Geometric Group Theory

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1 Free groups

1.1 Free groups and freely generating sets

Definition 1.1 (Free groups). *Let S be a set, called an* alphabet*, and let S^{-1} be the set of formal inverses of elements of S, i.e. $S^{-1} = \{s^{-1},\, s \in S\}$. A* word *in the alphabet S is a finite sequence* *of elements of $S \cup S^{-1}$ (with repetitions allowed), or the empty word. A word is said to be* reduced *if it does not contain occurrences of ss^{-1} or $s^{-1}s$ for some $s \in S$. Given a word, we can reduce it by repeatedly removing any such subword. This induces an equivalence relation such that there is a unique reduced word in each class.*

We introduce the notations $bb = b^2$, etc.

The free group *on the set S, denoted by $F(S)$, is the set of reduced words in S, together with the operation of concatenation (followed by reduction if necessary).*

The cardinality of S is called the rank *of $F(S)$ and denoted by $\operatorname{rk} F(S)$.*

Theorem 1.2 (Universal property of free groups)**.** *Let $F(S)$ be the free group on S, and let $\iota : S \hookrightarrow F(S)$ be the natural inclusion. Whenever G is a group and $\varphi : S \to G$ is a function (of sets), there is a unique group homomorphism $\overline{\varphi} : F(S) \to G$ extending φ, i.e. making the following diagram commute:*

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\iota"'] & G \\
F(S) \arrow[ur, dashed, "\overline{\varphi}"'] &
\end{tikzcd}$$

In other words, homomorphisms $F(S) \to G$ are in one-to-one correspondance with functions $S \to G$.

Proof. Given $\varphi : S \to G$, define $\overline{\varphi} : F(S) \to G$ by

$$\overline{\varphi}\left(s_1^{\alpha_1} \cdots s_n^{\alpha_n}\right) = \varphi\left(s_1\right)^{\alpha_1} \cdots \varphi\left(s_n\right)^{\alpha_n}.$$

□

Corollary 1.3. *If $|S| = |T|$, then $F(S) \cong F(T)$.*

Proof. If $|S| = |T|$, then there is a bijection $\theta : S \to T$. By the universal property of free groups, the composite $\iota \circ \theta : S \to F(T)$ induces a group homomorphism $\overline{\theta} : F(S) \to F(T)$ making the following diagram commute:

$$\begin{tikzcd}
S \arrow[r, "\theta"] \arrow[d, "\iota"'] & T \arrow[d, "\iota"'] \\
F(S) \arrow[r, dashed, "\overline{\theta}"] & F(T)
\end{tikzcd}$$

Similarly, there is a group homomorphism $\overline{\theta^{-1}} : F(T) \to F(S)$ such that $\overline{\theta^{-1}} \circ \iota = \iota \circ \theta^{-1}$. Now the homomorphism $\overline{\theta^{-1}} \circ \overline{\theta} : F(S) \to F(S)$ extends $\iota : S \to F(S)$, so by uniqueness in the universal property, it must be equal to $\mathrm{id}_{F(S)}$. This proves that $\overline{\theta^{-1}} \circ \overline{\theta} = \mathrm{id}_{F(S)}$, and likewise $\overline{\theta} \circ \overline{\theta^{-1}} = \mathrm{id}_{F(T)}$. □

Notation 1.4. *We write F_n for the isomorphism class of $F(S)$ with $|S| = n$.*

Proposition 1.5. *$F_m \cong F_n$ if and only if $m = n$.*

Corollary 1.6. *Every group is a quotient of a free group.*

Proof. Given a group G, consider the free group $F(G)$. By the universal property, there is a homomorphism $\pi : F(G) \to G$ extending $\mathrm{id}_G : G \to G$, i.e. such that $\pi \circ \iota = \mathrm{id}_G$. This implies that π is onto, and therefore $G \cong F(G)/\operatorname{Ker} \pi$. □

Definition 1.7 (Subgroup generated by a subset)**.** *Given a group G and a subset $A \subseteq G$, the* subgroup generated by *A, denoted by $\langle A \rangle$, is the intersection of all subgroups of G containing A, i.e. the unique smallest subgroup containing A.*

We say that G is generated by *A, or that A is a* generating set *for G, if $\langle A \rangle = G$.*

We say that G is finitely generated *if it has a finite generating set.*

We shall write $\langle a_1, \ldots, a_n \rangle$ for $\langle \{a_1, \ldots, a_n\} \rangle$, given $a_1, \ldots, a_n \in G$.

Example 1.8. (i) $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ *and* $\mathbb{Z}$ *can be generated by one element only (they are called* cyclic*).*

(ii) $\mathbb{Z}^n$ *can be generated by* n *elements (and no less).*

(iii) $F_2 = \langle a, b \rangle = \langle a, ab \rangle$.
Note that generating sets are not unique.

Definition 1.9 (Freely generated group)**.** *A group* F *is* freely generated *by a subset* $S \subseteq F$ *if for any group* G *and any map* $\varphi : S \to G$*, there is a unique homomorphism* $\widetilde{\varphi} : F \to G$ *extending* φ*.*

Lemma 1.10. *If a group* F *is freely generated by a subset* S*, then* F *is generated by* S*.*

1.2 Subgroups of free groups

Remark 1.11. *We wish to understand subgroups of free groups.*

- *Given any* $w \in F_n \backslash \{e\}$, $\langle w \rangle \cong \mathbb{Z}$*.*
- *Given* $T \subseteq S$, $\langle T \rangle$ *is a free subgroup of* $F(S)$ *of rank* $|T|$*.*
- *If* $S = \{a, b\}$*, the set* $\{a^{-n}ba^n,\ n \in \mathbb{N}\}$ *freely generates a subgroup of* F_2 *that is isomorphic to* F_∞*.*

Note that the last example shows that subgroups of finitely generated groups are not necessarily finitely generated.

Proposition 1.12. *If* $Y \subseteq X$ *is a closed and simply connected of a "nice" space* X*, then*

$$\pi_1 X \cong \pi_1(X/Y).$$

Corollary 1.13. *The fundamental group of a graph with* v *vertices and* e *edges is a free group of rank* $(e - v + 1)$*.*

Proof. If X is a graph, then it has a maximal spanning tree T. Since T is a tree, it has v vertices and $v - 1$ edges. Now X/T is a bouquet of circles, with one circle for each edge of $X \backslash T$, i.e. $(e - v + 1)$ edges in total. By Proposition 1.12, $\pi_1 X \cong \pi_1 (X/T) \cong F_{e-v+1}$. □

Theorem 1.14 (Nielsen-Schreier)**.** *Let* H *be a subgroup of the free group* F_n*. Then* H *is free; moreover, if* H *has finite index in* F_n*, then*

$$(\operatorname{rk} H - 1) = (\operatorname{rk} F_n - 1)\,[F_n : H]\,.$$

Proof. Let X be a bouquet of n circles and let $\widetilde{X}$ be the universal cover of X, i.e. an infinite tree where each vertex has degree $2n$. We have $\pi_1 X \cong F_n$. By the Galois Correspondance, there exists a covering space $\overline{X}$ over X such that $\pi_1 \overline{X} \cong H$. Since X is a graph, $\overline{X}$ is also a graph. Therefore, Corollary 1.13 implies that H is free.

Now if $[F_n : H] < \infty$, then the degree of $\overline{X}$ as a covering space is $[F_n : H]$, so $\overline{X}$ has $[F_n : H]$ vertices. Moreover, each vertex in $\overline{X}$ has degree $2n$ (because the only vertex of X has degree $2n$), so the number of edges of $\overline{X}$ is $\frac{1}{2}\left(2n \cdot [F_n : H]\right) = n\,[F_n : H]$. It follows that

$$\operatorname{rk} H = n\,[F_n : H] - [F_n : H] + 1 = (\operatorname{rk} F_n - 1)\,[F_n : H] + 1.$$ □

Remark 1.15. *If we define the rank of a group as the cardinality of a minimal generating set, then for any finitely generated group* G *and for any finite index subgroup* H*, we can prove that* $(\operatorname{rk} H - 1) \leqslant (\operatorname{rk} G - 1)\,[G : H]$*.*

2 Group presentations and constructions

2.1 Group presentations

Definition 2.1 (Normal closure). *If G is a group, the* normal closure *of a subset $A \subseteq G$ is the unique smallest normal subgroup of G containing A, denoted by $\langle\langle A \rangle\rangle$ or $\langle\langle A \rangle\rangle^G$.*

It is the subgroup of G generated by $\{g^{-1}ag,\ g \in G,\ a \in A\}$.

Definition 2.2 (Presentation of a group). *Given a free group $F(S)$ and a set $R \subseteq F(S)$, we define*

$$\langle S \mid R \rangle = F(S) / \langle\langle R \rangle\rangle .$$

The elements of S are called generators *and the elements of R are called* relators.

A presentation *of a group G is an isomorphism of G with a group of the form $\langle S \mid R \rangle$. We say that G is* finitely presented *if it admits a presentation $\langle S \mid R \rangle$ with S, R finite.*

Example 2.3. (i) $\langle S \mid \varnothing \rangle \cong F(S)$.

(ii) $\langle a \mid a^n \rangle \cong \mathbb{Z}_n$.

(iii) $\langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2$.

(iv) *More generally, finitely generated abelian groups are always finitely presented.*

This is also true of nilpotent groups*, i.e. groups G such that the lower central series defined by $G_{i+1} = [G_i, G]$ terminates in a finite number of steps in the trivial group.*

Proof. (iii) Note that $\mathbb{Z}^2 = \{c^m d^n,\ m, n \in \mathbb{Z}\}$, where $c = (1, 0)$ and $d = (0, 1)$. There is a surjective homomorphism $\varphi : F(a, b) \twoheadrightarrow \mathbb{Z}^2$ defined by $a \mapsto c$ and $b \mapsto d$. It is clear that $\langle\langle aba^{-1}b^{-1} \rangle\rangle \subseteq \operatorname{Ker} \varphi$ because $cd = dc$. Therefore, there is a surjection $\langle a, b \mid aba^{-1}b^{-1} \rangle \twoheadrightarrow \mathbb{Z}^2$. But $\langle a, b \mid aba^{-1}b^{-1} \rangle$ is 2-generated and abelian, so the classification of finitely generated abelian groups allows us to conclude that it has to be isomorphic to $\mathbb{Z}^2$ (otherwise it could not surject to $\mathbb{Z}^2$). □

Remark 2.4. *We have $\langle a, b \mid aba^{-1}b^{-2}, a^{-2}b^{-1}ab \rangle = \{1\}$.*

It is difficult in general to tell which group is given by a particular presentation. Indeed, there does not exist an algorithm that, upon input of a presentation, can determine whether the corresponding group is trivial.

This is called the word problem*; it was introduced by Dehn in the early twentieth century, and the classes of groups for which it does have a solution are often geometric.*

Remark 2.5. *There are uncountably many isomorphism classes of finitely generated (or even 2-generated) groups (c.f. [4]) but only countably many isomorphism classes of finitely presented groups.*

Theorem 2.6. *Given a (not necessarily finite) presentation $\langle S \mid R \rangle$ of a finitely presented group G, there exists a finite subset $S_0 \subseteq S$ and a finite set $\widetilde{R}$ of elements of the free group $F(S_0)$ such that $\left\langle S_0 \mid \widetilde{R} \right\rangle$ is a finite presentation of G.*

Proof. See [4]. □

2.2 Finite index subgroups of finitely presented groups

Remark 2.7. *We now aim to prove that finite index subgroups of finitely generated (resp. presented) groups are finitely generated (resp. presented).*

Definition 2.8 (Schreier transversal). *Let $F(S)$ be a free group and $H \leqslant F(S)$ be a subgroup. A* (right) Schreier transversal *for H in $F(S)$ is a set J of reduced words such that each right coset of H in G contains exactly one word of J (called a* representative *of this class) and all initial segments of these words are also in J (for instance, if $aba^{-1}b^2 \in J$, then $a, ab, aba^{-1}, \dots \in J$).*

In this setting, for each $g \in F(S)$, we denote by $\overline{g}$ the element of J such that $Hg = H\overline{g}$.

Theorem 2.9. *Let $H \leqslant F(S)$.*

(i) *There is a Schreier transversal J for H.*

(ii) *Moreover, H is freely generated by the set $\left\{ts\left(\overline{ts}\right)^{-1},\ t \in J,\ s \in S,\ ts\left(\overline{ts}\right)^{-1} \neq 1\right\}$.*

Proof. Let X be a bouquet of circles indexed by S, so that $\pi_1 X \cong F(S)$. Let $\overline{X}$ be the cover of X corresponding to the subgroup $H \leqslant F(S)$. Hence, $\pi_1 \overline{X} \cong H$, and $\overline{X}$ is a graph. The vertices of $\overline{X}$ correspond to cosets of H in $F(S)$, and choosing a path from a fixed basepoint to a vertex gives a coset representative for that coset.

Now pick a maximal spanning tree T in $\overline{X}$. Choosing the unique paths to each vertex in T gives coset representatives such that initial segments are also such paths. The group $H \cong \pi_1 \overline{X}$ is freely generated by the set of loops with exactly one edge not in T, which corresponds to $\left\{ts\left(\overline{ts}\right)^{-1},\ t \in J,\ s \in S,\ ts\left(\overline{ts}\right)^{-1} \neq 1\right\}$. □

Remark 2.10. *The above proof shows that the set of Schreier transversals for $H \leqslant F(S)$ is in bijection with the set of maximal spanning trees in $\overline{X}$.*

Remark 2.11. *Let $H \leqslant F(S)$. For $t \in J$ and $s \in S$, we write $\gamma(t,s) = ts\left(\overline{ts}\right)^{-1}$; hence, the set $B = \{\gamma(t,s),\ t \in J,\ s \in S,\ \gamma(t,s) \neq 1\}$ freely generates H. Now given $h \in H$ written as $s_1 s_2 \cdots s_n$, with $s_i \in S \cup S^{-1}$, we can rewrite*

$$h = \gamma\left(1, s_1\right)\gamma\left(\overline{s_1}, s_2\right)\cdots\gamma\left(\overline{s_1 \cdots s_{n-1}}, s_n\right),$$

because $\gamma\left(t, s^{-1}\right) = \gamma\left(\overline{ts^{-1}}, s\right)^{-1}$. This is called the Reidermeister-Schreier rewriting process.

Theorem 2.12. *Let G be a group with a presentation $\langle S \mid R\rangle$ and let $\varphi : F(S) \twoheadrightarrow G$ be the homomorphism corresponding to this presentation.*

Let $G_1 \leqslant G$ and let H be the subgroup of $F(S)$ containing $\operatorname{Ker}\varphi$ *such that $\varphi(H) = G_1$. Then G_1 has the presentation*

$$G_1 \cong \left\langle \gamma(t,s),\ t \in J,\ s \in S,\ \gamma(t,s) \neq 1 \mid trt^{-1},\ t \in J,\ r \in R \right\rangle,$$

where J is a Schreier transversal for H in $F(S)$.

Proof. We have $G_1 = H/\langle\langle R\rangle\rangle^{F(S)}$ but we would like to have $G_1 = H/\langle\langle R\rangle\rangle^{H}$. By Theorem 2.9, $\{\gamma(t,s),\ t \in J,\ s \in S\}$ freely generates H. Now the subgroup $\langle\langle R\rangle\rangle^{F(S)}$ is generated by the set $\{grg^{-1},\ g \in G,\ r \in R\}$. Each $g \in G$ may be written as $h_g \overline{g}$ with $h_g \in H$ and $\overline{g} \in J$. Hence,

$$grg^{-1} = \left(h_g \overline{g}\right) r \left(h_g \overline{g}\right)^{-1} = h_g \overline{g} r \overline{g}^{-1} h_g^{-1},$$

and therefore $\langle\langle R\rangle\rangle^{F(S)} = \langle\langle trt^{-1},\ t \in J,\ r \in R\rangle\rangle^{H}$. □

Corollary 2.13. *A finite index subgroup of a finitely generated (resp. finitely presented) group is finitely generated (resp. finitely presented).*

Proof. If $G_1 \leqslant_{fi} G$, then $H \leqslant_{fi} F(S)$, so the Schreier transversal J is finite. □

2.3 Free products and the Ping-pong Lemma

Definition 2.14 (Free product of groups)**.** *Given groups A, B such that $A \cap B = \{1\}$, a* normal form *is an expression of the form $g_1 \cdots g_n$ where $n \geqslant 0$, $g_i \in A \cup B \backslash \{1\}$, and consecutive elements g_i, g_{i+1} do not lie in the same group. We call n the* length *of the normal form. We define a multiplication of normal forms by induction on the length:*

- $(g_1 \cdots g_n) \cdot 1 = 1 \cdot (g_1 \cdots g_n) = g_1 \cdots g_n$,

- *For $m, n \geqslant 2$,*

$$(g_1 \cdots g_m)(h_1 \cdots h_n) = \begin{cases} g_1 \cdots g_m h_1 \cdots h_n & \text{if } g_m, h_1 \text{ in different groups} \\ g_1 \cdots (g_m h_1) \cdots h_n & \text{if } g_m, h_1 \text{ in the same group and } g_m h_1 \neq 1 \\ (g_1 \cdots g_{m-1})(h_2 \cdots h_n) & \text{if } g_m, h_1 \text{ in the same group and } g_m h_1 = 1 \end{cases}.$$

The set of normal form with this multiplication is a group $A * B$, called the free product *of A and B.*

Remark 2.15. *Let A, B be groups with $A \cap B = \{1\}$.*

(i) *The groups A, B embed naturally into $A * B$.*

(ii) *If A, B are subgroups of a group G such that any $g \neq 1$ can be represented in a unique way as a product $g = g_1 \cdots g_n$ with $g_i \in A \cup B \backslash \{1\}$ and consecutive g_i, g_{i+1} not in the same group, then $G = A * B$.*

Theorem 2.16. *If $A = \langle S_A \mid R_A \rangle$ and $B = \langle S_B \mid R_B \rangle$ with $S_A \cap S_B = \varnothing$, then*

$$A * B = \langle S_A \cup S_B \mid R_A \cup R_B \rangle .$$

Proof. Let $\varphi_A : F(S_A) \twoheadrightarrow A$ and $\varphi_B : F(S_B) \twoheadrightarrow B$ be the homomorphisms corresponding to the given presentations. Let $\theta : F(S_A \cup S_B) \to A * B$ be the unique homomorphism coinciding with φ_A on S_A and with φ_B on S_B. We need to show that $\operatorname{Ker} \theta = \langle\langle R_A \cup R_B \rangle\rangle^{F(S_A \cup S_B)}$. The inclusion $(\supseteq)$ is clear. For $(\subseteq)$, consider $g = g_1 \cdots g_n \in \operatorname{Ker} \theta$, with $g_i \in F(S_A) \cup F(S_B) \backslash \{1\}$ and consecutive g_i, g_{i+1} not in the same group (we use here the fact that $F(S_A \cup S_B) = F(S_A) * F(S_B)$. Hence

$$1 = \theta(g) = \theta(g_1) \cdots \theta(g_n) .$$

By definition of $A * B$, there must therefore exist i such that $\theta(g_i) = 1$, and so $g_i \in \langle\langle R_A \rangle\rangle^{F(S_A)} \cup \langle\langle R_B \rangle\rangle^{F(S_B)} \subseteq \langle\langle R_A \cup R_B \rangle\rangle^{F(S_A \cup S_B)}$. Now $\theta(g_1 \cdots g_{i-1} g_{i+1} \cdots g_n) = 1$, so by induction on n, $g_1 \cdots g_n \in \langle\langle R_A \cup R_B \rangle\rangle^{F(S_A \cup S_B)}$. □

Example 2.17. *The group D_∞ of graph automorphisms of the infinite line D_∞ has the presentation*

$$D_\infty = \left\langle a, b \mid a^2 = 1,\ a^{-1} b a = b^{-1} \right\rangle .$$

Note that D_∞ is generated by a and $c = ba$. We can check that $(ca)^n, (ca)^n c, a(ca)^n c, a(ca)^n$ all give different elements of D_∞. By Remark 2.15,

$$D_\infty = \left\langle a, c \mid a^2 = c^2 = 1 \right\rangle \cong \left\langle a \mid a^2 \right\rangle * \left\langle c \mid c^2 \right\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

In fact, $\mathbb{Z}_2 * \mathbb{Z}_2$ is the only free product of nontrivial groups that does not contain a nonabelian free subgroup. For instance, $F_2 \cong [\mathbb{Z}_2, \mathbb{Z}_3] \leqslant \mathbb{Z}_2 * \mathbb{Z}_3$.

Theorem 2.18 (Ping-pong Lemma)**.** *Let G be a group acting on a set X. Let $H_1, H_2 \leqslant G$ such that $|H_1| \geqslant 3$, $|H_2| \geqslant 2$ and let $H = \langle H_1, H_2 \rangle$. Suppose that there are nonempty sets $X_1, X_2 \subseteq X$ with $X_2 \nsubseteq X_1$ such that*

$$\forall h_1 \in H_1 \backslash \{e\},\ h_1(X_2) \subseteq X_1, \qquad \text{and} \qquad \forall h_2 \in H_2 \backslash \{e\},\ h_2(X_1) \subseteq X_2.$$

Then $H \cong H_1 * H_2$.

Proof. Let w be a nonempty reduced word in the alphabet $(H_1 \backslash \{e\}) \amalg (H_2 \backslash \{e\})$. We need to show that the element defined by w in G is not e.

- If $w = a_1 b_1 a_2 b_2 \cdots b_{k-1} a_k$ with $a_i \in H_1 \backslash \{e\}$ and $b_i \in H_2 \backslash \{e\}$, then

  $$w(X_2) = a_1 b_1 a_2 b_2 \cdots b_{k-1} a_k (X_2) \subseteq a_1 b_1 a_2 b_2 \cdots b_{k-1} (X_1) \subseteq \cdots \subseteq a_1 (X_2) \subseteq X_1.$$

  Since $X_2 \not\subseteq X_1$, it follows that $w \neq e$ in G.

- If $w = b_1 a_2 b_2 \cdots a_k b_k$, apply the first case to awa^{-1}, with $a \in H_1 \backslash \{e\}$.
- If $w = a_1 b_1 a_2 b_2 \cdots a_k b_k$, apply the first case to awa^{-1}, with $a \in H_1 \backslash \{e, a_1^{-1}\}$.
- If $w = b_1 a_2 b_2 \cdots a_k$, apply the first case to awa^{-1}, with $a \in H_1 \backslash \{e, a_k\}$. $\square$

Example 2.19. *$SL_2\mathbb{Z} \geqslant \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle * \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle \cong F_2$.*

Actually, F_2 has finite index in $SL_2\mathbb{Z}$.

Proof. Consider $H_1 = \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ and $H_2 = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$. Make $SL_2\mathbb{Z}$ act on $\mathbb{R}^2$ in the usual way. Set

$$X_1 = \left\{(x,y) \in \mathbb{R}^2,\ |x| < |y|\right\} \subseteq \mathbb{R}^2 \quad \text{and} \quad X_2 = \left\{(x,y) \in \mathbb{R}^2,\ |y| < |x|\right\} \subseteq \mathbb{R}^2.$$

Check that $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} X_2 \subseteq X_1$ and similarly $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} X_1 \subseteq X_2$, and apply the Ping-pong Lemma. $\square$

2.4 Amalgamated free products and HNN extensions

Definition 2.20 (Amalgamated free product)**.** *Let $A \leqslant G$, $B \leqslant H$ be subgroups with $\varphi : A \xrightarrow{\cong} B$. The* free product of G and H with amalgamation of A *(*via φ*) is the group*

$$G *_A H = (G * H) \,/\, \left\langle\!\left\langle \varphi(a)a^{-1},\ a \in A \right\rangle\!\right\rangle.$$

The groups G, H embed as subgroups of $G *_A H$, and each element of $G *_A H$ has a normal form.

If $G = \langle S_G \mid R_G \rangle$ and $H = \langle S_H \mid R_H \rangle$, then

$$G *_A H = \left\langle S_G \cup S_H \mid R_G \cup R_H \cup \left\{\varphi(a)a^{-1},\ a \in A\right\}\right\rangle.$$

The notion of amalgamated free product corresponds to gluing G and H along A.

Example 2.21. *$SL_2\mathbb{Z} \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.*

Definition 2.22 (HNN extension)**.** *Let $A, B \leqslant G$ be subgroups, with $\varphi : A \xrightarrow{\cong} B$. The* HNN extension *of G (with respect to A, B, φ) is*

$$G*_\varphi = (G * \langle t \rangle) \,/\, \left\langle\!\left\langle t^{-1} a t \varphi(a)^{-1},\ a \in A \right\rangle\!\right\rangle.$$

The group G embeds as a subgroup of $G*_\varphi$, and each element of $G*_\varphi$ has a normal form.

If $G = \langle S \mid R \rangle$, then

$$G*_\varphi = \left\langle S \cup \{t\} \mid R \cup \left\{t^{-1} a t \varphi(a)^{-1},\ a \in A\right\}\right\rangle.$$

Example 2.23. *The fundamental group of a surface bundle over $\mathbb{S}^1$ is a HNN extension.*

2.5 Semidirect products and wreath products

Definition 2.24 (Semidirect product)**.** *Let G, N, H be groups. We say that G is the* semidirect product *of N by H if one of the following equivalent conditions is satisfied:*

(i) *$H, N \leqslant G$ and $N \cap H = \{e\}$ and $G = NH$.*

(ii) *$H \leqslant G$ and there is a surjective homomorphism $\varphi : G \twoheadrightarrow H$ s.t. $\operatorname{Ker}\varphi = N \geqslant H$.*

(iii) *There is an exact sequence $1 \to N \to G \xrightarrow{\pi} H \to 1$ that splits, i.e. there exists $\psi : H \to G$ such that $\pi \circ \psi = \mathrm{id}_H$.*

We then write $G = N \rtimes H$.

Let H, N be two groups with a morphism $\alpha : H \to \operatorname{Aut}(N)$. The semidirect product *$N \rtimes_\alpha H$ is the group whose underlying set is $N \times H$, and with multiplication defined by*

$$(n_1, h_1)(n_2, h_2) = (n_1 \cdot \alpha(h_1)(n_2), h_1 h_1).$$

The subgroups $N \times 1$ and $1 \times H$ then satisfy the conditions above. Conversely, given a semidirect product $G = N \rtimes H$ as above, we can recover G as $N \rtimes_\alpha H$ by setting $\alpha(h)(n) = hnh^{-1}$.

Example 2.25. (i) *A direct product is also a semidirect product.*

(ii) *$D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$.*

(iii) *The fundamental group of the Klein bottle is a direct product $\mathbb{Z} \rtimes \mathbb{Z}$.*

(iv) *More generally, a* group extension *is a group G together with an exact sequence $1 \to N \to G \to H \to 1$.*

Remark 2.26. *An exact sequence $1 \to N \to G \to H \to 1$ always splits if H is free.*

However, not all exact sequence splits. For instance, $1 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 1$ doesn't.

Definition 2.27 (Wreath product)**.** *Given two groups G, H, the* wreath product *of G and H is*

$$G \wr H = \left(\bigoplus_{h \in H} G \right) \rtimes H,$$

where we think of $\bigoplus_{h\in H} G$ as the group of finitely-supported functions $H \to G$, and H acts on it by

$$(h \cdot f)(h_1) = f\left(h^{-1} h_1\right).$$

Example 2.28. *The* lamplighter group *is $\mathbb{Z}_2 \wr \mathbb{Z}$. It is not finitely presented.*

Theorem 2.29 (Kaloujnine-Krasner)**.** *If D, Q are two groups with Q finite, then $D \wr Q$ contains an isomorphic copy of every extension of D by Q (i.e. every group G with an exact sequence $1 \to D \to G \to Q \to 1$).*

Proof. Consider an exact sequence $1 \to D \to G \xrightarrow{\pi} Q \to 1$, and view $D = \operatorname{Ker}\pi$ as a subgroup of G. We choose a *transversal* (i.e. a set of coset representatives) for D in G, which we view as a map $T : Q \to G$ such that $\pi \circ T = \mathrm{id}_Q$ (but T may not be a group homomorphism). For $a \in G$, define a map $f_a : Q \to D$ by

$$f_a(x) = T(x)^{-1} a T\left(\pi(a)^{-1} x\right).$$

Since $\pi \circ f_a = 1$, we have indeed $\operatorname{Im} f_a \subseteq \operatorname{Ker}\pi = D$. Now for $a, b \in G$ and $x \in Q$, we have

$$\begin{aligned}
f_a(x)\left(\pi(a) \cdot f_b\right)(x) &= f_a(x) f_b\left(\pi(a)^{-1} x\right) \\
&= T(x)^{-1} a T\left(\pi(a)^{-1} x\right) T\left(\pi(a)^{-1} x\right)^{-1} b T\left(\pi(b)^{-1} \pi(a)^{-1} x\right) \\
&= T(x)^{-1} ab T\left(\pi(ab)^{-1} x\right) = f_{ab}(x),
\end{aligned}$$

so $f_a(\pi(a)\cdot f_b) = f_{ab}$. Therefore, the map

$$\varphi : a \in G \longmapsto (f_a, \pi(a)) \in D \wr Q = \left(\bigoplus_{q\in Q} D\right) \rtimes Q$$

is a group homomorphism. This map is injective: if $a \in \operatorname{Ker}\varphi$, then in particular $a \in \operatorname{Ker}\pi = D$, and also $f_a = 1$, from which it follows that $a = 1$. Hence, ϕ is an embedding $G \hookrightarrow D \wr H$. □

Remark 2.30. *The proof of Theorem 2.29 would also work when Q is infinite if one replaces the direct sum by a direct product in the definition of the wreath product.*

2.6 Bass-Serre Theory

Theorem 2.31. *Let $G = G_1 *_A G_2$. Then G acts without inversion of edges on a tree X such that the quotient graph $G\backslash X$ is a segment. Moreover, this segment lifts to a segment of X such that the stabilisers of its vertices are G_1, G_2 and the stabiliser of the edge is A.*

Sketch of proof. Define a graph X by

$$V(X) = (G/G_1) \amalg (G/G_2) \qquad \text{and} \qquad \vec{E}(X) = G/A,$$

with the edge gA going from gG_1 to gG_2. Then $G \curvearrowright X$ by left multiplication.

To prove that X is connected, it suffices to show that gG_1 is connected to G_1 for all g (then G_1 has a direct edge to G_2, which is connected to hG_2 for all h). Express g as $g_1 \cdots g_n$, with $g_i \in G_1 \amalg G_2$, and g_i, g_{i+1} not in the same group. Then either $g_1 \cdots g_n G_1 = g_1 \cdots g_{n-1} G_1$ or $g_1 \cdots g_n G_2 = g_1 \cdots g_{n-1} G_2$; in both cases, $g_1 \cdots g_n G_1$ is connected to $g_1 \cdots g_{n-1} G_1$. By induction on n, gG_1 is connected to G_1 and therefore X is connected.

The fact that X is acyclic follows from the uniqueness of the normal form in amalgamated free products; therefore X is a tree. □

Remark 2.32. *Bass-Serre Theory gives a correspondance as follows:*

- *Given a group G, there is an action $G \curvearrowright X$ on a tree without inversion of edges.*
- *Given an amalgamated free product $G = G_1 *_A G_2$, $G\backslash X$ is a segment.*
- *Given a HNN extension $G = H*_\varphi$, $G\backslash X$ is a loop.*
- *Given a group G, $G\backslash X$ is a graph such that G is the fundamental group $\pi_1(\mathbb{G}, Y)$ of a graph of loops.*

Corollary 2.33 (Kuvosh Subgroup Theorem)**.** *Let $G = A * B$. Then any subgroup of G can be written as the free product of a free group, some conjugates of subgroups of A, and some conjugates of subgroups of B.*

3 Cayley graphs

3.1 Cayley graphs and word metrics

Definition 3.1 (Cayley graph)**.** *Let G be a group together with a finite generating set $S \subseteq G$. The* Cayley graph *of G with respect to S is the graph* $\operatorname{Cay}(G,S)$ *given by*

$$\begin{aligned} V(\operatorname{Cay}(G,S)) &= G, \\ E(\operatorname{Cay}(G,S)) &= \{(g, gs),\ g \in G,\ s \in S\}. \end{aligned}$$

Remark 3.2. (i) $\operatorname{Cay}(G,S)$ *is a $2|S|$-regular graph.*

(ii) $\mathrm{Cay}(G,S)$ *is connected if* $G = \langle S \rangle$.

(iii) *Relators in* S *give rise to cycles in* $\mathrm{Cay}(G,S)$.

(iv) *Paths from* e *to* g *in* $\mathrm{Cay}(G,S)$ *correspond to words in* S *representing* g.

Definition 3.3 (Word metric)**.** *Let* G *be a group with a finite generating set* S. *The* word metric d_S *is defined as follows:* $d_S(g,h)$ *is the length of a shortest path in* $\mathrm{Cay}(G,S)$ *from* g *to* h. *We also have a* word length *defined by* $|g| = d_S(e,g)$.

Remark 3.4. $G \curvearrowright \mathrm{Cay}(G,S)$ *isometrically by left multiplication.*
It follows that $d_S(g,h) = |g^{-1}h|$.

Theorem 3.5. $\mathrm{Cay}\,(F(S),S)$ *is a tree.*

Remark 3.6. *Let* $\overline{X}$ *be the covering space of a bouquet* X *of* $|S|$ *circles corresponding to the normal subgroup* $N \trianglelefteq F(S)$. *Then* $\overline{X}$ *is exactly* $\mathrm{Cay}\,(F(S)/N, \pi(S))$, *where* π *is the projection* $F(S) \twoheadrightarrow F(S)/N$.

Remark 3.7.

- *Non-isomorphic groups can have the same Cayley graph, e.g.* $(\mathbb{Z}_6, \{2,3\})$ *and* $(\mathfrak{S}_3, \{(1\,2),(1\,2\,3)\})$.
- *There can be non-isomorphic Cayley graphs for a group* G *with (minimal) generating sets, e.g.* $(\mathfrak{S}_3, \{(1\,2),(2\,3)\})$ *and* $(\mathfrak{S}_3, \{(1\,2),(1\,2\,3)\})$.

3.2 Quasi-isometry

Definition 3.8 (Quasi-isometry)**.** *Let* (X,d_X) *and* (Y,d_Y) *be metric spaces. A map* $f : X \to Y$ *is a* quasi-isometric embedding *if there exists* $\lambda \geqslant 1$ *and* $c \geqslant 0$ *such that, for all* $x, x' \in X$,

$$\frac{1}{\lambda} d_X(x,x') - c \leqslant d_Y(f(x), f(x')) \leqslant \lambda d_X(x,x') + c.$$

We say that f *is a* quasi-isometry *(or a* (λ, c, D)*-quasi-isometry) if in addition there exists* $D \geqslant 0$ *such that*

$$\forall y \in Y,\ \exists x \in X,\ d_Y(f(x), y) \leqslant D.$$

We then write $X \simeq_{qi} Y$.
Intuitively, quasi-isometry preserves the large-scale structure of our space.

Proposition 3.9. *Quasi-isometry is an equivalence relation on metric spaces.*

Example 3.10. (i) *A non-empty bounded metric space is quasi-isometric to a point.*

(ii) $\mathbb{R} \times [0,1] \simeq_{qi} \mathbb{R}$.

(iii) *if* $\mathbb{Z}^n = \langle S \rangle$, *then* $\mathrm{Cay}\,(\mathbb{Z}^n, S) \simeq_{qi} \mathbb{R}^n$.

(iv) *The* 3*-regular tree* T_3 *is quasi-isometric to the* 4*-regular tree* T_4 *(colour the edges of* T_3 *with three different colours and contract all edges of one specific colour).*

Example 3.11. *To prove that two spaces are not quasi-isometric, we often use quasi-isometry invariants.*

(i) $\mathbb{R} \not\simeq_{qi} \{*\}$ *because boundedness is a quasi-isometry invariant.*

(ii) $\mathbb{R} \not\simeq_{qi} [0,\infty)$.

(iii) $\mathbb{R}^2 \not\simeq_{qi} \mathbb{R}$.

(iv) $T_3 \not\simeq_{qi} \mathbb{R}$.

Proof. (ii) We shall actually prove the following (stronger) result: $\mathbb{R}$ does not quasi-isometrically embed into $[0,\infty)$. Suppose there is a (λ, c)-quasi-isometric embedding $\varphi : \mathbb{R} \to [0,\infty)$. Note that

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to-\infty} \varphi(t) = \infty.$$

Hence, for any $x \in [0,\infty)$, we can define

$$\begin{aligned} M_x &= \max\{n \in \mathbb{Z},\ \varphi(n) < x\}, \\ N_x &= \min\{n \in \mathbb{Z},\ \varphi(n) < x\}. \end{aligned}$$

So we have $\varphi(M_x) < x \leqslant \varphi(M_x + 1)$ and $\varphi(N_x) < x \leqslant \varphi(N_x - 1)$. Hence,

$$\begin{aligned} d_{[0,\infty)}\left(\varphi(M_x), \varphi(N_x)\right) &\leqslant d_{[0,\infty)}\left(\varphi(M_x), x\right) + d_{[0,\infty)}\left(x, \varphi(N_x)\right) \\ &\leqslant d_{[0,\infty)}\left(\varphi(M_x), \varphi(M_x+1)\right) + d_{[0,\infty)}\left(\varphi(N_x - 1), \varphi(N_x)\right) \\ &\leqslant \lambda d_{\mathbb{R}}\left(M_x, M_x+1\right) + c + \lambda d_{\mathbb{R}}\left(N_x - 1, N_x\right) + c \\ &= 2\lambda + 2c. \end{aligned}$$

But we can check that $d_{\mathbb{R}}(M_x, N_x) \xrightarrow[n\to\infty]{} \infty$. This is a contradiction. □

Proposition 3.12. *Let G be a finitely generated group and S, S' be two finite generating sets for G. Then*

$$\mathrm{Cay}(G,S) \simeq_{qi} \mathrm{Cay}(G,S').$$

Proof. Let $\varphi : (G, d_S) \to (G, d_{S'})$ be the identity map. Let

$$\lambda = \max_{a\in S} |a|_{S'} \qquad \text{and} \qquad \lambda' = \max_{b\in S'} |b|_S.$$

Then $d_{S'}(\varphi(g), \varphi(h)) \leqslant \lambda d_S(g,h)$, and similarly $d_S(g,h) \leqslant \lambda' d_{S'}(\varphi(g), \varphi(h))$. □

3.3 The fundamental theorem of geometric group theory

Definition 3.13 (Proper and geodesic metric spaces)**.** *A metric space is called* proper *if all closed balls are compact. It is called* geodesic *if for any two points, there is a path realising the distance between them.*

Definition 3.14 (Proper action)**.** *An action $G \curvearrowright X$ is called* proper *if for any compact set $K \subseteq X$, the set*

$$\{g \in G,\ gK \cap K \neq \varnothing\}$$

is finite. This implies that X/G is Hausdorff and locally compact.

Theorem 3.15 (Švarc-Milnor)**.** *Let X be a proper geodesic metric space and let $G \curvearrowright X$ properly by isometries on X with compact quotient. Then G is finitely generated, and picking $x_0 \in X$ defines a quasi-isometry*

$$\varphi_{x_0} : \left| \begin{array}{l} G \longrightarrow X \\ g \longmapsto gx_0 \end{array} \right. .$$

In particular, $G \simeq_{qi} X$.

Proof. Since the action is cocompact, there exists a closed ball $\overline{B} = \overline{B}(x_0, D)$ such that $G\overline{B} = X$. Since X is proper, $\overline{B}$ is compact. Define

$$S = \left\{ s \in G \backslash \{e\},\ s\overline{B} \cap \overline{B} \neq \varnothing \right\}.$$

The set S is finite since the action is proper. For $A, B \subseteq X$, define $d(A,B) = \inf_{(a,b)\in A\times B} d_X(a,b)$, and consider

$$2d = \inf_{g\in G\setminus(S\cup\{e\})} d\left(\overline{B}, g\overline{B}\right).$$

Pick some $h_1 \in G\setminus(S\cup\{e\})$ and let $R = d\left(\overline{B}, h_1\overline{B}\right) > 0$. Let

$$H = \left\{h \in G\setminus(S\cup\{e\}),\ d\left(\overline{B}, h\overline{B}\right) \leqslant R\right\} \subseteq \left\{g \in G,\ g\overline{B}\left(x_0, D+R\right) \cap \overline{B}\left(x_0, D+R\right) \neq \varnothing\right\},$$

so H is finite since the action is proper, and nonempty because $h_1 \in H$. And

$$2d = \inf_{g\in G\setminus(S\cup\{e\})} d_X\left(\overline{B}, g\overline{B}\right) = \inf_{h\in H} d\left(\overline{B}, g\overline{B}\right),$$

so the infimum is achieved at some $h_0 \in H$. Note that, if $d\left(\overline{B}, g\overline{B}\right) < 2d$, then $g \in S\cup\{e\}$.

To prove that $G = \langle S\rangle$, take $g \in G$ and consider a geodesic segment $[x_0, gx_0]$. Let $k = \left\lfloor \frac{1}{d} d_X\left(x_0, gx_0\right)\right\rfloor$. Take a sequence $x_0 = y_0, y_1, \ldots, y_k, y_{k+1} = gx_0$ in $[x_0, gx_0]$ such that $d_X\left(y_i, y_{i+1}\right) \leqslant d$ for all i. Take a sequence $e = h_0, h_1, \ldots, h_k, h_{k+1} = g$ in G such that $y_i \in h_i\overline{B}$ for all i. Now we have

$$d\left(h_i\overline{B}, h_{i+1}\overline{B}\right) \leqslant d_X\left(y_i, y_{i+1}\right) \leqslant d,$$

so $d\left(\overline{B}, h_i^{-1}h_{i+1}\overline{B}\right) \leqslant d$ and therefore $h_i^{-1}h_{i+1} \in S\cup\{e\}$. Hence,

$$g = \prod_{i=0}^{k}\left(h_i^{-1}h_{i+1}\right) \in \langle S\rangle.$$

This shows that $G = \langle S\rangle$.

Now we equip G with d_S (using the fact that all word metrics on G are quasi-isometric, c.f. Proposition 3.12). It is clear that the $2D$-neighbourhood of the image of $\varphi_{x_0} : x \mapsto gx_0$ is X, so it remains to show that φ_{x_0} is a quasi-isometric embedding. The above implies that

$$|g|_S \leqslant k+1 \leqslant \frac{1}{d} d_X\left(x_0, gx_0\right) + 1.$$

Conversely, if $|g|_S = m$, write $g = t_1 \cdots t_m$ with $t_i \in S$. Then

$$d\left(x_0, gx_0\right) \leqslant \sum_{i=1}^{m} d\left(t_1\cdots t_{i-1}x_0, t_1\cdots t_i x_0\right) = \sum_{i=1}^{m} \underbrace{d\left(x_0, t_i x_0\right)}_{\leqslant 2D} \leqslant 2Dm = 2D\,|g|_S,$$

where the inequality $d\left(x_0, t_i x_0\right) \leqslant 2D$ comes from the fact that $\overline{B}\left(x_0, D\right) \cap \overline{B}\left(t_i x_0, D\right) \neq \varnothing$ by definition of S. Thus,

$$d \cdot d_S\left(e, g\right) - d \leqslant d_X\left(x_0, gx_0\right) \leqslant 2D \cdot d_S(e,g).$$

By left-invariance of d_S and d_X under the action of G, φ_{x_0} is a quasi-isometric embedding. □

Example 3.16. (i) *If M is a compact connected Riemannian manifold, then $\pi_1 M$ is finitely generated and quasi-isometric to the universal cover $\widetilde{M}$.*

(ii) *If G is a connected real Lie group and Γ is a cocompact lattice in G, then Γ is finitely generated and quasi-isometric to G.*

Definition 3.17 (Commensurable groups)**.** *Two groups G, H are said to be* commensurable *if there exist subgroups of finite index $K_1 \leqslant_{fi} G$ and $K_2 \leqslant_{fi} H$ such that $K_1 \cong K_2$.*

Corollary 3.18. *Let G be a finitely generated group.*

(i) *If H is a finite index subgroup of G, then $H \simeq_{qi} G$.*

(ii) *If* $N \trianglelefteq G$, *then* $G \simeq_{qi} G/N$.

(iii) *If* G, H *are commensurable finitely generated groups, then* $G \simeq_{qi} H$.

Proof. (i) Apply Theorem 3.15 to $H \curvearrowright \mathrm{Cay}(G)$. (ii) Apply Theorem 3.15 to $G \curvearrowright \mathrm{Cay}(G/N)$. (iii) Use (i). □

Corollary 3.19. *All finitely generated free groups are commensurable, hence quasi-isometric.*

Proof. Note that F_n embeds as a subgroup of finite index of F_2 for all $n \geqslant 2$. □

3.4 Rigidity properties

Example 3.20. *Consider*

$$G = \mathbb{Z}_4 \wr \mathbb{Z} \qquad \textit{and} \qquad H = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbb{Z}.$$

Then G *and* H *are quasi-isometric but not commensurable.*

Proof. Not commensurable. Note that the only elements of finite order in any finite index subgroup of H are of order 2. On the other hand, any finite index subgroup $K \leqslant_{fi} G$ necessarily contains elements of order 4: indeed,

$$\left[\bigoplus_{\mathbb{Z}} \mathbb{Z}_4 : K \cap \bigoplus_{\mathbb{Z}} \mathbb{Z}_4\right] = \left[K \oplus \bigoplus_{\mathbb{Z}} \mathbb{Z}_4 : K\right] < \infty,$$

so $K \cap \bigoplus_{\mathbb{Z}} \mathbb{Z}_4$ is a finite index subgroup in $\bigoplus_{\mathbb{Z}} \mathbb{Z}_4$.

Quasi-isometric. Take

$$S_G = \left\{(\overline{0}, 1), (f_1, 0), (f_2, 0), (f_3, 0)\right\},$$

where $\overline{0}$ is the zero function, thinking of $\bigoplus_{\mathbb{Z}} \mathbb{Z}_4$ as functions $\mathbb{Z} \to \mathbb{Z}_4$, and $f_i : \mathbb{Z} \to \mathbb{Z}_4$ is given by $f_i(0) = i$ and $f_i(n) = 0$ for $n \neq 0$. Similarly, let

$$S_H = \left\{(\overline{0}, 1), \left(f_{(1,0)}, 0\right), \left(f_{(0,1)}, 0\right), \left(f_{(1,1)}, 0\right)\right\}.$$

Then $\mathrm{Cay}(G, S_G)$ and $\mathrm{Cay}(H, S_H)$ are in fact isometric. □

Remark 3.21. *We wish to find additional assumptions such that quasi-isometry implies commensurability.*

Definition 3.22 (Virtually)**.** *A group is* virtually P *it if has a finite index subgroup that is* P.

Example 3.23. (i) $\mathbb{Z}_2 \times \mathbb{Z}$ *is virtually* $\mathbb{Z}$.

(ii) $SL_2\mathbb{Z}$ *is virtually free.*

Theorem 3.24. *Let* G *be a finitely generated group. If* $G \simeq_{qi} \mathbb{Z}$, *then* G *is virtually* $\mathbb{Z}$.

Proof. To construct an element of infinite order in G, we will find $g \in G$ and $A \subseteq G$ such that $gA \subsetneq A$; it will follow that $g^n \neq e$ for all n. Let $\varphi : G \to \mathbb{R}$ be a quasi-isometry (because $\mathbb{Z} \simeq_{qi} \mathbb{R}$). Since $G \curvearrowright \mathrm{Cay}(G)$ by isometries, any $g \in G$ determines a quasi-isometry $\psi_g : \mathbb{R} \to \mathbb{R}$ making the diagram

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "\varphi"'] & G \arrow[d, "\varphi"'] \\ \mathbb{R} \arrow[r, "\psi_g"] & \mathbb{R} \end{tikzcd}$$

commute. Consider $[0, \infty) \subseteq \mathbb{R}$. Then $\psi_g([0, \infty))$ is either at a bounded distance from $[\psi_g(0), \infty)$ or from $(-\infty, \psi_g(0)]$. In the first case for example, set

$$A = \varphi^{-1}([0, \infty)) \subseteq G.$$

If $\psi_g(0) \gg 0$, then $gA \subsetneq A$.

Therefore, we need to find g such that $\psi_g([0, \infty))$ is at a bounded distance from $[\psi_g(0), \infty)$ and $\psi_g(0) \gg 0$. To find such a g, take $h, k \in G$ such that e, h, k are far apart in G (or equivalently, $\varphi(e), \varphi(h), \varphi(k)$ are far apart in $\mathbb{R}$). Consider images of $[0, \infty)$ under ψ_e, ψ_h, ψ_k – at least two of these images will be at a bounded distance from each other, so at least two of A, hA, kA are nested; therefore, we can take g to be one of the elements $h^{\pm 1}, k^{\pm 1}, (h^{-1}k)^{\pm 1}, (k^{-1}h)^{\pm 1}$.

Now let $H = \langle g \rangle \cong \mathbb{Z}$. We want $[G : H] < \infty$. We have $\lim_{n \to \pm\infty} d(e, g^n) = \infty$ and $d(g^m, g^n) = d(e, g^{n-m})$. Using the quasi-isometry $\varphi : G \to \mathbb{R}$, we define $f : \mathbb{Z} \to \mathbb{R}$ by

$$f(n) = \varphi(g^n).$$

The map f has the following properties:

(i) $\exists M \geqslant 0, \forall n \in \mathbb{Z}, |f(n) - f(n-1)| \leqslant M$.

(ii) $\forall r \geqslant 0, \exists k \in \mathbb{N}, \forall m, n \in \mathbb{Z}, |f(m) - f(n)| \leqslant r \Longrightarrow |m - n| \leqslant k$.

We can check that the image of a map satisfying the above has the properties that there exists $C > 0$ such that for all $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$, such that $|x - f(n)| \leqslant C$, and hence there exists $C' > 0$ such that for all $g' \in G$, there exists $n \in \mathbb{Z}$, such that $d(g', g^n) \leqslant C'$. This shows that $[G : H]$ is finite. $\square$

Example 3.25. *Examples of rigidity properties:*

(i) *If both groups are virtually abelian, quasi-isometry implies commensurability.*

This is also true, though much deeper, if only one of the groups is assumed to be virtually abelian.

(ii) *If both groups are virtually free, quasi-isometry implies commensurability (in this case, the two groups are in fact always commensurable).*

This is also true, though much deeper, if only one of the groups is assumed to be virtually free.

4 Geometric properties of groups

4.1 Growth

Notation 4.1. *Given $f, g : X \to \mathbb{R}$ (with $X \subseteq \mathbb{R}$), we write $f \preccurlyeq g$ if there exist $a, b > 0$ and $x_0 \in X$ such that, for all $x \geqslant x_0$,*

$$f(x) \leqslant a g(bx).$$

If $f \preccurlyeq g$ and $g \preccurlyeq f$, we write $f \asymp g$.

Definition 4.2 (Growth function)**.** *Let X be a discrete metric space with a basepoint $x_0 \in X$. The* growth function *of X at x_0 is the function $\beta_{X,x_0} : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ given by*

$$\beta_{X,x_0}(r) = \left|\overline{B}_X(x_0, r)\right|,$$

where $\overline{B}_X(x_0, r)$ is the closed ball of radius r centred at x_0 in X.

Lemma 4.3. *Let G be a group with a finite generating set S.*

(i) *If* $g, h \in G$, *then* $\beta_{\mathrm{Cay}(G,S),g} = \beta_{\mathrm{Cay}(G,S),h}$; *this will be denoted by* $\beta_{G,S}$.

(ii) *The equivalence class under* $\asymp$ *of* $\beta_{G,S}$ *is a quasi-isometry invariant. We write* β_G *for the equivalence class of* $\beta_{G,S}$.

Proposition 4.4. *Let* G *be a group with a finite generating set* S.

(i) *If* G *is infinite, then* $\beta_{G,S_{|\mathbb{N}}}$ *is strictly increasing.*

(ii) $\beta_{G,S}(r+t) \leqslant \beta_{G,S}(r)\beta_{G,S}(t)$.

(iii) $\beta_{G,S}(r) \leqslant (2|S|)^r$.

Lemma 4.5 (Fekete)**.** *If* $(a_n)_{n\geqslant 1}$ *is a subadditive sequence of real numbers, then* $\lim_{n\to\infty} \frac{a_n}{n}$ *exists.*

Corollary 4.6. *If* G *is a group with a finite generating set* S, *then* $\lim_{n\to\infty} \beta_{G,S}(n)^{1/n}$ *exists and is at least* 1.

Definition 4.7 (Exponential and polynomial growth)**.** *Let* G *be a finitely generated group.*

(i) G *is said to have* exponential growth *if* $\lim_{n\to\infty} \beta_G(n)^{1/n} > 1$. *Otherwise,* G *has* subexponential growth.

(ii) G *is said to have* polynomial growth *if there exists* d *such that* $\beta_G(r) \preccurlyeq r^d$.

Example 4.8. (i) $\beta_{\mathbb{Z}^k}(r) \asymp r^k$, *so* $\mathbb{Z}^k$ *has polynomial growth.*

(ii) $\beta_{F_k}(r) \asymp (2k)^r$, *so* F_k *has exponential growth.*

Proposition 4.9. *Let* G *be a finitely generated group.*

(i) *If* H *is a finitely generated subgroup of* G, *then* $\beta_H \preccurlyeq \beta_G$.

(ii) *If* H *is a finite index subgroup of* G, *then* $\beta_H \asymp \beta_G$.

(iii) *If* $N \trianglelefteq G$, *then* $\beta_{G/N} \preccurlyeq \beta_G$.

(iv) *If* $N \trianglelefteq G$ *with* N *finite, then* $\beta_{G/N} \asymp \beta_G$.

Proof. (ii) and (iv) are easy consequences of Lemma 4.3 and the Švarc-Milnor Lemma (Theorem 3.15).

(i) Take $H = \langle T \rangle$ with T finite, and let $S \supseteq T$ such that $G = \langle S \rangle$. Then $\mathrm{Cay}(H,T)$ is a subgraph of $\mathrm{Cay}(G,S)$, so $d_S(e,h) \leqslant d_T(e,h)$ for all $h \in H$, and therefore $\overline{B}_{\mathrm{Cay}(G,S)}(e,r) \supseteq \overline{B}_{\mathrm{Cay}(H,T)}(e,r)$.

(iii) Take $G = \langle S \rangle$ with S finite, and let $T = \{sN,\ s \in S \backslash N\}$. Then $G/N = \langle T \rangle$ and $\pi : G \twoheadrightarrow G/N$ maps $\overline{B}_{\mathrm{Cay}(G,S)}(e,r)$ onto $\overline{B}_{\mathrm{Cay}(G/N,T)}(e,r)$. □

4.2 Groups of polynomial growth

Remark 4.10. *Since* $\mathbb{Z}^k$ *has polynomial growth, finitely generated virtually abelian groups have polynomial growth.*

Proposition 4.11. *Let* G *be a 2-step finitely generated nilpotent group, i.e.*

$$G \geqslant [G,G] \geqslant [G,[G,G]] = \{e\}.$$

Then G *has polynomial growth.*

Proof. Suppose $G = \langle g_1, \ldots, g_m \rangle$. Since G is 2-step nilpotent, $[G, G]$ is central in G. Note that, given $g, h \in G$, we have
$$gh = hgg^{-1}h^{-1}gh = hg[g, h].$$
Therefore, given a product of n generators, we can exchange two of the generators and produce a commutator on the right. Since commutators lie in the centre, we can move them to the right at no cost, thus arranging the generators in order. After at most n^2 such moves, we get
$$g_1^{\alpha_1} \cdots g_m^{\alpha_m} C,$$
where C is a product of at most n^2 commutators. But note that, in this case, $[G, G]$ is finitely generated by commutators $\left[g_i^{\pm 1}, g_j^{\pm 1}\right]$; since $[G, G]$ is also abelian, it has polynomial growth, say of degree d (by Remark 4.10). This implies that G has polynomial growth of degree at most $m+2d$. □

Theorem 4.12. *All finitely generated virtually nilpotent groups have polynomial growth.*

Proof. Induction on nilpotency step. □

Definition 4.13 (Solvable group)**.** *A group G is* solvable *if the derived series $\left(G^{(n)}\right)_{n \geqslant 0}$ terminates in a finite number of steps, where $G^{(0)} = G$ and $G^{(n+1)} = \left[G^{(n)}, G^{(n)}\right]$ for all $n \geqslant 0$.*

Note that abelian implies nilpotent, which implies solvable.

Remark 4.14. *There do exist solvable groups of exponential growth, for instance the* lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

Theorem 4.15 (Gromov, 1981)**.** *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Proof. Uses asymptotic cones, as well as the Tits alternative. See for example Wilkie-Van der Dries, or Kleiner, or Ozawa. There is also a recent proof by Tao using approximate groups. For more on asymptotic cones, see Druțu-Kapovich. □

Corollary 4.16. *Being virtually nilpotent is a quasi-isometry invariant.*

Theorem 4.17 (Grigorchuk, 1983)**.** *There exists a finitely generated group G, together with constants $0 < \alpha_1 < \alpha_2 < 1$, such that*
$$2^{r^{\alpha_1}} \preccurlyeq \beta_G(r) \preccurlyeq 2^{r^{\alpha_2}}.$$

Proof. See de la Harpe. □

4.3 Ends

Definition 4.18 (Proper map)**.** *A map $f : X \to Y$ between topological spaces is said to be* proper *if $f^{-1}(C) \subseteq X$ is compact whenever $C \subseteq Y$ is compact.*

Definition 4.19 (Rays)**.** *A* ray *in a topological space X is a continuous proper map $r : [0, \infty) \to X$.*

Two rays $r_1, r_2 : [0, \infty) \to X$ are said to converge to the same end *if for every compact $C \subseteq X$, there exists $N \in \mathbb{N}$ large enough so that $r_1([N, \infty))$ and $r_2([N, \infty))$ are contained in the same path-component of $X \backslash C$.*

This defines an equivalence relation on rays; the set of equivalence classes is called the set of ends *of X, denoted by* $\mathrm{Ends}(X)$*. Given a ray r, we write* $\mathrm{end}(r)$ *for the class of r in* $\mathrm{Ends}(X)$.

If $|\mathrm{Ends}(X)| = m$, we say that X has m ends.

If $(r_n)_{n \geqslant 1}$ and r are rays in X, we say that $\mathrm{end}(r_n) \xrightarrow[n \to \infty]{} \mathrm{end}(r)$ *if for all $C \subseteq X$ compact, there exists a sequence $(N_n)_{n \geqslant 1}$ of natural numbers such that for all n sufficiently large, $r_n([N_n, \infty))$ and $r([N_n, \infty))$ lie in the same path-component of $X \backslash C$. This defines a topology on* $\mathrm{Ends}(X)$.

Definition 4.20 (k-path)**.** *A* k-path *from x to y in a metric space X is a sequence of points $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_{i-1}, x_i) \leqslant k$ for all $i \in \{1, \ldots, n\}$.*

Lemma 4.21. *Let X be a proper geodesic metric space. We denote by $\mathcal{G}_{x_0}(X)$ the set of geodesic rays in X starting at $x_0 \in X$.*

(i) *Given two rays r_1, r_2 in X, and $k > 0$, we have $\operatorname{end}(r_1) = \operatorname{end}(r_2)$ if and only if for all $R > 0$, there exists $T > 0$ such that for all $t \geqslant T$, $r_1(t)$ can be connected to $r_2(t)$ by a k-path in $X \backslash B(x_0, R)$.*

(ii) *The natural map $\mathcal{G}_{x_0}(X) \to \operatorname{Ends}(X)$ is surjective.*

Sketch of proof. (i) Every compact subset of X is contained in an open ball about x_0 and *vice-versa*. Moreover, given a k-path, we can concatenate geodesics from x_{i-1} to x_i to get a continuous path from x to y.

(ii) Let $r : [0, \infty) \to X$ be a ray. Let $c_n : [0, d_n] \to X$ be a geodesic from x_0 to $r(n)$. We can extend c_n to $[0, \infty)$ by setting $c_n(t) = r(t)$ for $t \geqslant d_n$. By Arzelà-Ascoli, there is a convergent subsequence of $(c_n)_{n \geqslant 1}$, with limit $c : [0, \infty) \to X$, a geodesic ray with $\operatorname{end}(c) = \operatorname{end}(r)$. □

Lemma 4.22. *Let X be a metric space. Given $f, g : X \to X$, we say that $f \sim g$ if*

$$\sup_{x \in X} d_X(f(x), g(x)) < \infty.$$

The set of equivalence classes of quasi-isometries $X \to X$ forms a group denoted by $\mathrm{QI}(X)$, and any quasi-isometry $\varphi : X \to Y$ induces an isomorphism

$$\varphi_* : \mathrm{QI}(X) \xrightarrow{\cong} \mathrm{QI}(Y).$$

Proposition 4.23. *Let $f : X \to Y$ be a quasi-isometry between proper geodesic metric spaces. Then f induces a homeomorphism*

$$\overline{f} : \operatorname{Ends}(X) \xrightarrow{\cong} \operatorname{Ends}(Y).$$

Moreover, the map $\mathrm{QI}(X) \to \operatorname{Homeo}(\operatorname{Ends}(X))$ given by $f \mapsto \overline{f}$ is a group homomorphism.

Proof. Given a geodesic ray r in X from x_0, let $f_* r$ be the ray in Y obtained by concatenating some choice of geodesic segments $[f(r(n)), f(r(n+1))]$ for $n \in \mathbb{N}$. Since f is a quasi-isometry, $f_* r$ is a ray and $\operatorname{end}(f_* r)$ is independent of the choices of geodesic segments.

This allows one to define $\overline{f} : \operatorname{Ends}(X) \to \operatorname{Ends}(Y)$ by $\operatorname{end}(r) \mapsto \operatorname{end}(f_* r)$. The image of a k-path under f is a $(\lambda k + c)$-path, so $\overline{f}$ is well-defined, and continuous by Lemma 4.21.(i). Moreover, Lemma 4.21.(ii) ensures that $\overline{f}$ is defined on all of $\operatorname{Ends}(X)$. □

Definition 4.24 (Ends of a group)**.** *Given a finitely generated group G, we define*

$$\operatorname{Ends}(G) = \operatorname{Ends}(\operatorname{Cay}(G)).$$

Theorem 4.25. *Let G be a finitely generated group.*

(i) *G has zero, one, two, or infinitely many ends.*

(ii) *G has zero end* iff *G is finite.*

(iii) *G has two ends* iff *G is virtually $\mathbb{Z}$.*

(iv) *G has infinitely many ends* iff *G can be expressed as $A *_C B$ or $A *_C$, with C finite, $[A : C] \geqslant 3$ and $[B : C] \geqslant 3$.*

Proof. (i) Fix a finite generating set S for G. The action $G \curvearrowright \mathrm{Cay}(G,S)$ is by isometries and therefore gives a homomorphism

$$G \to \mathrm{Isom}\left(\mathrm{Cay}(G,S)\right) \subseteq \mathrm{QI}\left(\mathrm{Cay}(G,S)\right) \to \mathrm{Homeo}\left(\mathrm{Ends}\left(\mathrm{Cay}(G,S)\right)\right)$$

by Proposition 4.23. Let $H \trianglelefteq G$ be the kernel of the above composite.

Assume for contradiction that $\mathrm{Ends}(G)$ is finite but contains three distinct ends η_0, η_1, η_2. Fix geodesic rays $r_1, r_2 : [0,\infty) \to \mathrm{Cay}(G,S)$ with $r_1(0) = r_2(0) = e$ in G and $\mathrm{end}\,(r_i) = \eta_i$. Now the fact that $|\mathrm{Ends}(G)| < \infty$ implies that $[G:H] < \infty$, so there exists $\mu > 0$ such that for all $g \in G$, there exists $h \in H$ with $d(g,h) \leqslant \mu$. Therefore, there is a ray $r_0 : [0,\infty) \to \mathrm{Cay}(G,S)$ with $\mathrm{end}\,(r_0) = \eta_0$, $r_0(n) \in H$ and $d\left(r_0(n), e\right) \geqslant n$ for all n. Set $h_n = r_0(n) \in H$ for all $n \in \mathbb{N}$.

There exists $N \in \mathbb{N}$ s.t. $r_0\left([N,\infty)\right), r_1\left([N,\infty)\right), r_2\left([N,\infty)\right)$ lie in different path-components of $\mathrm{Cay}(G,S) \backslash B\left(e,N\right)$. If $t, t' > 2N$, then

$$d\left(r_1(t), r_2\left(t'\right)\right) > 2N \tag{$*$}$$

since any path joining $r_1(t)$ and $r_2\left(t'\right)$ must pass through $B(e,N)$. Moreover, H acts trivially on $\mathrm{Ends}^(G)$ by definition, so $\mathrm{end}\left(h_n r_i\right) = \mathrm{end}\left(r_i\right)$ for all i. Let $n > 3N$. Then $h_n r_i(0) = h_n$ lies in a path-component of $\mathrm{Cay}(G,S) \backslash B(e,N)$ different from that of $r_i\left([N,\infty)\right)$ for $i = 1,2$, so $h_n r_i$ must pass through $B(e,N)$. Therefore, there exist $t, t' > 2N$ such that $h_n r_1(t) \in B(e,N)$ and $h_n r_2\left(t'\right) \in B(e,N)$. Since H acts by isometries, $d\left(r_1(t), r_2\left(t'\right)\right) \leqslant 2N$, contradicting $(*)$. ☐

Remark 4.26. (i) *Being virtually free is a geometric property (equivalent to being quasi-isometric to a tree), c.f. Antolin.*

(ii) *Being finitely presentable is a geometric property, c.f. Bridson-Haefliger.*

5 Amenability

5.1 Paradoxical decompositions

Definition 5.1 (Equidecompositions)**.** *Let G be a group acting on a set X. Let $A, B \subseteq X$. We say that A, B are* (finitely) G-equidecomposable, *and we write $A \sim B$, if there are partitions*

$$A = A_1 \amalg A_2 \amalg \cdots \amalg A_n \quad \textit{and} \quad B = B_1 \amalg B_2 \amalg \cdots \amalg B_n,$$

and there are $g_1, \ldots, g_n \in G$, such that $g_i A_i = B_i$ for all i.

If $A \sim C$ for some $C \subseteq B$, we write $A \lesssim B$.

A realisation *h of $A \sim B$ is a bijection $h : A \to B$ such that there is a decomposition as above with $h\left(a_i\right) = g_i a_i$ for all i and for all $a_i \in A_i$. Note that, if $h : A \to B$ is a realisation of $A \sim B$, then $S \sim h(S)$ for all $S \subseteq A$.*

For fixed $G \curvearrowright X$, $\sim$ is an equivalence relation on $\mathcal{P}(X)$.

Theorem 5.2. *Let $G \curvearrowright X$ and $A, B \subseteq X$. If $A \lesssim B$ and $A \gtrsim B$, then $A \sim B$.*

Proof. Since $A \lesssim B$ and $A \gtrsim B$, there are realisations $f : A \to B_1$ and $g : A_1 \to B$, with $A_1 \subseteq A$ and $B_1 \subseteq B$. Define inductively $C_0 = A \backslash A_1$ and $C_{n+1} = g^{-1} f\left(C_n\right)$ for $n \geqslant 0$, and set

$$C = \bigcup_{n=0}^{\infty} C_n.$$

Then

$$g\left(A \backslash C\right) = g\left(\bigcap_{n=0}^{\infty} A \backslash C_n\right) \subseteq \bigcap_{n=0}^{\infty} B \backslash g\left(C_n\right) \subseteq \bigcap_{n=0}^{\infty} B \backslash g\left(C_{n+1}\right) = \bigcap_{n=0}^{\infty} B \backslash f\left(C_n\right) = B \backslash f(C).$$

Similarly, $B \backslash f(C) \subseteq g(A \backslash C)$. It follows that $A \backslash C \sim B \backslash f(C)$ *via* g. Since we also have $C \sim f(C)$ *via* f, we get $A \sim B$. ☐

Definition 5.3 (Paradoxical set)**.** *Let $G \curvearrowright X$. Then the following assertions are equivalent:*

(i) *There exist proper disjoint subsets $A, B \subseteq X$ such that $A \sim X \sim B$.*

(ii) *There exists a partition $X = A \amalg B$ such that $A \sim X \sim B$.*

In this case, we say that X is (finitely) G-paradoxical*.*

Proof. (ii) $\Rightarrow$ (i) Obvious.

(i) $\Rightarrow$ (ii) Since $X \sim B \subseteq X \backslash A$, we have $X \lesssim X \backslash A$. Moreover, the inclusion $X \backslash A \subseteq X$ implies that $X \backslash A \lesssim X$. It follows by Theorem 5.2 that $A \sim X \sim X \backslash A$. □

Proposition 5.4. (i) *F_2 is F_2-paradoxical (for the action of F_2 on itself by left multiplication).*

(ii) *If $F_2 \curvearrowright X$ freely, then X is F_2-paradoxical.*

Proof. (i) Let $\{a, b\}$ be a freely generating susbet of F_2. Given $y \in F_2$, denote by $\mathcal{W}(y)$ the set of reduced words starting in y. Then

$$\begin{aligned} F_2 &= \{e\} \amalg \mathcal{W}(a) \amalg \mathcal{W}\left(a^{-1}\right) \amalg \mathcal{W}(b) \amalg \mathcal{W}\left(b^{-1}\right) \\ &= \mathcal{W}(a) \amalg a\mathcal{W}\left(a^{-1}\right) \\ &= \mathcal{W}(b) \amalg b\mathcal{W}\left(b^{-1}\right). \end{aligned}$$

Hence, if $A = \mathcal{W}(a) \amalg \mathcal{W}\left(a^{-1}\right)$ and $B = \mathcal{W}(b) \amalg \mathcal{W}\left(b^{-1}\right)$, then $A \sim X \sim B$ and A, B are disjoint, so F_2 is F_2-paradoxical.

(ii) Let M be a set of representatives of F_2-orbits of X (note that we are using the Axiom of Choice). For $y \in F_2$, set

$$X_y = \{zm,\ z \in \mathcal{W}(y),\ m \in M\}.$$

Then $X_a, X_{a^{-1}}, X_b, X_{b^{-1}}$ are disjoint (because the action is free), and

$$X = X_a \amalg aX_{a^{-1}} = X_b \amalg bX_{b^{-1}},$$

giving the desired decomposition. □

Proposition 5.5. *The matrices*

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

generate a free subgroup of rank 2 inside $SO(3, \mathbb{R})$.

Theorem 5.6 (Hausdorff Paradox)**.** *There is a countable set $D \subseteq \mathbb{S}^2$ such that $\mathbb{S}^2 \backslash D$ is $SO(3, \mathbb{R})$-paradoxical.*

Proof. Given $\varphi \in SO(3, \mathbb{R}) \setminus \{\mathrm{id}\}$, let $F(\varphi) = \{x \in \mathbb{S}^2,\ \varphi(x) = x\}$. Note that $|F(\varphi)| = 2$ if $\varphi \neq \mathrm{id}$, and set

$$D = \bigcup_{\varphi \in F_2 \setminus \{1\}} F(\varphi),$$

where $F_2 \leqslant SO(3, \mathbb{R})$ as in Proposition 5.5. Then F_2 acts freely on $\mathbb{S}^2 \backslash D$, so we are done by Proposition 5.4. □

Proposition 5.7. *If $D \subseteq \mathbb{S}^2$ is countable, then $\mathbb{S}^2$ and $\mathbb{S}^2 \backslash D$ are $SO(3, \mathbb{R})$-equidecomposable.*

Proof. Let ℓ be a line through 0 that does not intersect D. There exists $\theta \in \mathbb{R}$ such that for all $n \geqslant 1$, the image $\rho^n(D)$ of D under the rotation ρ^n by $n\theta$ about ℓ does not intersect D. Set $\overline{D} = \bigcup_{n=0}^{\infty} \rho^n(D)$. Then

$$\mathbb{S}^2 = \overline{D} \amalg \left(\mathbb{S}^2 \backslash \overline{D}\right) \sim \rho\left(\overline{D}\right) \amalg \rho\left(\mathbb{S}^2 \backslash \overline{D}\right) \sim \rho\left(\overline{D}\right) \amalg \rho\left(\mathbb{S}^2 \backslash \overline{D}\right) \sim \rho\left(\overline{D}\right) \amalg \left(\mathbb{S}^2 \backslash \overline{D}\right) = \mathbb{S}^2 \backslash D. \qquad \square$$

Theorem 5.8 (Banach-Tarski)**.** *$\mathbb{S}^2$ is $SO(3, \mathbb{R})$-paradoxical.*

Proof. This is a corollary of Theorem 5.6 and Proposition 5.7. $\square$

Remark 5.9. (i) *We can prove similarly that if $E(3)$ is the group of isometries of $\mathbb{R}^3$, then any solid ball in $\mathbb{R}^3$ is $E(3)$-paradoxical and so is $\mathbb{R}^3$ itself.*

(ii) *By the Banach-Tarski paradox, we cannot put a finitely-additive probability measure that is invariant under rotation on subsets of $\mathbb{S}^2$.*

Theorem 5.10 (Tarski)**.** *Let $G \curvearrowright X$ and $E \subseteq X$. Then E is not G-paradoxical if and only if there is a G-invariant finitely-additive measure $\mu : \mathcal{P}(X) \to [0, \infty]$ with $\mu(E) = 1$.*

5.2 Non-paradoxical groups and amenability

Definition 5.11 (Amenable group)**.** *Let G be a discrete (or locally compact) group. A* measure *on G is a finitely-additive measure μ on $\mathcal{P}(G)$ (or on the Borel σ-algebra of G) with $\mu(G) = 1$ and which is G-invariant, i.e.*

$$\mu(gA) = \mu(A)$$

for all $g \in G$ and $A \subseteq G$. If G has such a measure, we say that G is amenable*.*

Example 5.12. *All finite groups are amenable.*

Remark 5.13. *If the action $G \curvearrowright G$ by left multiplication is paradoxical, then G is not amenable.*
In particular, F_2 is not amenable, nor is any group containing F_2.

Conjecture 5.14 (Von-Neumann)**.** *All non-amenable groups contain F_2 as a subgroup.*

Remark 5.15. *The Von-Neumann Conjecture was disproved by Olshankii, using the example of the* Tarski monster*.*

Definition 5.16 (Left-invariant mean)**.** *Let G be a finitely generated group. A* left-invariant mean *on G is a linear functional $M : \ell^\infty(G) \to \mathbb{R}$ such that*

(i) *If $\lambda(g) \geqslant 0$ for all $g \in G$, then $M(\lambda) \geqslant 0$,*

(ii) *$M(\mathbb{1}_G) = 1$,*

(iii) *$M(g \cdot \lambda) = M(\lambda)$ for all $\lambda \in \ell^\infty(G)$ and $g \in G$, where $(g \cdot \lambda)(h) = \lambda(gh)$.*

Proposition 5.17. *G is amenable if and only if G admits a left-invariant mean.*

Proof. ($\Rightarrow$) If μ is a measure on G, define

$$M(f) = \int_G f \, \mathrm{d}\mu.$$

($\Leftarrow$) If M is a left-invariant mean on G, define $\mu(A) = M(\mathbb{1}_A)$. $\square$

Proposition 5.18. *Let G be an amenable group acting on a set X. Then there is a finitely-additive probability measure on $\mathcal{P}(X)$ that is G-invariant.*
In particular, X is not G-paradoxical.

Proof. Let μ be a measure on G realising amenability. Fix $x_0 \in X$ and define $\nu : \mathcal{P}(X) \to [0,1]$ by

$$\nu(A) = \mu\left(\{g \in G,\ g(x_0) \in A\}\right). \qquad \square$$

Theorem 5.19. *Let G be a finitely generated group. Then the following assertions are equivalent.*

(i) *G is amenable.*

(ii) *G admits a left-invariant mean.*

(iii) *G is not paradoxical.*

Proposition 5.20. *Let G be a discrete (or locally compact) group.*

(i) *If G is amenable, then every subgroup of G is amenable.*

(ii) *If G is amenable, then every quotient of G is amenable.*

(iii) *If $N \trianglelefteq G$ is such that N and G/N are amenable, then G is amenable.*

(iv) *If $G = \varinjlim_{j\in J} G_j$, where $(G_j)_{j\in J}$ is a directed system of amenable groups, then G is amenable.*

Proof. (i) Let $\mu : \mathcal{P}(G) \to [0,1]$ be a G-invariant finitely-additive probability measure on $\mathcal{P}(G)$. Given a subgroup $H \leqslant G$, let M be a right transversal of H in G. Define $\nu : \mathcal{P}(H) \to [0,1]$ by $\nu(A) = \mu(AM)$.

(ii) Define $\lambda : \mathcal{P}(G/N) \to [0,1]$ by $\lambda(A) = \mu\left(\pi^{-1}(A)\right)$ where $\pi : G \twoheadrightarrow G/N$ is the projection.

(iii) Let $\nu_1 : \mathcal{P}(N) \to [0,1]$ and $\nu_2 : \mathcal{P}(G/N) \to [0,1]$ be invariant finitely-additive probability measures. For $A \subseteq G$, define

$$f_A : g \in G \longmapsto \nu_1\left(N \cap g^{-1}A\right) \in \mathbb{R}.$$

Note that $f_A(gn) = f_A(g)$ for all $n \in N$. Therefore, f_A induces $f_A : G/N \to \mathbb{R}$. Now define $\mu : \mathcal{P}(G) \to [0,1]$ by

$$\mu(A) = \int_{G/N} f_A \,\mathrm{d}\nu_2;$$

this is a finitely-additive probability measure. To check left-invariance, note that, for $h \in G$,

$$\mu(hA) = \int_{G/N} f_{hA}(g)\,\mathrm{d}\nu_2(g) = \int_{G/N} f_A\left(h^{-1}g\right)\mathrm{d}\nu_2(g) = \int_{G/N}\left(h^{-1}\cdot f_A\right)\mathrm{d}\nu_2.$$

Now the action of G on functions $G/N \to \mathbb{R}$ is *via* G/N, i.e. $(hn)^{-1}\cdot f_A = h^{-1}\cdot f_A$ for $h \in G$ and $n \in N$. It follows that

$$\mu(hA) = \int_{G/N}\left(h^{-1}\cdot f_A\right)\mathrm{d}\nu_2 = \int_{G/N} f_A\,\mathrm{d}\nu_2 = \mu(A). \qquad \square$$

5.3 Geometric interpretation of amenability

Definition 5.21 (Følner condition)**.** *A finitely generated group G satisfies the* (left) Følner condition *if for any finite subset $A \subseteq G$ and for any $\varepsilon > 0$, there is a finite nonempty subset $F \subseteq G$ such that*

$$\frac{|aF \triangle F|}{|F|} \leqslant \varepsilon$$

for all $a \in A$.

Theorem 5.22. *A finitely generated group G is amenable if and only if it satisfies the Følner condition.*

Proof. ($\Rightarrow$) See Druțu-Kapovich or Juschenko.

($\Leftarrow$) For $A \subseteq G$ finite and $\varepsilon > 0$, define

$$M_{A,\varepsilon} = \{\mu \text{ finitely-additive probability measure on } G,\ \forall B \subseteq G,\ \forall a \in A,\ |\mu(B) - \mu(aB)| \leqslant \varepsilon\}.$$

Note that $M_{A,\varepsilon}$ is a closed subset of $[0,1]^{\mathcal{P}(G)}$ for the product topology. Moreover, if F is a Følner set given by A, ε, and $\mu(B) = \frac{|B\cap F|}{|F|}$, then we have

$$\begin{aligned}|\mu(B) - \mu(aB)| &= \left|\frac{|B\cap F|}{|F|} - \frac{|aB\cap F|}{|F|}\right| = \left|\frac{|B\cap F|}{|F|} - \frac{|B\cap a^{-1}F|}{|F|}\right| \\ &\leqslant \frac{|F\triangle a^{-1}F|}{|F|} = \frac{|aF\triangle F|}{|F|} \leqslant \varepsilon,\end{aligned}$$

so $\mu \in M_{A,\varepsilon}$. Therefore, $(M_{A,\varepsilon})_{A,\varepsilon}$ is a collection of nonempty closed subsets of $[0,1]^{\mathcal{P}(G)}$, such that the intersection of any finite subcollection is nonempty:

$$\bigcap_{i=1}^{k} M_{A_i,\varepsilon_i} \supseteq M_{A_1\cup\cdots\cup A_k, \min\{\varepsilon_1,\dots,\varepsilon_k\}} \supsetneq \varnothing.$$

It follows by compactness of $[0,1]^{\mathcal{P}(G)}$ that $\bigcap_{A,\varepsilon} M_{A,\varepsilon} \neq \varnothing$; now any element $\mu \in \bigcap_{A,\varepsilon} M_{A,\varepsilon}$ is a finitely-additive G-invariant probability measure on G. $\square$

Definition 5.23 (Cheeger constant)**.** *Given a graph X, the* Cheeger constant *of X is defined by*

$$h(X) = \inf_{\substack{A\subseteq V(X) \text{ finite} \\ 1\leqslant |A| \leqslant \frac{1}{2}|V(X)|}} \frac{|\partial A|}{|A|},$$

where $\partial A = \{x \in V(X)\backslash A,\ \exists a \in A,\ xa \in E(X)\}$.

Remark 5.24. *A finitely generated group G satisfies the left Følner condition if and only if satisfies the right Følner condition: for any finite subset $A \subseteq G$ and for any $\varepsilon > 0$, there is a finite nonempty subset $F \subseteq G$ such that $\frac{|Fa\triangle F|}{|F|} \leqslant \varepsilon$ for all $a \in A$.*

Proof. Given $A \subseteq G$ and $\varepsilon > 0$, let F be the Følner set corresponding to $A^{-1} = \{a^{-1},\ a \in A\}$. Then F^{-1} satisfies

$$\frac{|F^{-1}a\triangle F^{-1}|}{|F^{-1}|} = \frac{|a^{-1}F\triangle F|}{|F|} \leqslant \varepsilon.$$ $\square$

Proposition 5.25. *Given a finitely generated group G, the following assertions are equivalent:*

(i) *G satisfies the Følner condition.*

(ii) *For any generating set S, $h(\mathrm{Cay}(G,S)) = 0$.*

(iii) *There is a generating set S such that $h(\mathrm{Cay}(G,S)) = 0$.*

Proof. (ii) $\Rightarrow$ (iii) Obvious.

(i) $\Rightarrow$ (ii) Let S be a finite generating set for G. Let $\varepsilon > 0$. Then there is a right Følner set F corresponding to $S^{\pm1}$ and ε, i.e. $\frac{|Fs\triangle F|}{|F|} \leqslant \varepsilon$ for all $s \in S^{\pm1}$. Then

$$\begin{aligned}h(\mathrm{Cay}(G,S)) &\leqslant \frac{|\partial F|}{|F|} = \frac{|\{gs,\ g\in G,\ s\in S^{\pm1},\ gs\notin F\}|}{|F|} \\ &\leqslant \frac{1}{|F|}\left|\bigcup_{s\in S^{\pm1}} (Fs\triangle F)\right| \leqslant \sum_{s\in S^{\pm1}} \frac{|Fs\triangle F|}{|F|} \leqslant 2|S|\,\varepsilon.\end{aligned}$$

Therefore, $h(\mathrm{Cay}(G,S)) = 0$. $\square$

Proposition 5.26. *Having Cheeger constant equal to zero is a quasi-isometry invariant.*

Corollary 5.27. *Amenability is a quasi-isometry invariant.*

Corollary 5.28. *All finitely generated groups of subexponential growth are amenable.*

Proof. Let G be a group of subexponential growth with a finite generating set S. If there existed $\varepsilon > 0$ such that $\frac{|B(k+1)|}{|B(k)|} > 1+\varepsilon$ for all $k \geqslant 0$, then we would have $|B(k+1)| > (1+\varepsilon)^k |B(1)|$, contradicting the fact that G has subexponential growth. Therefore, for all $\varepsilon > 0$, there exists $k \geqslant 0$ such that $\frac{|B(k+1)|}{|B(k)|} \leqslant 1+\varepsilon$, so that

$$h\left(\mathrm{Cay}(G,S)\right) \leqslant \frac{|\partial B\left(k\right)|}{|B(k)|} = \frac{|B(k+1)| - |B(k)|}{|B(k)|} \leqslant \varepsilon.$$

Hence $h\left(\mathrm{Cay}(G,S)\right) = 0$. □

Corollary 5.29. *All solvable groups are amenable.*

Proof. Corollary 5.28 implies that all abelian groups are amenable (because they have polynomial growth). Now Proposition 5.20 allows one to use an induction argument to show that all solvable groups are amenable. □

Remark 5.30. *Consider the closure of the class of finite groups and abelian groups under the operations of Proposition 5.20; this is called the class of* elementary amenable groups. *It is strictly contained within the class of all amenable groups (for example, Grigorchuk's group is non-elementary amenable).*

It is an open question to know whether or not elementary amenability is a quasi-isometry invariant.

Definition 5.31 (Ponzi scheme)**.** *A* Ponzi scheme *on a metric graph X is a map $\rho : X \to X$ such that:*

(i) $\exists R \geqslant 0,\ \forall x \in X,\ d\left(x, \rho x\right) \leqslant R,$

(ii) $\forall x \in X,\ |\rho^{-1}\left(\{x\}\right)| \geqslant 2.$

Proposition 5.32. *A group G has a Ponzi scheme if and only if it is non-amenable.*

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