*

Proof. See [BH99], Theorem II.5.18. □

4.3 Coxeter groups are CAT(0)

We now reach this report's main theorem. A group is said to be CAT(0) if it acts properly cocompactly by isometries on a proper CAT(0) space (according to Theorem 2.4, this implies that it is finitely generated and quasi-isometric to a CAT(0) space). Firstly, we make the following observation:

Proposition 4.7. *Let (W,S) be a Coxeter system and let Σ be its Davis complex. Then W acts on Σ properly cocompactly by isometries.*

We can prove that a special class of Coxeter groups are CAT(0):

Theorem 4.8. *Let (W,S) be a right-angled Coxeter system, i.e. a Coxeter system such that the Coxeter matrix has entries in $\{1,2,\infty\}$. Then W is* CAT(0)*.*

Proof. It suffices to prove that the Davis complex Σ, a proper simply connected geodesic space, is CAT(0). According to Corollary 3.4, Σ is CAT(0) if and only if it is nonpositively curved; according to Corollary 3.6, Σ is nonpositively curved if and only if the link of each vertex is CAT(1). But the link of each vertex is isomorphic to the nerve $\mathscr{N}$ of (W,S), so in the light of Lemma 4.6, it is enough to prove that $\mathscr{N}$ is a flag complex. To do this, let $T\subseteq \mathrm{Vert}\,(\mathscr{N})=S$ be a finite set of pairwise adjacent vertices. Since (W,S) is right-angled, we have $m_{s,t}=2$ for all $s,t\in T$ with $s\neq t$. Therefore, we see that $\langle T\rangle\simeq(\mathbb{Z}/2\mathbb{Z})^{|T|}$, which is finite, so T spans a simplex in $\mathscr{N}$, and $\mathscr{N}$ is a flag complex. □

In his PhD thesis, Moussong generalised Theorem 4.8 to any Coxeter system. The method used consists in generalising Gromov's Lemma to be able to apply it to the nerve of any Coxeter system.

Theorem 4.9 (Moussong)**.** *The Davis complex of any Coxeter group is* CAT(0)*. In particular, Coxeter groups are* CAT(0)*.*

Proof. See [Mou88]. □

4.4 Solvability of the conjugacy problem for Coxeter groups

Using Theorem 4.9, a geometric result, we are going to give an answer to an algebraic question which we asked at the beginning of this report, namely the **conjugacy problem**. In Theorem 1.6, we have already seen that a Coxeter system (W, S) has a solvable **word problem**, i.e. one can decide whether two words on S represent the same element of W.

A finitely generated group Γ is said to have the **quasi-monotone conjugacy property** if for any finite generating set S, there is a constant $K \geq 1$ such that if two words $u, v \in F(S)$ are conjugate in Γ, then there exists a word $w = s_1 \cdots s_n$ with $s_i \in S \cup S^{-1}$ such that $w^{-1}uw \overset{\Gamma}{=} v$ and $\ell_S\left((s_1 \cdots s_i)^{-1} u (s_1 \cdots s_i)\right) \leq K \max\{\ell_S(u), \ell_S(v)\}$ for all $1 \leq i \leq n$. Due to Proposition 2.2, it is enough to check that there exists one finite generating set S satisfying the above condition.

Proposition 4.10. *If* Γ *is a* CAT(0) *group, then* Γ *has the quasi-monotone conjugacy property.*

Proof. Let X be a proper CAT(0) space on which Γ acts properly cocompactly by isometries. Fix $x_0 \in X$ and $r > 0$ s.t. $X = \Gamma \cdot B\left(x_0, \frac{r}{3}\right)$. Set $S = \{\gamma \in \Gamma,\ \gamma B(x_0, r+1) \cap B(x_0, r+1) \neq \varnothing\}$. According to Lemma 2.3, S is a finite generating set for Γ.

For $\gamma \in \Gamma$, let $c_\gamma : [0, \ell] \to X$ be a geodesic segment from x_0 to γx_0. For $k \in \mathbb{N}^*$ with $k < \ell$, set $w_k \in \Gamma$ such that $d(c_\gamma(k), w_k x_0) < \frac{r}{3}$, and define $w_0 = 1$ and $w_k = \gamma$ if $k \geq \ell$. Hence, for all $k \in \mathbb{N}^*$, $s_k = w_{k-1}^{-1} w_k \in S$. Define $\sigma_\gamma \overset{F(S)}{=} s_1 \cdots s_{\lceil \ell \rceil} \overset{\Gamma}{=} \gamma$.

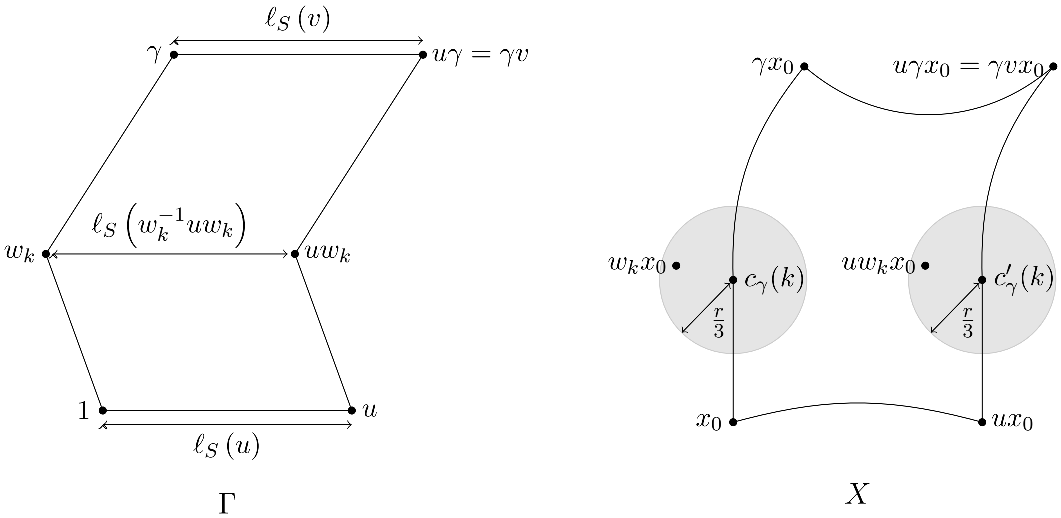

Figure 5: The quasi-monotone conjugacy property in a CAT(0) group.

By the Švarc-Milnor Theorem (Theorem 2.4), there are constants $\lambda \geq 1$, $\varepsilon \geq 0$ such that $\gamma \mapsto \gamma x_0$ is a (λ, ε)-quasi-isometric-embedding, as in Section 2.3.

Let $u, v \in F(S)$ such that there exists $\gamma \in \Gamma$ with $v \overset{\Gamma}{=} \gamma^{-1} u \gamma$. We consider the geodesic paths $c_\gamma : [0, \ell] \to X$ and $c'_\gamma = u c_\gamma$ joining respectively x_0 to γx_0 and $u x_0$ to $u\gamma x_0 = \gamma v x_0$. Set

$w \overset{F(S)}{=} \sigma_\gamma \overset{F(S)}{=} s_1 \cdots s_m$, with $m = \lceil \ell \rceil$. We have $v \overset{\Gamma}{=} \gamma^{-1} u \gamma \overset{\Gamma}{=} w^{-1} u w$. Now let $k \in \{1, \ldots, m\}$ and consider $w_k = s_1 \cdots s_k$. We have:

$$\begin{aligned} \ell_S\left(w_k^{-1} u w_k\right) &\leq \lambda d\left(w_k x_0, u w_k x_0\right) + \lambda\varepsilon \leq \lambda d\left(c_\gamma(k), c'_\gamma(k)\right) + \lambda\varepsilon + \frac{2}{3}\lambda r \\ &\leq \lambda \max\left\{d\left(c_\gamma(0), c'_\gamma(0)\right), d\left(c_\gamma(\ell), c'_\gamma(\ell)\right)\right\} + \lambda\varepsilon + \frac{2}{3}\lambda r \\ &\qquad \text{because } X \text{ is CAT(0), so } d \text{ is convex} \\ &\leq \lambda^2 \max\left\{\ell_S(u), \ell_S(v)\right\} + 2\lambda\varepsilon + \frac{2}{3}\lambda r. \end{aligned}$$

□

Using the preceding lemma, we are ready to prove that Coxeter groups have a solvable conjugacy problem:

Proposition 4.11. *Let Γ be a group with a finite generating set S. If Γ has a quasi-monotone conjugacy property with a computable constant K for the set S and if Γ has a solvable word problem, then Γ has a solvable conjugacy problem.*

Proof. For $n \in \mathbb{N}$, let $V(n) = \{w \in F(S), \ \ell_S(w) \leq n\}$. We define a relation $\mathfrak{C}$ on $V(n)$ by $v_1 \mathfrak{C} v_2$ if and only if there exists $a \in S \cup S^{-1}$ such that $a^{-1} v_1 a = v_2$. The group Γ has a solvable word problem and S is finite; therefore, given $v_1, v_2 \in V(n)$, one can decide whether $v_1 \mathfrak{C} v_2$. Thus, we can algorithmically construct a finite graph $\mathcal{G}(n)$ with vertex set $V(n)$ that has an edge joining v_1 and v_2 if and only if $v_1 \mathfrak{C} v_2$.

Now fix $u, v \in F(S)$. According to the quasi-monotone conjugacy property, u and v are conjugate in Γ if and only if there exists a path from u to v in the graph $\mathcal{G}(n)$, with $n \geq K \max\{\ell_S(u), \ell_S(v)\}$. Therefore, we can decide whether u and v are conjugate in Γ. □

Theorem 4.12. *Coxeter systems have a solvable conjugacy problem.*

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