Category Theory

Lectures by Peter Johnstone
Notes by Alexis Marchand

University of Cambridge
Michaelmas 2019
Part III course

Contents

1 Definitions and examples

1.1 Categories and functors

Definition 1.1 (Category). *A* category $\mathbf{C}$ *consists of:*

(i) *A collection* $\operatorname{ob}\mathbf{C}$ *of* objects $A, B, C, \ldots,$

(ii) *A collection* $\operatorname{mor}\mathbf{C}$ *of* morphisms $f, g, h, \ldots,$

(iii) *Two operations* dom *and* cod *from* $\operatorname{mor}\mathbf{C}$ *to* $\operatorname{ob}\mathbf{C}$*: we write* $A \xrightarrow{f} B$ *to mean* $f \in \operatorname{mor}\mathbf{C}$, $\operatorname{dom} f = A$ *and* $\operatorname{cod} f = B$*,*

(iv) *An operation* $A \mapsto 1_A$ *from* $\operatorname{ob}\mathbf{C}$ *to* $\operatorname{mor}\mathbf{C}$ *s.t.* $A \xrightarrow{1_A} A$*,*

(v) *A partial binary operation* $(f, g) \mapsto fg$ *on* $\operatorname{mor}\mathbf{C}$ *defined* iff $\operatorname{dom} f = \operatorname{cod} g$ *and satisfying* $\operatorname{dom}(fg) = \operatorname{dom} g$ *and* $\operatorname{cod}(fg) = \operatorname{cod} f$*,*

satisfying:

(vi) $1_B f = f = f 1_A$ *for all* $A \xrightarrow{f} B$*,*

(vii) $f(gh) = (fg)h$ *for all* $A \xrightarrow{h} B \xrightarrow{g} C \xrightarrow{f} D$*.*

Remark 1.2.
- *We don't require* $\operatorname{ob}\mathbf{C}$ *and* $\operatorname{mor}\mathbf{C}$ *to be sets. If they are,* $\mathbf{C}$ *is called* small.
- *We could formulate the definition of categories with 'morphisms' as the only primitive notion, identifying 'objects' with identity morphisms. However, in practice, the objects are often logically prior to the morphisms.*

Example 1.3. (i) *The category* **Set** *has all sets as objects, and all functions between them as morphisms. Formally, morphisms are pairs* (f, B) *where* f *is a set-theoretic function and* $B = \operatorname{cod}(f, B)$*.*

(ii) **Gp** *is the category of groups and group homomorphisms. Likewise,* **Rng** *is the category of rings and ring homomorphisms, and for any given ring* R*,* $\mathbf{Mod}_R$ *is the category of* R*-modules and* R*-linear maps.*

(iii) **Top** *(resp.* **Unif***,* **Mf***) is the category of topological spaces (resp. uniform spaces, smooth manifolds) and continuous (resp. uniformly continuous, smooth) maps.*

(iv) *The category* **Htpy** *has the same objects as* **Top***, but morphisms are homotopy classes of continuous maps. More generally, given an equivalence relation* $\sim$ *on* $\operatorname{mor}\mathbf{C}$ *satisfying*

- $f \sim g \Longrightarrow \operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$,
- $f \sim g \Longrightarrow fh \sim gh$ and $kf \sim kg$ *whenever the composites are defined,*

we have a new quotient category $\mathbf{C}/\sim$ *with the same objects as* $\mathbf{C}$*, but with* $\sim$*-equivalence classes as morphisms.*

(v) *The category* **Rel** *has the same objects as* **Sets***, but morphisms* $A \to B$ *are all relations* $R \subseteq A \times B$*, with composition of* $A \xrightarrow{R} B \xrightarrow{S} C$ *defined to be*

$$SR = \{(a, c) \in A \times C,\ \exists b \in B,\ (a, b) \in R \text{ and } (b, c) \in S\}.$$

(vi) *For any category* $\mathbf{C}$*, the* opposite category $\mathbf{C}^{\mathrm{op}}$ *has the same objects and morphisms as* $\mathbf{C}$*, but* dom *and* cod *are interchanged, and* fg *in* $\mathbf{C}^{\mathrm{op}}$ *is* gf *in* $\mathbf{C}$*. Hence, we have the* duality principle*: if* P *is a true statement about categories, so is the* dual statement P^* *obtained from* P *by reversing all arrows.*

(vii) *A category with only one object* $*$ *has* $\operatorname{dom} f = \operatorname{cod} f = *$ *for all morphisms* f*, so composition is defined everywhere. Hence, a small category with one object may be identified with a* monoid*. In particular, a* group *is a small category with one object, in which all morphisms are isomorphisms.*

(viii) *A (Brandt)* groupoid *is a category in which all morphisms are isomorphisms. For instance, for any category* $\mathbf{C}$*, the* core category $\operatorname{core}(\mathbf{C})$ *has the same objects as* $\mathbf{C}$*, but only the isomorphisms of* $\mathbf{C}$ *are morphisms of* $\operatorname{core}(\mathbf{C})$*. Also, for any topological space* X*, the fundamental groupoid* $\pi(X)$ *has points of* X *as objects and morphisms* $x \to y$ *are homotopy classes of paths* $u : [0, 1] \to X$ *with* $u(0) = x$ *and* $u(1) = y$*. If* $x \xrightarrow{u} y \xrightarrow{v} z$*,* $[v][u]$ *is the homotopy class of the concatenation* $u \cdot v$ *of* u *and* v*.*

(ix) *A* discrete category *is a category whose only morphisms are identities. More generally, if* $\mathbf{C}$ *is a category with at most one morphism between any two objects* A *and* B*, we can view the morphisms as a reflexive and transitive relation on* $\operatorname{ob}\mathbf{C}$*. We call such a relation a* preorder*. In particular, a* partial order *is a preorder in which the only isomorphisms are identities.*

(x) *Let* K *be a field. The category* $\mathbf{Mat}_K$ *has natural numbers as objects, and morphisms* $n \to p$ *are* $p \times n$ *matrices with entries in* K*.*

Definition 1.4 (Functor)**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. A functor* $F : \mathbf{C} \to \mathbf{D}$ *consists of*

(i) *A mapping* $A \mapsto FA$ *from* $\operatorname{ob}\mathbf{C}$ *to* $\operatorname{ob}\mathbf{D}$*,*

(ii) *A mapping* $f \mapsto Ff$ *from* $\operatorname{mor}\mathbf{C}$ *to* $\operatorname{mor}\mathbf{D}$*,*

satisfying $F(\operatorname{dom} f) = \operatorname{dom}(Ff)$*,* $F(\operatorname{cod} f) = \operatorname{cod}(Ff)$*,* $F(1_A) = 1_{FA}$ *and* $F(fg) = (Ff)(Fg)$ *whenever* fg *is defined.*

Example 1.5. *We write* **Cat** *for the category whose objects are small categories and whose morphisms are the functors between them.*

Example 1.6. (i) *We have forgetful functors* **Gp** $\to$ **Set***,* **Top** $\to$ **Set***,* **Rg** $\to$ **AbGp***,* **Mf** $\to$ **Top***, etc.*

(ii) *We have a functor* $F : \mathbf{Set} \to \mathbf{Gp}$*, where* FA *is the* free group *on a set* A*, and we have an injection* $\eta_A : A \to FA$ *s.t. for any function* $f : A \to G$*, where* G *has a group structure, there is a unique group homomorphism* $\overline{f} : FA \to G$ *s.t.* $\overline{f}\eta_A = f$*. Given a function* $f : A \to B$*, we define* $Ff : FA \to FB$ *to be* $Ff = \overline{\eta_B \circ f}$*. This is functorial.*

(iii) *Given a set A, we write PA for the* power-set *of A, i.e. the set of all subsets of A. We can make P into a functor* $\mathbf{Set} \to \mathbf{Set}$*: given $f : A \to B$, we define $Pf : PA \to PB$ by $A' \mapsto \{f(a),\ a \in A'\}$. But we also have a functor $P^* : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ defined by $P^*A = PA$ and $P^*f : B' \mapsto \{a \in A,\ f(a) \in B'\}$*

Remark 1.7. *We sometimes use the term* contravariant functor $\mathbf{C} \to \mathbf{D}$ *for a functor* $\mathbf{C} \to \mathbf{D}^{\mathrm{op}}$ *(or* $\mathbf{C}^{\mathrm{op}} \to \mathbf{D}$*). In this context, we also say* covariant functor *for one which does not reverse arrows.*

Example 1.8. (i) *Given a vector space V over a field K, the* dual space V^* *consists of all linear forms $V \to K$. If $f : V \to W$ is linear, it induces a linear map $f^* : W^* \to V^*$ given by $\theta \mapsto \theta f$. This defines a functor* $\mathbf{Mod}_K \to \mathbf{Mod}_K^{\mathrm{op}}$.

(ii) *We have a functor* $\mathrm{op} : \mathbf{Cat} \to \mathbf{Cat}$*, with the identity operation on morphisms. Note that this functor is covariant!*

(iii) *A functor between monoids is exactly a monoid homomorphism.*

(iv) *A functor between preorders is exactly an order-preserving map.*

(v) *Let G be a group (viewed as a category with $\mathrm{ob}\, G = \{*\}$). A functor $F : G \to$* $\mathbf{Set}$ *consists of a set $A = F*$ together with a mapping $g \cdot (-) : A \to A$ for all $g \in G$, such that $1 \cdot a = a$ and $g \cdot (h \cdot a) = (gh) \cdot a$ for all $a \in A$ and $g, h \in G$. This is just a* permutation representation *of G (or a G-set). Similarly, if K is a field, a functor $G \to$* $\mathbf{Mod}_K$ *is a* K-linear representation *of G.*

(vi) *The* fundamental groupoid *defines a functor* $\pi : \mathbf{Top} \to \mathbf{Cat}$*. If* $\mathbf{Top}_*$ *denotes the category of spaces with chosen basepoint, then π_1 defines a functor* $\pi_1 : \mathbf{Top}_* \to \mathbf{Gp}$.

1.2 Natural transformations and equivalence of categories

Definition 1.9 (Natural transformation)**.** *Let $\mathbf{C}$ and $\mathbf{D}$ be two categories and $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ be two functors. A* natural transformation $\alpha : F \to G$ *consists of a mapping $A \mapsto \alpha_A$ from* $\mathrm{ob}\,\mathbf{C}$ *to* $\mathrm{mor}\,\mathbf{D}$ *s.t.* $\mathrm{dom}\,\alpha_A = FA$, $\mathrm{cod}\,\alpha_A = GA$ *and the following diagram commutes for every $A \xrightarrow{f} B$ in* $\mathrm{mor}\,\mathbf{C}$*:*

$$
\begin{array}{c}
A \\
\downarrow{\scriptstyle f} \\
B
\end{array}
\qquad
\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FB \arrow[r, "\alpha_B"] & GB
\end{tikzcd}
$$

Example 1.10. *We denote by $[\mathbf{C}, \mathbf{D}]$ the category whose objects are functors $\mathbf{C} \to \mathbf{D}$ and whose morphisms are natural transformations.*

Example 1.11. (i) *Let K be a field. For any vector space V over K, we have a linear map $\alpha_V : V \to V^{**}$ given by $\alpha_V(x) \cdot \lambda = \lambda(x)$. This defines a natural transformation* $1_{\mathbf{Mod}_K} \to **$.

(ii) *For any set A, we have a mapping $A \xrightarrow{\{\cdot\}_A} PA$ given by $a \mapsto \{a\}$. This defines a natural transformation* $1_{\mathbf{Set}} \to P$.

(iii) *Let $F :$* $\mathbf{Set} \to \mathbf{Gp}$ *be the free group functor and $U :$* $\mathbf{Gp} \to \mathbf{Set}$ *be the forgetful functor. For every set A, we have a mapping $\eta_A : A \to UFA$, defining a natural transformation* $1_{\mathbf{Set}} \to UF$.

(iv) *Let G, H be groups (considered as categories) and $f, g : G \rightrightarrows H$ be two group homomorphisms. A natural transformation $\alpha : f \to g$ consists of an element $h = \alpha_* \in H$ such that, for any $x \in G$, we have $hf(x) = g(x)h$. In other words, a natural transformation $f \to g$ is a conjugacy from f to g.*

(v) *Let G be a group and A, B be two G-sets considered as functors $G \rightrightarrows \mathbf{Set}$. A natural transformation $\alpha : A \to B$ is a mapping $A \xrightarrow{f} B$ which is G-equivarient, i.e. $f(g \cdot a) = g \cdot f(a)$ for all $g \in G$, $a \in A$.*

Lemma 1.12. *Let $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ be two functors between categories and let $\alpha : F \to G$ be a natural transformation. Then α is an isomorphism in the category $[\mathbf{C}, \mathbf{D}]$* iff *for all $A \in \operatorname{ob}\mathbf{C}$, α_A is an isomorphism in the category $\mathbf{D}$.*

In that case, we say that $\alpha : F \to G$ is a natural isomorphism *and we write $F \cong G$.*

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) Assume that for all $A \in \operatorname{ob}\mathbf{C}$, there exists $\beta_A : GA \to FA$ which is an inverse of $\alpha_A : FA \to GA$. It suffices to prove that the morphisms β_A define a natural transformation. To do this, consider a morphism $A \xrightarrow{f} B$ in $\mathbf{C}$. Then we have

$$\beta_B\,(Gf) = \beta_B\,(Gf)\,\alpha_A\beta_A = \beta_B\alpha_B\,(Ff)\,\beta_A = (Ff)\,\beta_A,$$

as required. □

Definition 1.13 (Equivalence of categories)**.** *An* equivalence *between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ together with natural isomorphisms $\alpha : 1_{\mathbf{C}} \to GF$ and $\beta : FG \to 1_D$.*

We write $\mathbf{C} \simeq \mathbf{D}$ if there exists such an equivalence.

A property $\mathcal{P}$ of categories is said to be a categorical property *if it is invariant under equivalence: if $\mathbf{C}$ has $\mathcal{P}$ and $\mathbf{C} \simeq \mathbf{D}$, then $\mathbf{D}$ has $\mathcal{P}$.*

Example 1.14. *The properties of being a groupoid or a preorder are categorical, but those of being a group or a partial order are not.*

Example 1.15. (i) *The category* **Part** *of sets and partial functions is equivalent to the category $\mathbf{Set}_*$ of pointed sets.*

We have a functor $F : \mathbf{Set}_* \to \mathbf{Part}$ defined by $F(A, a) = A\backslash\{a\}$ and, given $(A, a) \xrightarrow{f} (B, b)$, Ff is the mapping defined on $A\backslash f^{-1}(\{b\})$ by $Ff(x) = f(x)$. Likewise, there is a functor $G : \mathbf{Part} \to \mathbf{Set}$ defined by $G(A) = A \cup \{A\}$ and, given $A \xrightarrow{f} B$, Gf is defined by $Gf(x) = f(x)$ if f is defined at x, $Gf(x) = A$ otherwise.

Hence, $FG = 1_{\mathbf{Set}}$ and $GF \cong 1_{\mathbf{Set}}$.

Note that in **Part***, the object $\varnothing$ is not isomorphic to any other object, whereas in $\mathbf{Set}_*$, each isomorphism class of objects has many elements. Thus, $\mathbf{Part} \not\cong \mathbf{Set}_*$.*

(ii) *The category $\mathbf{fdMod}_K$ of finite-dimensional K-vector spaces is equivalent to $\mathbf{fdMod}_K^{\mathrm{op}}$, where F and G are both the duality functor, and both α and β are the natural isomorphism $V \to V^{**}$.*

(iii) *The category $\mathbf{fdMod}_K$ is equivalent to $\mathbf{Mat}_K$.*

We have a functor $F : \mathbf{Mat}_K \to \mathbf{fdMod}_K$ given by $F(n) = K^n$, and if A is a $p \times n$ matrix, then $F(A)$ is the linear map $K^n \to K^p$ represented by A in the standard basis. To define $G : \mathbf{fdMod}_K \to \mathbf{Mat}_K$, choose a basis for each finite-dimensional vector space and define $G(V) = \dim V$ and $G\left(V \xrightarrow{f} W\right)$ is the matrix representing f in the chosen bases.

Thus, $GF = 1_{\mathbf{Mat}_K}$ (if one has chosen the canonical basis for K^n) and $FG \cong 1_{\mathbf{fdMod}_K}$ via *the isomorphism $K^{\dim V} \to V$ induced by the choice of basis on V.*

Definition 1.16 (Faithful, full and essentially surjective functors)**.** *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor.*

(i) *We say that F is* faithful *if, given $f, g : A \rightrightarrows B$ in $\mathbf{C}$, $Ff = Fg$ implies $f = g$.*

(ii) *We say that F is* full *if, given $FA \xrightarrow{g} FB$ in $\mathbf{D}$, there exists $A \xrightarrow{f} B$ in $\mathbf{C}$ with $Ff = g$.*

(iii) *We say that F is* essentially surjective *if, given $B \in \mathrm{ob}\,\mathbf{D}$, there exists $A \in \mathrm{ob}\,\mathbf{C}$ with $FA \cong B$.*

(iv) *We say that a subcategory $\mathbf{C}' \subseteq \mathbf{C}$ is* full *if the inclusion functor $\mathbf{C}' \to \mathbf{C}$ is full.*

Example 1.17. **Gp** *is a full subcategory of the category* **Mon** *of monoids, but* **Mon** *is not a full subcategory of the category* **Sgp** *of semi-groups.*

Lemma 1.18. *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Then F is part of an equivalence $\mathbf{C} \simeq \mathbf{D}$ if and only if F is full, faithful and essentially surjective.*

Proof. ($\Rightarrow$) Suppose there is a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\alpha : 1_{\mathbf{C}} \to GF$ and $\beta : FG \to 1_{\mathbf{D}}$. For any $B \in \mathrm{ob}\,\mathbf{D}$, there is an isomorphism $\beta_B : FGB \to B$, so F is essentially surjective. Consider $f, g : A \rightrightarrows B$ with $Ff = Fg$. Then $GFf = GFg$, and f is the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{GFf} GFB \xrightarrow{\alpha_B^{-1}} B,$$

so $f = g$ and F is faithful. Now, suppose given $FA \xrightarrow{g} FB$. Let f be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFB \xrightarrow{\alpha_B^{-1}} B.$$

Then Ff and g have the same image under G because we have the following naturality square:

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "f"'] & GFA \arrow[d, "Gg = GFf"] \\ B \arrow[r, "\alpha_B"] & GFB \end{tikzcd}$$

And G is faithful for the same reason as F, so $Ff = g$.

($\Leftarrow$) Suppose F is full, faithful and essentially surjective. For each $B \in \mathrm{ob}\,\mathbf{D}$, choose an object $GB \in \mathrm{ob}\,\mathbf{C}$ and an isomorphism $\beta_B : FGB \to B$. Note that we are using the Axiom of Choice. Given $B \xrightarrow{g} C$, we define $Gg : GB \to GC$ to be the unique morphism whose image under F is the composite

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} C \xrightarrow{\beta_C^{-1}} FGC.$$

The faithfulness of F implies the functoriality of G: given $C \xrightarrow{h} D$, $G(hg)$ and $(Gh)(Gg)$ have the same image under F, so they are equal. Thus, G defines a functor $G : \mathbf{D} \to \mathbf{C}$, and β is a natural isomorphism $\beta : FG \to 1_{\mathbf{D}}$. Given $A \in \mathrm{ob}\,\mathbf{C}$, we now define $\alpha_A : A \to GFA$ to be the unique morphism mapped by F to $\beta_{FA}^{-1} : FA \to FGFA$. This is an isomorphism, with α_A^{-1} the unique morphism s.t. $F\left(\alpha_A^{-1}\right) = \beta_{FA}$. Finally, α defines a natural transformation $1_{\mathbf{C}} \to GF$ because each naturality square for α is mapped by F to a naturality square for β, so it commutes. Hence, $\alpha : 1_{\mathbf{C}} \to GF$ is a natural isomorphism. $\square$

Definition 1.19 (Skeletal category and skeletons)**.** (i) *We say that a category $\mathbf{C}$ is* skeletal *if every isomorphism f in $\mathbf{C}$ satisfies* $\mathrm{dom}\, f = \mathrm{cod}\, f$.

(ii) *A* skeleton *of a category $\mathbf{C}$ is a full subcategory containing exactly one object from each isomorphism class of objects of $\mathbf{C}$.*

Note that a skeleton of a category is a skeletal category.

Example 1.20. *The category $\mathbf{Mat}_K$ is skeletal.*

Corollary 1.21. *Any category is equivalent to any of its skeletons.*

Moreover, an equivalence between skeletal categories must be an isomorphism.

Remark 1.22. *It may be tempting to restrict one's attention to skeletal categories, but that would require heavy use of the Axiom of Choice.*

Definition 1.23 (Monomorphisms and epimorphisms)**.** *Let $A \xrightarrow{f} B$ be a morphism in a category $\mathbf{C}$.*

(i) *We say that f is a* monomorphism *if, given $h, k : C \rightrightarrows A$ with $fh = fk$, we have $h = k$.*

In that case, we may write $f : A \rightarrowtail B$.

If there exists $B \xrightarrow{g} A$ with $gf = 1_A$, we say that f is a split monomorphism*.*

(ii) *We say that f is a (split) epimorphism if it is a (split) monomorphism in $\mathbf{C}^{\mathrm{op}}$.*

In that case, we may write $f : A \twoheadrightarrow B$.

(iii) *We say that the category $\mathbf{C}$ is* balanced *if every morphism which is both monic and epic is an isomorphism.*

Example 1.24. (i) *In **Set**, monic is equivalent to injective and epic is equivalent to surjective. The category **Set** is balanced.*

(ii) *In **Gp**, monic is equivalent to injective and epic is equivalent to surjective. The category **Gp** is balanced.*

(iii) *In **Rg**, monic is equivalent to injective but the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic and not surjective. The category **Rg** is not balanced.*

(iv) *In **Top**, monic is equivalent to injective and epic is equivalent to surjective. The category **Top** is not balanced because not all bijective continuous maps are homeomorphisms.*

(v) *In a preorder, every morphism is both monic and epic, so a balanced preorder is a groupoid.*

2 The Yoneda Lemma

2.1 The Yoneda Lemma

Definition 2.1 (Locally small category)**.** *We say that a category $\mathbf{C}$ is* locally small *if, for any pair (A, B) of objects of $\mathbf{C}$, the morphisms $A \to B$ in $\mathbf{C}$ form a set $\mathbf{C}(A, B)$.*

In that case, if we fix A, the assignment $B \mapsto \mathbf{C}(A, B)$ becomes a functor $\mathbf{C}(A, -) : \mathbf{C} \to \mathbf{Set}$: given $B \xrightarrow{g} C$, we define $\mathbf{C}(A, B) \xrightarrow{\mathbf{C}(A,g)} \mathbf{C}(A, C)$ by $f \mapsto gf$.

Similarly, for fixed B, $\mathbf{C}(-, B)$ defines a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Lemma 2.2 (Yoneda Lemma)**.** *Let $\mathbf{C}$ be a locally small category. Consider $A \in \operatorname{ob} \mathbf{C}$ and $F : \mathbf{C} \to \mathbf{Set}$ a functor. Then:*

(i) *There is a bijection between natural transformations $\mathbf{C}(A, -) \to F$ and elements of FA.*

(ii) *This bijection is natural in both A and F.*

Proof. (i) Given $\alpha : \mathbf{C}(A, -) \to F$, we define

$$\Phi(\alpha) = \alpha_A (1_A) \in FA.$$

Conversely, given $x \in FA$, we define $\Psi(x) : \mathbf{C}(A, -) \to F$ by

$$\Psi(x)_B \left(A \xrightarrow{f} B \right) = Ff(x) \in FB.$$

$\Psi(x)$ is natural: given $g : B \to C$, we have

$$(Fg)\Psi(x)_B(f) = FgFf(x) = F(gf)(x) = \Psi(x)_C(gf) = \Psi(x)_C \mathbf{C}(A, g) f.$$

Moreover, we have $\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x$, and $\Psi\Phi(\alpha) = \alpha$: given any $A \xrightarrow{f} B$, we have $\Psi\Phi(\alpha)_B(f) = \Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) = \alpha_B \mathbf{C}(A, f)(1_A) = \alpha_B(f)$.

(ii) Suppose first that $\mathbf{C}$ is small, so that $[\mathbf{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \to \mathbf{Set}$: the first one sends (A, F) to FA and $\left(A \xrightarrow{f} B, F \xrightarrow{\alpha} G\right)$ to the diagonal of the naturality square of α at f; while the second one is the composite

$$\mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathbf{C}, \mathbf{Set}]^{\mathrm{op}} \times [\mathbf{C}, \mathbf{Set}] \xrightarrow{[\mathbf{C},\mathbf{Set}](-,-)} \mathbf{Set},$$

where $\mathbf{C}^{\mathrm{op}} \xrightarrow{Y} [\mathbf{C}, \mathbf{Set}]$ is the Yoneda embedding, given by $A \mapsto \mathbf{C}(A, -)$. Then Φ defines a natural isomorphism between these two functors. If we turn this into an elementary statement about Φ, it does not require $\mathbf{C}$ to be small. To verify it, suppose given $A \xrightarrow{f} B$, $\mathbf{C}(A, -) \xrightarrow{\alpha} F$ and $F \xrightarrow{\beta} G$. Then

$$\beta_B Ff(\Phi(\alpha)) = \beta_B(Ff)(\alpha_A(1_A)) = \beta_B \alpha_B \mathbf{C}(-, f)_A(1_A) = \beta_B \alpha_B \mathbf{C}(f, -)_B(1_B). \qquad \square$$

Corollary 2.3. *For any locally small category* $\mathbf{C}$*, the assignment* $A \mapsto \mathbf{C}(A, -)$ *defines a full and faithful functor* $\mathbf{C}^{\mathrm{op}} \xrightarrow{Y} [\mathbf{C}, \mathbf{Set}]$.

Similarly, $B \mapsto \mathbf{C}(-, B)$ *defines a full and faithful functor* $\mathbf{C} \xrightarrow{Y} [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$.

We call either of these functors the Yoneda embedding.

Proof. Applying the Yoneda Lemma to $F = \mathbf{C}(B, -)$ gives a bijection between elements of $\mathbf{C}(B, A)$ and morphisms $\mathbf{C}(A, -) \to \mathbf{C}(B, -)$ in $[\mathbf{C}, \mathbf{Set}]$. We may verify that this sends $B \xrightarrow{f} A$ to the natural transformation whose value at $A \xrightarrow{g} C$ is gf; therefore it is functorial, and clearly full and faithful. $\square$

2.2 Representable functors

Definition 2.4 (Representable functor)**.** *We say that a functor* $\mathbf{C} \to \mathbf{Set}$ *is* representable *if it is isomorphic to* $\mathbf{C}(A, -)$ *for some* A.

By a representation of $F : \mathbf{C} \to \mathbf{Set}$*, we mean a pair* (A, x)*, with* $x \in FA$ *such that* $\Psi(x)$ *is a natural isomorphism* $\mathbf{C}(A, -) \to F$*. We also call* x *a* universal element *of* F*.*

Corollary 2.5. *If* (A, x) *and* (B, y) *are two representations of* F*, then there is a unique isomorphism* $f : A \to B$ *s.t.* $Ff(x) = y$*.*

Proof. Consider the isomorphism

$$\mathbf{C}(B, -) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathbf{C}(A, -).$$

By the Yoneda Lemma (Corollary 2.3), this is of the form $\mathbf{C}(f, -)$ for a unique isomorphism $A \xrightarrow{f} B$. In other words, $\Psi(y) = \Psi(x) \circ \mathbf{C}(f, -)$, i.e. $Ff(x) = y$. $\square$

Example 2.6. (i) *The forgetful functor* $\mathbf{Gp} \to \mathbf{Set}$ *is representable by* $(\mathbb{Z}, 1)$.

The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ *is representable by* $(\{*\}, *)$.

(ii) *The functor* $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ *is representable by* $(\{0, 1\}, \{1\})$.

(iii) *The composite* $\mathbf{Mod}_K^{\mathrm{op}} \xrightarrow{*} \mathbf{Mod}_K \xrightarrow{U} \mathbf{Set}$ *is representable by* (K, id_K).

(iv) *If* G *is a group, the unique functor* $G(*, -) : G \to \mathbf{Set}$ *is the* Cayley representation *of* G*, i.e. the set* G *together with* G*-action by multiplication.*

Definition 2.7 (Products and coproducts)**.** *Suppose given two objects A, B in a locally small category* $\mathbf{C}$*. Consider the functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ whose value at $\mathbf{C}$ is $\mathbf{C}(C, A) \times \mathbf{C}(C, B)$. A representation of this functor, if it exists, is called a* product *of A and B: it consists of an object $A \times B$ together with morphisms $A \times B \xrightarrow{\pi_1} A$ and $A \times B \xrightarrow{\pi_2} B$ such that, given any pair of morphisms $C \xrightarrow{f} A$, $C \xrightarrow{g} B$, there exists a unique $C \xrightarrow{h} A \times B$ s.t. $\pi_1 h = f$ and $\pi_2 h = g$.*

Note that this definition still makes sense if $\mathbf{C}$ is not locally small.

Dually, we have a notion of coproduct *$A + B$ equipped with maps $A \xrightarrow{\nu_1} A + B$ and $B \xrightarrow{\nu_2} A + B$.*

Definition 2.8 (Equalizers and coequalizers)**.** *Suppose given a parallel pair of arrows $f, g : A \rightrightarrows B$ in a locally small category $\mathbf{C}$. We have a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ sending C to $\{h : C \to A,\ fh = gh\}$. A representation of this functor, if it exists, is called an* equalizer *of (f, g): it consists of $E \xrightarrow{e} A$ satisfying $fe = ge$ and such that any $h : C \to A$ with $fh = gh$ factors uniquely as ek for some $C \xrightarrow{k} E$.*

Dually, we have a notion of coequalizer*, namely a morphism $B \xrightarrow{d} D$ s.t. $df = dg$ and any h with $hf = hg$ factors uniquely as kd.*

Note that if $E \xrightarrow{e} A$ is an equalizer of (f, g), then it is monic (and dually, coequalizers are epic).

We say that a monomorphism (resp. an epimorphism) is regular *if it occurs as an equalizer (resp. as a coequalizer). Note that split monomorphisms (resp. epimorphisms) are always regular.*

2.3 Separating and detecting families

Definition 2.9 (Separating and detecting families)**.** *Let $\mathbf{C}$ be a category and $\mathcal{G}$ be a class (not necessarily a set) of objects of $\mathbf{C}$.*

(i) *We say that $\mathcal{G}$ is a* separating family *if, given $f, g : A \rightrightarrows B$ s.t. $fh = gh$ for all $\mathcal{G} \ni G \xrightarrow{h} A$, we have $f = g$.*

(ii) *We say that $\mathcal{G}$ is a* detecting family *if, given $A \xrightarrow{f} B$ s.t. every $\mathcal{G} \ni G \xrightarrow{h} B$ factors uniquely through f, f is an isomorphism.*

If $\mathcal{G} = \{G\}$, we call G a separator *(resp. a* detector*) for $\mathbf{C}$.*

Remark 2.10. *Assume that $\mathbf{C}$ is locally small and let $G \in \mathrm{ob}\,\mathbf{C}$. Then G is a separator if and only if the functor $\mathbf{C}(G, -) : \mathbf{C} \to \mathbf{Set}$ is faithful.*

Lemma 2.11. (i) *If $\mathbf{C}$ has equalizers, then every detecting family is separating.*

(ii) *If $\mathbf{C}$ is balanced, then every separating family is detecting.*

Proof. (i) Suppose $\mathcal{G}$ is detecting. Let $f, g : A \rightrightarrows B$ s.t. $fh = gh$ for all $\mathcal{G} \ni G \xrightarrow{h} A$. Let $E \xrightarrow{e} A$ be an equalizer of (f, g). Then every $\mathcal{G} \ni G \xrightarrow{h} A$ factors uniquely through e, so e is an isomorphism because $\mathcal{G}$ is detecting, and therefore $f = g$.

(ii) Suppose $\mathcal{G}$ is separating. Let $A \xrightarrow{f} B$ s.t. every $\mathcal{G} \ni G \xrightarrow{h} B$ factors uniquely through f. The morphism f is monic: if $g, h : C \rightrightarrows A$ satisfy $fg = fh$, then any $\mathcal{G} \ni G \xrightarrow{k} C$ must satisfy $gk = hk$ because both are factorisations of fgk through f, and therefore $g = h$ because $\mathcal{G}$ is separating. Likewise, f is epic, and $\mathbf{C}$ is balanced so f is an isomorphism. □

Example 2.12. (i) *In $\mathbf{Set}$, $\{*\}$ is a separator because it represents the identity functor (which is faithful), and it is also a detector.*

(ii) *In $\mathbf{Gp}$, $\mathbb{Z}$ is a separator because it represents the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$, and also a detector.*

(iii) *In $\mathbf{Set}$, $\{0, 1\}$ is a coseparator because it represents the functor $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, and also a codetector.*

(iv) *In* **Top**, $\{*\}$ *is a separator since it represents the forgetful functor* **Top** $\to$ **Set**.

However, a cardinality argument shows that **Top** *has no detecting set of objects (even though it could have a detecting class): if there were such a set* $\{G_i,\ i \in I\}$*, choose a set* X *such that* $\operatorname{card} X > \operatorname{card} G_i$ *for all* i*, and define* $\mathcal{T}_1$ *to be the discrete topology on* X *and* $\mathcal{T}_2$ *the topology in which the closed subsets are* X *and all subsets of cardinality* $< \operatorname{card} X$*. Then the identity mapping* $(X, \mathcal{T}_1) \xrightarrow{j} (X, \mathcal{T}_2)$ *is continuous but not a homeomorphism; and any continuous mapping* $G_i \to (X, \mathcal{T}_2)$ *factors (uniquely) through* j *because it is continuous as a mapping* $G_i \to (X, \mathcal{T}_1)$*, a contradiction.*

(v) *Let* **C** *be the category of pointed connected CW-complexes and homotopy classes of maps.*

Whitehead showed that if $X \xrightarrow{f} Y$ *induces isomorphisms* $\pi_n X \cong \pi_n Y$ *for all* n*, then it is a homotopy equivalence. This says that* $\{\mathbb{S}^n,\ n \geqslant 0\}$ *is a detecting set for* **C**.

Freyd showed that there is no faithful functor **C** $\to$ **Set**, *hence* **C** *cannot have a separating set.*

(vi) *In* $[\mathbf{C}, \mathbf{Set}]$ *with* **C** *locally small, the family* $\{\mathbf{C}(A, -),\ A \in \operatorname{ob} \mathbf{C}\}$ *is separating and detecting as a consequence of the Yoneda Lemma.*

2.4 Projective objects

Remark 2.13. *In a locally small category* **C**, *any functor of the form* $\mathbf{C}(A, -)$ *preserves monomorphisms.*

Definition 2.14 (Projective object)**.** *We say that an object* P *of a category* **C** *is* projective *if, given a morphism* $f : P \to B$ *and an epimorphism* $e : A \twoheadrightarrow B$ *in* **C**, *there exists* $g : P \to A$ *with* $eg = f$:

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "g"'] \\ A \arrow[r, two heads, "e"] & B \end{tikzcd}$$

Dually, P *is* injective *if it is projective in* $\mathbf{C}^{\mathrm{op}}$.

If $\mathcal{E}$ *is a class of epimorphisms, we say that* P *is* $\mathcal{E}$*-projective if it satisfies the above condition whenever* $e \in \mathcal{E}$.

Lemma 2.15. *Let* **C** *be a locally small category. Representable functors are* $\mathcal{E}$*-projective in* $[\mathbf{C}, \mathbf{Set}]$*, where* $\mathcal{E}$ *is the class of pointwise epimorphisms.*

Proof. Suppose given $\alpha : \mathbf{C}(A, -) \to G$ and $\varepsilon : F \twoheadrightarrow G$. Then by the Yoneda Lemma, α corresponds to some $x \in GA$. But ε_A is surjective, so we can choose $y \in FA$ s.t. $\varepsilon_A(y) = x$. The corresponding $\beta : \mathbf{C}(A, -) \to F$ satisfies $\varepsilon\beta = \alpha$. $\square$

Remark 2.16. *We will show later (c.f. Corollary 4.10) that all epimorphisms are pointwise epis in* $[\mathbf{C}, \mathbf{Set}]$*. Therefore, representable functors are projective.*

3 Adjunctions

3.1 Definition and examples

Definition 3.1 (Adjunction)**.** *Let* $\mathbf{C} \xrightarrow{F} \mathbf{D}$ *and* $\mathbf{D} \xrightarrow{G} \mathbf{C}$ *be functors. By an* adjunction *between* F *and* G*, we mean a bijection*

$$\mathbf{D}(FA, B) \cong \mathbf{C}(A, GB),$$

which is natural in A *and* B.

If **C** *and* **D** *are locally small, this says that the functors* $\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ *sending* (A, B) *to* $\mathbf{D}(FA, B)$ *and* $\mathbf{C}(A, GB)$ *are naturally isomorphic.*

We then say that F *is* left adjoint *to* G*, or that* G *is* right adjoint *to* F*, and we write* $F \dashv G$.

Example 3.2. (i) *The free functor* $F : \mathbf{Set} \to \mathbf{Gp}$ *is left adjoint to the forgetful functor* $U : \mathbf{Gp} \to \mathbf{Set}$.

(ii) *The forgetful functor* $U : \mathbf{Top} \to \mathbf{Set}$ *has a left adjoint* D *sending* A *to the discrete space on* A. *It also has a right adjoint* I *sending* A *to the space* A *with the topology* $\{\varnothing, A\}$.

(iii) *The functor* $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$ *has a left adjoint* D *sending* A *to the discrete category with object set* A *(and identities as only morphisms), and a right adjoint sending* A *to the trivial groupoid on* A *(i.e. with exactly one morphism* $a \to b$ *for each* $(a, b) \in A \times A$*).*

Moreover, D *itself has a left adjoint* π_0 *sending* $\mathbf{C}$ *to the quotient of* $\mathrm{ob}\,\mathbf{C}$ *by the smallest equivalence relation identifying* $\mathrm{dom}\,f$ *and* $\mathrm{cod}\,f$ *for all* $f \in \mathrm{mor}\,\mathbf{C}$ *(the equivalence classes are called* connected components *of* $\mathbf{C}$*).*

(iv) *Let* $\mathbf{Idem}$ *be the category whose objects are pairs* (A, e)*, where* A *is a set and* $e : A \to A$ *satisfies* $ee = e$*, and whose morphisms* $(A, e) \to (B, d)$ *are functions* $f : A \to B$ *satisfying* $df = fe$*. Define functors* $F : \mathbf{Set} \to \mathbf{Idem}$ *and* $G : \mathbf{Idem} \to \mathbf{Set}$ *by* $FA = (A, 1_A)$ *and* $G(A, e) = \{a \in A,\ e(a) = a\} = \{e(a),\ a \in A\}$*. Then we have* $F \dashv G \dashv F$*, even though* $FG \not\cong 1_{\mathbf{Idem}}$.

(v) *Consider the category* $\mathbf{1}$ *with only one object and no non-identity morphism. The unique functor* $\mathbf{C} \to \mathbf{1}$ *has a left adjoint* iff $\mathbf{C}$ *has an* initial object I*, i.e. an object for which there is a unique morphism* $I \to A$ *for every* $A \in \mathrm{ob}\,\mathbf{C}$.

Similarly, a right adjoint for $\mathbf{C} \to \mathbf{1}$ *corresponds to a* terminal object *of* $\mathbf{C}$.

(vi) *Suppose given* $A \xrightarrow{f} B$ *in* $\mathbf{Set}$*. Viewing the power-sets* PA *and* PB *as posets, we have functors* $PA \xrightarrow{Pf} PB$ *and* $PA \xleftarrow{P^*f} PB$*, satisfying* $Pf \dashv P^*f$*. Indeed,* $Pf\,(A') \subseteq B' \iff A' \subseteq P^*f\,(B')$.

(vii) *Suppose given two sets* A, B *and a relation* $R \subseteq A \times B$*. We have order-reversing maps* $PA \xrightarrow{(-)^r} PB$ *and* $PA \xleftarrow{(-)^\ell} PB$ *defined by*

$$\begin{aligned} (A')^r &= \{b \in B,\ \forall a \in A',\ (a, b) \in R\}\,, \\ (B')^\ell &= \{a \in A,\ \forall b \in B',\ (a, b) \in R\}\,. \end{aligned}$$

They have the property that $A' \subseteq (B')^\ell \iff B' \subseteq (A')^r$*. We say that the contravariant functors* $(-)^\ell$ *and* $(-)^r$ *are* adjoint on the right.

(viii) *The functor* $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ *is self-adjoint on the right: functions* $A \xrightarrow{f} P^*B$ *correspond to relations* $R \subseteq A \times B$ *and hence to functions* $B \xrightarrow{\overline{f}} P^*A$*. This means that*

$$\left(\mathbf{Set} \xrightarrow{P^*} \mathbf{Set}^{\mathrm{op}}\right) \dashv \left(\mathbf{Set}^{\mathrm{op}} \xrightarrow{P^*} \mathbf{Set}\right).$$

3.2 Characterisations of adjunction

Definition 3.3. *Given a functor* $G : \mathbf{D} \to \mathbf{C}$ *and an object* $A \in \mathrm{ob}\,\mathbf{C}$*, we denote by* $(A \downarrow G)$ *the category whose objects are pairs* (B, f) *with* $B \in \mathrm{ob}\,\mathbf{D}$ *and* $A \xrightarrow{f} GB$ *in* $\mathbf{C}$*, and whose morphisms* $(B, f) \to (B', f')$ *are morphisms* $B \xrightarrow{g} B'$ *in* $\mathbf{D}$ *s.t.* $(Gg)f = f'$.

Theorem 3.4. *Specifying a left adjoint for a functor* $G : \mathbf{D} \to \mathbf{C}$ *is equivalent to specifying initial objects of* $(A \downarrow G)$ *for all* $A \in \mathrm{ob}\,\mathbf{C}$.

Proof. ($\Rightarrow$) First suppose that $F \dashv G$. For each $A \in \mathrm{ob}\,\mathbf{C}$, let $A \xrightarrow{\eta_A} GFA$ be the morphism corresponding to $FA \xrightarrow{1_{FA}} FA$. Then (FA, η_A) is initial in $(A \downarrow G)$.

($\Leftarrow$) Suppose given an initial object (FA, η_A) of $(A \downarrow G)$ for all $A \in \mathrm{ob}\,\mathbf{C}$. Given $A \xrightarrow{f} A'$ in $\mathbf{C}$, we define $FA \xrightarrow{Ff} FA'$ to be the unique morphism $(FA, \eta_A) \to (FA', \eta_{A'} f)$ in $(A \downarrow G)$. Functoriality follows from uniqueness: given $A' \xrightarrow{f'} A''$, $(Ff')(Ff)$ and $F(f'f)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_{A''} f'f)$ in $(A \downarrow G)$, so they are equal. Note also that $A \mapsto \eta_A$ is a natural transformation $1_{\mathbf{C}} \to GF$. Given $FA \xrightarrow{g} B$, we map it to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$: this is bijective, with inverse sending $A \xrightarrow{f} GB$ to the unique morphism $(FA, \eta_A) \to (B, f)$ in $(A \downarrow G)$. This bijection is natural in A since η is natural, and in B because G is a functor. □

Corollary 3.5. *If F, F' are both left adjoints of G, then $F \cong F'$ in $[\mathbf{C}, \mathbf{D}]$.*

Proof. By Theorem 3.4, for each $A \in \mathrm{ob}\,\mathbf{C}$ there is a unique isomorphism $(FA, \eta_A) \xrightarrow{\alpha_A} (F'A, \eta'_A)$ in $(A \downarrow G)$, and this defines a natural isomorphism $F \to F'$. □

Lemma 3.6. *Consider $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{H} \mathbf{E}$ and $\mathbf{C} \xleftarrow{G} \mathbf{D} \xleftarrow{K} \mathbf{E}$ with $F \dashv G$ and $H \dashv K$. Then $HF \dashv GK$.*

Proof. Morphisms $HFA \to C$ correspond bijectively to morphisms $FA \to KC$ and hence to morphisms $A \to GKC$, and both bijections are natural in A and C. □

Corollary 3.7. *Consider a commutative square of categories and functors:*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathbf{D} \arrow[d, "H"] \\ \mathbf{E} \arrow[r, "K"] & \mathbf{F} \end{tikzcd}$$

If F, G, H, K have left adjoints, then the diagram of left adjoints commutes up to natural isomorphism.

Remark 3.8. *Given an adjunction $F \dashv G$, we have a natural transformation $\eta : 1_{\mathbf{C}} \to GF$ defined in the proof of Theorem 3.4. Dually, there is a natural transformation $\varepsilon : FG \to 1_{\mathbf{D}}$ s.t. ε_B corresponds to $GB \xrightarrow{1_{GB}} GB$.*

Theorem 3.9. *Suppose given functors $\mathbf{C} \xrightarrow{F} \mathbf{D}$ and $\mathbf{C} \xleftarrow{G} \mathbf{D}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathbf{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathbf{D}}$ satisfying the commutative triangles:*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

The natural transformations η and ε are called the unit *and* counit *of $F \dashv G$, and the two diagrams are called the* triangular identities.

Proof. ($\Rightarrow$) Given the adjunction, we need to verify the triangular identities. Note that ε_{FA} corresponds to $GFA \xrightarrow{1_{GFA}} GFA$, so $(\varepsilon_{FA})(F\eta_A)$ corresponds to the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$; hence $(\varepsilon_{FA})(F\eta_A) = 1_{FA}$; the other identity is dual.

($\Leftarrow$) Suppose given η and ε satisfying the triangular identities. Given $FA \xrightarrow{g} B$, we define $\Phi(g)$ to be the composite $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$, and given $A \xrightarrow{f} GB$, we define $\Psi(f)$ to be $FA \xrightarrow{Ff} FGB \xrightarrow{\varepsilon_B} B$. Now, $\Psi\Phi(g)$ is the composite given by the following diagram:

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, "1_{FA}"'] & FGFA \arrow[r, "FGg"] \arrow[d, "\varepsilon_{FA}"] & FGB \arrow[d, "\varepsilon_B"] \\ & FA \arrow[r, "g"] & B \end{tikzcd}$$

The diagram commutes by the naturality of ε and the first triangular identity; hence $\Psi\Phi(g) = g$. The proof that $\Phi\Psi(f) = f$ is dual; and both Φ and Ψ are natural in A and B. □

3.3 Adjunctions and equivalence of categories

Lemma 3.10. *Suppose given an equivalence of categories* $\mathbf{C} \xrightarrow{F} \mathbf{D}$, $\mathbf{C} \xleftarrow{G} \mathbf{D}$ *with natural isomorphisms* $1_\mathbf{C} \xrightarrow{\alpha} GF$ *and* $FG \xrightarrow{\beta} 1_\mathbf{D}$*. Then there exist natural isomorphisms* $1_\mathbf{C} \xrightarrow{\alpha'} GF$ *and* $FG \xrightarrow{\beta'} 1_\mathbf{D}$ *which satisfy the triangular identities. Therefore* $F \dashv G$ *and* $G \dashv F$.

Proof. Define $\alpha' = \alpha$, and take β' to be the composite

$$FG \xrightarrow{FG\beta^{-1}} FGFG \xrightarrow{F\alpha_G^{-1}} FG \xrightarrow{\beta} 1_\mathbf{D}.$$

Note that $FG\beta = \beta_{FG}$ by naturality and similarly $GF\alpha = \alpha_{GF}$. Now the following diagram commutes by naturality of β:

$$\begin{tikzcd}
F \arrow[r, "F\alpha"] \arrow[drr, "\beta_F^{-1}"'] & FGF \arrow[r, "\beta_{FGF}^{-1}"] & FGFGF \arrow[r, "F\alpha_{GF}^{-1}"] & FGF \arrow[r, "\beta_F"] & F \\
 & & FGF \arrow[u, "FGF\alpha"] \arrow[ur, "1_{FGF}"'] & &
\end{tikzcd}$$

Therefore, the composite is equal to 1_F, which shows that the first triangular identity is satisfied. For the second one, consider the following diagram:

$$\begin{tikzcd}
G \arrow[r, "\alpha_G"] \arrow[drr, "G\beta^{-1}"'] & GFG \arrow[r, "GFG\beta^{-1}"] & GFGFG \arrow[r, "GF\alpha_G^{-1}"] & GFG \arrow[r, "G\beta"] & G \\
 & & GFG \arrow[u, "\alpha_{GFG}"] \arrow[ur, "1_{GFG}"'] & &
\end{tikzcd}$$

□

Lemma 3.11. *Let* $G : \mathbf{D} \to \mathbf{C}$ *have a left adjoint* F *with counit* ε*. Then:*

(i) *G is faithful* iff ε *is pointwise epic.*

(ii) *G is full and faithful* iff ε *is an isomorphism.*

Proof. For both equivalences, note that, given $B \xrightarrow{g} B'$ in $\mathbf{D}$, the morphism $GB \xrightarrow{Gg} GB'$ in $\mathbf{C}$ corresponds under the adjunction to the composite $FGB \xrightarrow{\varepsilon_B} B \xrightarrow{g} B'$. □

Definition 3.12 (Reflection)**.** (i) *By a* reflection*, we mean an adjunction such that the counit is an isomorphism (or equivalently, the right adjoint is full and faithful).*

(ii) *We say that a subcategory* $\mathbf{C}' \subseteq \mathbf{C}$ *is* reflective *if it is full and the inclusion* $\mathbf{C}' \to \mathbf{C}$ *has a left adjoint.*

Definition 3.13. (i) $\mathbf{AbGp}$ *is a reflective subcategory of* $\mathbf{Gp}$*: the left adjoint to the inclusion sends G to its abelianisation.*

(ii) *The subcategory* $\mathbf{AbGp}_t$ *of torsion abelian groups is coreflective in* $\mathbf{AbGp}$*: the right adjoint to the inclusion sends G to its torsion subgroup.*

(iii) *Let* $\mathbf{KHaus} \subseteq \mathbf{Top}$ *be the full subcategory of compact Hausdorff spaces. This inclusion is reflective: the left adjoint to the inclusion is the Stone-Čech compactification.*

4 Limits

4.1 Definition

Definition 4.1 (Diagrams, cones and limits). (i) *Let* $\mathbf{J}$ *be a category (almost always small, often finite). By a* diagram of type $\mathbf{J}$ *in a category* $\mathbf{C}$*, we mean a functor* $D : \mathbf{J} \to \mathbf{C}$.

For example, if $\mathbf{J}$ *is the finite category*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

then a diagram of type $\mathbf{J}$ *is a commutative square in* $\mathbf{C}$.

The objects $D(j)$, $j \in \operatorname{ob} \mathbf{J}$, *are called* vertices *of* D, *and the morphisms* $D(\alpha)$, $\alpha \in \operatorname{mor} \mathbf{J}$, *are called* edges *of* D.

(ii) *Given a diagram* $D : \mathbf{J} \to \mathbf{C}$, *a* cone *over* D *consists of an object* A *of* $\mathbf{C}$ *(the* apex *of the cone) and a family* $(\lambda_j)_{j \in \operatorname{ob} \mathbf{J}}$ *of morphisms* $A \to D(j)$ *(the* legs *of the cone) s.t. the diagram*

$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"'] & & D(j') \end{tikzcd}$$

commutes for all $j \xrightarrow{\alpha} j'$ *in* $\operatorname{mor} \mathbf{J}$.

Given cones $\left(A, (\lambda_j)_{j \in \operatorname{ob} \mathbf{J}}\right)$ *and* $\left(B, (\mu_j)_{j \in \operatorname{ob} \mathbf{J}}\right)$, *a* morphism of cones *from the first to the second is a morphism* $A \xrightarrow{f} B$ *s.t. the diagram*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda_j"'] & & B \arrow[dl, "\mu_j"] \\ & D(j) & \end{tikzcd}$$

commutes for all $j \in \operatorname{ob} \mathbf{J}$. *We write* $\operatorname{Cone}(D)$ *for the category of cones over* D.

(iii) *By a* limit *for a diagram* D, *we mean a terminal object of* $\operatorname{Cone}(D)$, *if it exists.*

We say that $\mathbf{C}$ *has limits of shape* $\mathbf{J}$ *if every diagram of shape* $\mathbf{J}$ *has a limit.*

Dually, we have the notions of cone under a diagram *(or* cocone*) and* colimit *(i.e. initial cone under* D*).*

Remark 4.2. *If we write* $\Delta : \mathbf{C} \to [\mathbf{J}, \mathbf{C}]$ *for the functor sending* A *to the constant diagram with all vertices* A *and all edges* 1_A, *a cone over* D *with apex* A *is just a natural transformation* $\Delta A \to D$. *Therefore, the category of cones over* D *is just the arrow category* $(\Delta \downarrow D)$. *Hence,* $\mathbf{C}$ *has limits of shape* $\mathbf{J}$ iff *the functor* $\Delta : \mathbf{C} \to [\mathbf{J}, \mathbf{C}]$ *has a right adjoint.*

4.2 Examples

Example 4.3. (i) *Suppose* $\mathbf{J} = \varnothing$. *A cone over the unique diagram* $D : \varnothing \to \mathbf{C}$ *is just an object of* $\mathbf{C}$, *so* $\mathrm{Cone}(D) \cong \mathbf{C}$ *and a limit for* D *is a terminal object of* $\mathbf{C}$ *(dually, a colimit for* D *is an initial object of* $\mathbf{C}$*).*

(ii) *Suppose* $\mathbf{J}$ *is a two-object discrete category. A diagram of shape* $\mathbf{J}$ *is just a pair of objects* (A, B)*; a cone over it is a* span*:*

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

and a limit cone is a product*:*

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & B \end{tikzcd}$$

More generally, if $\mathbf{J}$ *is any discrete category, a diagram of shape* $\mathbf{J}$ *is a* $\mathbf{J}$*-indexed family of objects* $(A_j)_{j\in\mathbf{J}}$ *and a limit for it is a* product $\left(\prod_{j\in\mathbf{J}} A_j \xrightarrow{\pi_j} A_j\right)_{j\in\mathbf{J}}$. *Dually, we have* coproducts $\left(A_j \xrightarrow{\nu_j} \sum_{j\in\mathbf{J}} A_j\right)_{j\in\mathbf{J}}$ *as colimits of diagrams. See Definition 2.7 for more details.*

(iii) *Suppose* $\mathbf{J}$ *is the following category:*

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

A diagram of shape $\mathbf{J}$ *is a parallel pair of arrows* $f, g : A \rightrightarrows B$, *and a cone over it consists of a morphism* $C \xrightarrow{h} A$ *satisfying* $fh = gh$*; so a limit for* $f, g : A \rightrightarrows B$ *is an* equalizer *of* (f, g) *(and a colimit is a* coequalizer*). See Definition 2.8 for more details.*

(iv) *Suppose* $\mathbf{J}$ *is the following category:*

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Then a diagram of shape $\mathbf{J}$ *is a* cospan*:*

$$\begin{tikzcd} & A \arrow[d, "f"'] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

and a cone over it consists of a completion of the cospan to a commutative square:

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, "f"'] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

A limit cone is called a pullback *of* (f, g).

For instance, in **Set**, *we can take* $D = \{(a, b) \in A \times B,\ f(a) = g(b)\}$.

More generally, in any category with products and equalizers, we can construct the pullback as the equalizer of $f\pi_1, g\pi_2 : A \times B \rightrightarrows C$. *Dually, a colimit of shape*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

is called a pushout.

(v) *Suppose* $\mathbf{J} = \mathbb{N}$, *the ordered set of natural numbers. A diagram of shape* $\mathbb{N}$ *is a direct sequence*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots.$$

A colimit for this is an object A_∞ *together with morphisms* $g_n : A_n \to A_\infty$ *for all* n, *satisfying* $g_{n+1}f_n = g_n$, *and universal among such.*

Dually, a diagram of shape $\mathbb{N}^{\mathrm{op}}$ *is an inverse sequence*

$$\cdots \to A_2 \to A_1 \to A_0,$$

and a limit for it is an inverse limit.

4.3 Construction of limits from products and equalizers

Definition 4.4 (Complete category). *A category that has all small limits is called* complete.

Theorem 4.5. (i) *If* **C** *has equalizers and all small (resp. finite) products, then* **C** *has all small (resp. finite) limits.*

(ii) *If* **C** *has pullbacks and a terminal object, then* **C** *has all finite limits.*

Proof. (i) Suppose given $D : \mathbf{J} \to \mathbf{C}$ with $\mathbf{J}$ small (resp. finite). Form the products

$$P = \prod_{j \in \mathrm{ob}\,\mathbf{J}} D(j) \qquad \text{and} \qquad Q = \prod_{\alpha \in \mathrm{mor}\,\mathbf{J}} D(\mathrm{cod}\,\alpha).$$

We have morphisms $f, g : P \rightrightarrows Q$ defined by

$$\pi_\alpha f = \pi_{\mathrm{cod}\,\alpha} \qquad \text{and} \qquad \pi_\alpha g = D(\alpha)\pi_{\mathrm{dom}\,\alpha}.$$

Let $E \xrightarrow{e} P$ be an equalizer of the pair (f, g). We claim that $\left(E \xrightarrow{\pi_j e} D(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$ form a limit cone for D. They form a cone: if $j \xrightarrow{\alpha} j'$ in $\mathbf{J}$, then

$$D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_{j'} e.$$

Moreover, if $\left(A \xrightarrow{\lambda_j} D(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$ is any cone over D, then the $(\lambda_j)_{j \in \mathrm{ob}\,\mathbf{J}}$ induce a unique arrow $A \xrightarrow{\lambda} P$ with $\pi_j \lambda = \lambda_j$ for all j, and therefore

$$\pi_\alpha f \lambda = \pi_{\mathrm{cod}\,\alpha}\lambda = \lambda_{\mathrm{cod}\,\alpha} = D(\alpha)\lambda_{\mathrm{dom}\,\alpha} = \pi_\alpha g \lambda,$$

for all α, which means that $f\lambda = g\lambda$. Therefore, λ factors uniquely as $e\mu$ for some $A \xrightarrow{\mu} E$, so μ is the unique factorisation of $(\lambda_j)_{j \in \mathrm{ob}\,\mathbf{J}}$ through $(\pi_j e)_{j \in \mathrm{ob}\,\mathbf{J}}$.

(ii) If **C** has a terminal object 1, then a cone over

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

is the same thing as a cone over the discrete pair (A, B), so a pullback for the above diagram is a product $A \times B$. This allows one to build all finite products by $\prod_{i=1}^{n} A_i = \left(\prod_{i=1}^{n-1} A_i\right) \times A_n$. It remains to construct equalizers. Given $f, g : A \rightrightarrows B$, consider the diagram:

$$\begin{tikzcd} & A \arrow[d, "(1_A, f)"] \\ A \arrow[r, "(1_A, g)"] & A \times B \end{tikzcd}$$

A cone over it consists of $h, k : C \rightrightarrows A$ satisfying $1_A h = 1_A k$ and $fh = gk$, or equivalently of $C \xrightarrow{h} A$ with $fh = gh$. So a pullback for $(1_A, f)$ and $(1_A, g)$ is an equalizer of (f, g). □

4.4 Preservation, reflection and creation of limits

Definition 4.6 (Preservation, reflection and creation of limits)**.** *Let $F : \mathbf{C} \to \mathbf{D}$ be a functor.*

(i) *We say that F* preserves limits of shape $\mathbf{J}$ *if, given a diagram $D : \mathbf{J} \to \mathbf{C}$ and a limit cone $\left(L \xrightarrow{\lambda_j} D(j)\right)_{j \in \mathbf{J}}$ for it, the cone $\left(FL \xrightarrow{F\lambda_j} FD(j)\right)_{j \in \mathbf{J}}$ is a limit cone for $FD : \mathbf{J} \to \mathbf{C}$.*

(ii) *We say that F* reflects limits of shape $\mathbf{J}$ *if, given $D : \mathbf{J} \to \mathbf{C}$ and a cone $\left(L \xrightarrow{\lambda_j} D(j)\right)_{j \in \mathbf{J}}$ such that $\left(FL \xrightarrow{F\lambda_j} FD(j)\right)_{j \in \mathbf{J}}$ is a limit cone for FD, then the given cone is a limit for D.*

(iii) *We say that F* creates limits of shape $\mathbf{J}$ *if, given a diagram $D : \mathbf{J} \to \mathbf{C}$ and a limit cone $\left(M \xrightarrow{\mu_j} FD(j)\right)_{j \in \mathbf{J}}$ for FD, there exists $\left(L \xrightarrow{\lambda_j} D(j)\right)_{j \in \mathbf{J}}$ such that $\left(FL \xrightarrow{F\lambda_j} FD(j)\right)_{j \in \mathbf{J}}$ is isomorphic to $\left(M \xrightarrow{\mu_j} FD(j)\right)_{j \in \mathbf{J}}$ and any such cone is a limit for D.*

Remark 4.7.

- *Most textbooks define creation of limits in a stricter way: they require unique lifting of limit cones from $\mathbf{D}$ to $\mathbf{C}$. Most examples we will see satisfy this stronger condition, but equivalence functors don't in general.*

- *The definitions can behave oddly if we don't assume that $\mathbf{C}$ and/or $\mathbf{D}$ have limits of shape $\mathbf{J}$. However, if $\mathbf{D}$ has and $F : \mathbf{C} \to \mathbf{D}$ creates limits of shape $\mathbf{J}$, then $\mathbf{C}$ also has limits of shape $\mathbf{J}$ and F preserves and reflects them.*

- *In any of the statements of Theorem 4.5, we can replace '$\mathbf{C}$ has' by either '$\mathbf{C}$ has and $F : \mathbf{C} \to \mathbf{D}$ preserves' or '$\mathbf{D}$ has and $F : \mathbf{C} \to \mathbf{D}$ creates'.*

Example 4.8.

(i) *The forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$ creates all small limits: if $(G_i)_{i \in I}$ is a family of groups, then there is a unique group structure on $\prod_{i \in I} UG_i$ which makes the projections into homomorphisms, and it makes $\prod_{i \in I} G_i$ into a product in $\mathbf{Gp}$.*

But U does not preserve colimits: in particular, it does not preserve the initial object.

(ii) *The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ preserves all small limits and colimits, but does not reflect them: given a family of spaces $(X_i)_{i \in I}$, there is a unique topology on $\prod_{i \in I} UX_i$ which makes it a product in $\mathbf{Top}$, but there are other topologies making the projections continuous.*

(iii) *The inclusion functor $\mathbf{AbGp} \to \mathbf{Gp}$ reflects coproducts but does not preserve them: the coproduct of A and B in $\mathbf{AbGp}$ is their direct sum $A \oplus B$, while in $\mathbf{Gp}$ it is the free product $A * B$, but $A * B$ is never abelian unless one of A and B is trivial.*

Lemma 4.9. *Suppose* $\mathbf{D}$ *has limits of shape* $\mathbf{J}$*. Then any functor category* $[\mathbf{C},\mathbf{D}]$ *has limits of shape* $\mathbf{J}$*, and the forgetful functor* $[\mathbf{C},\mathbf{D}] \to [\mathbf{C}_0,\mathbf{D}]$ *creates them, where* $\mathbf{C}_0$ *is the discrete category with the same objects as* $\mathbf{C}$*.*

Proof. Suppose given a diagram $D : \mathbf{J} \to [\mathbf{C},\mathbf{D}]$, or equivalently $D : \mathbf{J} \times \mathbf{C} \to \mathbf{D}$. For each $A \in \operatorname{ob}\mathbf{C}$, we have a diagram $D(-,A) : \mathbf{J} \to \mathbf{D}$; let $\left(LA \xrightarrow{\lambda(j,A)} D(j,A)\right)_{j \in \operatorname{ob}\mathbf{J}}$ be a limit for it in $\mathbf{D}$.

$$\begin{tikzcd}
LA \arrow[r, dashed, "Lf"] \arrow[d, "\lambda(j,A)"] \arrow[dd, bend right=60, "\lambda(j',A)"'] & LB \arrow[d, "\lambda(j,B)"'] \arrow[dd, bend left=60, "\lambda(j',B)"] \\
D(j,A) \arrow[r, "D(j,f)"] \arrow[d, "D(\alpha,A)"] & D(j,B) \arrow[d, "D(\alpha,B)"'] \\
D(j',A) \arrow[r, "D(j',f)"] & D(j',B)
\end{tikzcd}$$

For each arrow $A \xrightarrow{f} B$ in $\mathbf{C}$, the composites $LA \xrightarrow{\lambda(j,A)} D(j,A) \xrightarrow{D(j,f)} D(j,B)$ form a cone over $D(-,B)$, so they induce a unique arrow $LA \xrightarrow{Lf} LB$. Functoriality follows from uniqueness. Moreover, this is the unique functor structure on $A \mapsto LA$ making the $\lambda(j,-)$ into morphisms $L \to D(j,-)$ in $[\mathbf{C},\mathbf{D}]$. Given another cone $\left(M \xrightarrow{\mu(j,-)} D(j,-)\right)_{j \in \operatorname{ob}\mathbf{J}}$ over D, each MA is the apex of a cone over $D(-,A)$, so we get unique morphisms $MA \xrightarrow{\nu_A} LA$ for each A, and they form a natural transformation $M \to L$ by uniqueness of factorisation through limits. □

Corollary 4.10. *Note that in any category* $\mathbf{C}$*, a morphism* $A \xrightarrow{f} B$ *is monic* iff *the square*

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

is a pullback.

It follows from Lemma 4.9 that if $\mathbf{D}$ *has pullbacks, then a morphism* $F \xrightarrow{\alpha} G$ *in* $[\mathbf{C},\mathbf{D}]$ *is monic* iff α_A *is monic for each* $A \in \operatorname{ob}\mathbf{C}$*.*

Lemma 4.11. *Suppose the functor* $G : \mathbf{D} \to \mathbf{C}$ *has a left adjoint. Then* G *preserves all limits which exist in* $\mathbf{D}$*.*

Proof (special case where $\mathbf{C}$ *and* $\mathbf{D}$ *have limits of shape* $\mathbf{J}$*).* Let $F \dashv G$. Then the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "\Delta"'] & \mathbf{D} \arrow[d, "\Delta"'] \\
{[\mathbf{J},\mathbf{C}]} \arrow[r, "{[\mathbf{J},F]}"] & {[\mathbf{J},\mathbf{D}]}
\end{tikzcd}$$

commutes, and all the functors in it have right adjoints (because $\mathbf{C}$ and $\mathbf{D}$ have limits of shape $\mathbf{J}$). By Corollary 3.7, the diagram of right adjoints commutes up to isomorphism:

$$\begin{tikzcd}
{[\mathbf{J},\mathbf{D}]} \arrow[r, "{[\mathbf{J},G]}"] \arrow[d, "\lim_{\mathbf{J}}"'] & {[\mathbf{J},\mathbf{C}]} \arrow[d, "\lim_{\mathbf{J}}"'] \\
\mathbf{D} \arrow[r, "G"] & \mathbf{C}
\end{tikzcd}$$

This says that G preserves limits of shape $\mathbf{J}$. $\square$

Proof (General case). Let $F \dashv G$. Suppose given $D : \mathbf{J} \to \mathbf{D}$ and a limit cone $\left(L \xrightarrow{\lambda_j} D(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$ for it. Given a cone $\left(A \xrightarrow{\mu_j} GD(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$, the transposes $FA \xrightarrow{\overline{\mu}_j} D(j)$ form a cone over D, so there is a unique morphism $FA \xrightarrow{\overline{\nu}} L$ s.t. $\lambda_j \overline{\nu} = \overline{\mu}_j$ for all j. Then the transpose $A \xrightarrow{\nu} GL$ is the unique morphism satisfying $(G\lambda_j)\,\nu = \mu_j$ for all j. $\square$

4.5 The Adjoint Functor Theorem

Lemma 4.12. *Suppose* $\mathbf{D}$ *has all limits of shape* $\mathbf{J}$ *and* $G : \mathbf{D} \to \mathbf{C}$ *preserves them. Then for each* $A \in \mathrm{ob}\,\mathbf{C}$, $(A \downarrow G)$ *has limits of shape* $\mathbf{J}$ *and the forgetful functor* $U : (A \downarrow G) \to \mathbf{D}$ *creates them.*

Proof. Suppose given a diagram $D : \mathbf{J} \to (A \downarrow G)$ and write $D(j)$ as $\left(UD(j), A \xrightarrow{f_j} GUD(j)\right)$. Let $\left(L \xrightarrow{\lambda_j} UD(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$ be a limit cone for UD. Since G preserves limits of shape $\mathbf{J}$, the image $\left(GL \xrightarrow{G\lambda_j} GUD(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$ is a limit cone for GUD. Now the edges $(D(\alpha))_{\alpha \in \mathrm{mor}\,\mathbf{J}}$ of D are morphisms in $(A \downarrow G)$, so the morphisms $A \xrightarrow{f_j} GUD(j)$ form a cone over GUD and there is a unique $A \xrightarrow{g} GL$ s.t. $(G\lambda_j)\,g = f_j$ for all j, i.e. such that the λ_js are morphisms $(L, g) \to D(j)$ in $(A \downarrow G)$. This proves that the limit cone in $\mathbf{D}$ lifts uniquely to a cone in $(A \downarrow G)$. Moreover, if $\left((M, h) \xrightarrow{\mu_j} D(j)\right)_{j \in \mathrm{ob}\,\mathbf{J}}$ is any cone over D, then there exists $M \xrightarrow{k} L$ with $\lambda_j k = \mu_j$ for all j, and the equation $(Gk)h = g$ follows from uniqueness of factorisation through GL. $\square$

Lemma 4.13. *A category* $\mathbf{C}$ *has an initial object* iff *the diagram* $\mathbf{C} \xrightarrow{1_{\mathbf{C}}} \mathbf{C}$ *has a limit.*

Proof. ($\Rightarrow$) Suppose I is initial. Then the unique morphisms $\left(I \xrightarrow{\iota_A} A\right)_{A \in \mathrm{ob}\,\mathbf{C}}$ form a cone over $1_{\mathbf{C}}$, and given any cone $\left(L \xrightarrow{\lambda_A} A\right)_{A \in \mathrm{ob}\,\mathbf{C}}$, we have a (unique) factorisation $L \xrightarrow{\lambda_I} I$ of the λ_As through the ι_As. ($\Leftarrow$) Suppose given a limit $\left(I \xrightarrow{\iota_A} A\right)_{A \in \mathrm{ob}\,\mathbf{C}}$ for $1_{\mathbf{C}}$. We have morphisms $I \to A$ for all A, it remains to show that they are unique. For any $I \xrightarrow{f} A$, we have $f\iota_I = \iota_A$. In particular, when $f = \iota_A$, we see that ι_I is a factorisation of the limit cone through itself, so $\iota_I = 1_I$. It follows that any morphism $I \xrightarrow{f} A$ satisfies $\iota_A = f\iota_I = f$. $\square$

Theorem 4.14 (Primeval Adjoint Functor Theorem)**.** *If* $\mathbf{D}$ *has all limits and* $G : \mathbf{D} \to \mathbf{C}$ *preserves them, then* G *has a left adjoint.*

Proof. This follows from Lemmas 4.12, 4.13 and Theorem 3.4. $\square$

Theorem 4.15 (General Adjoint Functor Theorem)**.** *Suppose* $\mathbf{D}$ *is locally small and complete. Then a functor* $G : \mathbf{D} \to \mathbf{C}$ *has a left adjoint* iff *it satisfies the following two conditions:*

(i) *G preserves all small limits,*

(ii) *For each $A \in \mathrm{ob}\,\mathbf{C}$, there exists a* weakly initial *set of objects (or* solution-set*) in $(A \downarrow G)$, i.e. a family $(B_i)_{i \in I}$ of objects of $(A \downarrow G)$ indexed by a set I, such that every $B \in \mathrm{ob}\,(A \downarrow G)$ has a morphism $B_i \to B$ in $(A \downarrow G)$ for some $i \in I$.*

Proof. ($\Rightarrow$) If G has a left adjoint F, then it preserves all small limits by Lemma 4.11, and the singleton $\left\{A \xrightarrow{\eta_A} GFA\right\}$ is weakly initial in $(A \downarrow G)$. ($\Leftarrow$) For each $A \in \mathrm{ob}\,\mathbf{C}$, the category $(A \downarrow G)$ is complete by Lemma 4.12, and locally small since $\mathbf{D}$ is. By Theorem 3.4, it suffices to prove the following statement: if a category $\mathbf{A}$ is complete, locally small, and has a weakly initial set of objects $\{A_i,\ i \in I\}$, then $\mathbf{A}$ has an initial object.

First form $P = \prod_{i\in I} A_i$. Note that P is weakly initial. Now form the limit $I \xrightarrow{i} P$ of the diagram $P \rightrightarrows P$ whose edges are all endomorphisms of P. To show that I is initial, suppose given $f, g : I \rightrightarrows B$. Form their equalizer $E \xrightarrow{e} I$; by construction, there exists a morphism $P \xrightarrow{h} E$. Now, $ieh, 1_P : P \rightrightarrows P$ are both edges of the diagram above, so $iehi = i$. But i is monic (because it is an equalizer), so $ehi = 1_P$. It follows that e is split epic, so $f = g$. □

Example 4.16. (i) *Consider the forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$. We have seen that $\mathbf{Gp}$ has and U preserves all small limits, and $\mathbf{Gp}$ is locally small. Moreover, given a set A, any function $A \xrightarrow{f} UG$ factors as $A \xrightarrow{f} UG' \hookrightarrow UG$, where G' is the subgroup of G generated by $f(A)$, and we have $|G'| \leqslant \max\{\aleph_0, |A|\}$, so we get a weakly initial set in $(A \downarrow U)$ by considering a fixed set B of this cardinality, all possible subsets B' of B, all possible group structures on these, and all possible functions $A \to B'$. By the General Adjoint Functor Theorem, U has a left adjoint.*

(ii) *Consider the forgetful functor $U : \mathbf{CLat} \to \mathbf{Set}$, where $\mathbf{CLat}$ is the category of complete lattices. Note that U creates all small limits, so $\mathbf{CLat}$ has them and U preserves them; moreover, $\mathbf{CLat}$ is locally small. But Hales proved that, for any cardinal κ, there exists a complete lattice of cardinality at least κ generated by a three-element subset $\{a, b, c\}$, so the second condition of the General Adjoint Functor Theorem fails for $A = \{a, b, c\}$.*

Lemma 4.17. *Suppose given a pullback square:*

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, tail, "f"'] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

If f is monic, then so is k.

Proof. Suppose given $\ell, m : E \rightrightarrows D$ with $k\ell = km$. Then $fh\ell = gk\ell = gkm = fhm$, but f is monic so $h\ell = hm$. Thus, ℓ and m are factorisations of the same cone through the pullback, and hence $\ell = m$. □

Definition 4.18 (Subobject)**.** *By a* subobject *of an object A of $\mathbf{C}$, we mean a monomorphism $A' \rightarrowtail A$. We say that $\mathbf{C}$ is* well-powered *if for every $A \in \operatorname{ob}\mathbf{C}$, there exists a* representative set *$\{A_i \rightarrowtail A,\ i \in I\}$ of subobjects of A s.t. every subobject $A' \rightarrowtail A$ is isomorphic to some $A_i \rightarrowtail A$.*

Dually, we have a notion of quotient objects *and we say that $\mathbf{C}$ is* well-copowered *if every object has a representative set of quotients.*

Example 4.19. $\mathbf{Set}$ *is well-powered by the power-set axiom. So are* $\mathbf{Gp}$*,* $\mathbf{Top}$*, etc.*

Theorem 4.20 (Special Adjoint Functor Theorem)**.** *Suppose that $\mathbf{C}$ and $\mathbf{D}$ are both locally small, that $\mathbf{D}$ is well-powered and complete and has a coseparating set of objects. Then a functor $G : \mathbf{D} \to \mathbf{C}$ has a left adjoint* iff *G preserves all small limits.*

Proof. ($\Rightarrow$) Immediate from Lemma 4.11. ($\Leftarrow$) For each $A \in \operatorname{ob}\mathbf{C}$, the category $(A \downarrow G)$ is complete by Lemma 4.12, locally small since $\mathbf{D}$ is, and well-powered since the subobjecs of (B, f) in $(A \downarrow G)$ are just those subobjects $B' \rightarrowtail B$ in $\mathbf{D}$ for which f factors through $GB' \rightarrowtail GB$. Moreover, if $\{S_i,\ i \in I\}$ is a coseparating set for $\mathbf{D}$, then $\{(S_i, f),\ i \in I,\ f \in \mathbf{C}(A, GS_i)\}$ is a coseparating set for $(A \downarrow G)$: given arrows $g, h : (B; f) \rightrightarrows (B', f')$ s.t. $g \neq h$, there exists an arrow $B' \xrightarrow{k} S_i$ with $kg \neq kh$, and then k can be seen as a morphism $(B', f') \xrightarrow{k} (S_i, (Gk)f')$ in $(A \downarrow G)$. By Theorem 3.4, it suffices to prove the following: if a category $\mathbf{A}$ is complete, locally small, well-powered and has a coseparating set, then $\mathbf{A}$ has an initial object.

First form

$$P = \prod_{\lambda\in\Lambda} S_\lambda,$$

where $\{S_\lambda,\ \lambda \in \Lambda\}$ is a coseparating set for $\mathbf{A}$. Now form the limit I of the diagram

$$\begin{tikzcd}
P' \arrow[ddrr, tail] & & \\
P'' \arrow[drr, tail] & & \\
\vdots & & P \\
P^{(n)} \arrow[urr, tail] & & \\
\vdots & &
\end{tikzcd}$$

whose edges are a representative set of subobjects of P. By an obvious generalisation of Lemma 4.17, the legs $I \to P^{(n)}$ of the limit cone are monic, hence so is $I \to P$, and it is the smallest subobject of P. This implies that any monomorphism $I' \rightarrowtail I$ must be an isomorphism. Thus, given any two arrows $f, g : I \rightrightarrows A$, we must have $f = g$ (because the equalizer of (f, g) will have to be a monomorphism to I and therefore an isomorphism).

Now, given $A \in \mathrm{ob}\,\mathbf{A}$, it remains to construct an arrow $I \to A$. Form the product

$$Q = \prod_{\substack{\lambda \in \Lambda \\ f \in \mathbf{A}(A, S_\lambda)}} S_\lambda.$$

We have a morphism $A \xrightarrow{g} Q$ defined by $\pi_{\lambda,f} g = f$ for all pairs (λ, f). This morphism g is monic: given $x, y : B \rightrightarrows A$ with $x \neq y$, we can find $A \xrightarrow{f} S_i$ with $fx \neq fy$ (because the set $\{S_i,\ i \in I\}$ is coseparating), and hence $gx \neq gy$. There is also a morphism $P \xrightarrow{h} Q$ defined by $\pi_{i,f} h = \pi_i$. Now form the pullback C of g and h:

$$\begin{tikzcd}
 & C \arrow[r, "k"] \arrow[dd, tail, "\ell"'] & A \arrow[dd, tail, "g"] \\
I \arrow[ur, dashed, tail] \arrow[dr, tail] & & \\
 & P \arrow[r, "h"] & Q
\end{tikzcd}$$

The morphism ℓ is monic by Lemma 4.17, so there exists $I \xrightarrow{m} C$ (by construction of I) and hence we have a morphism $I \xrightarrow{m} C \xrightarrow{k} A$. This concludes the proof that I is initial in $\mathbf{A}$. □

Example 4.21. *Consider the inclusion $I : \mathbf{KHaus} \to \mathbf{Top}$ from the category of compact Hausdorff topological spaces to the category of all topological spaces.*

- **KHaus** *has and I preserves all small products by Tychonoff's Theorem.*
- **KHaus** *has equalizers: if X, Y are compact Hausdorff, the equalizer of $f, g : X \rightrightarrows Y$ is a closed, hence compact, subspace of X. Therefore,* **KHaus** *is complete.*
- **KHaus** *and* **Top** *are locally small.*
- **KHaus** *is well-powered (take the set of inclusions of closed subspaces of X).*
- **KHaus** *has a coseparator $[0, 1]$ by Uryson's Lemma.*

By the Special Adjoint Functor Theorem, I has a left adjoint β, which is the Stone-Čech compactification.

Čech's original construction of β was essentially the same as the above proof of the Special Adjoint Functor Theorem.

We could also have used the General Adjoint Functor Theorem to prove the existence of β.

5 Monads

5.1 Definition and examples

Remark 5.1. *Suppose we are given an adjunction $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$. How much of the structure of this adjunction can we describe purely in terms of $\mathbf{C}$?*

We have the composite $T = GF : \mathbf{C} \to \mathbf{C}$, the natural transformations $\eta : 1_{\mathbf{C}} \to T$ and $\mu = G\varepsilon_F : TT \to T$, satisfying some relations given by the triangular identities and the naturality of ε.

Definition 5.2 (Monad)**.** *A* monad $\mathbb{T}$ *on a category $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and natural transformations $\eta : 1_{\mathbf{C}} \to T$, $\mu : TT \to T$, satisfying the commutative diagrams:*

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & TT \arrow[d, "\mu"] & T \arrow[l, "\eta_T"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd}$$

Example 5.3. (i) *An adjunction $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$ induces a monad $(GF, \eta, G\varepsilon_F)$ on $\mathbf{C}$ (and a comonad $(FG, \varepsilon, F\eta_G)$ on $\mathbf{D}$).*

(ii) *Let M be a monoid. The functor $M \times (-) : \mathbf{Set} \to \mathbf{Set}$ carries a monad structure, with $\eta_A : A \to M \times A$ given by $a \mapsto (1, a)$ and $\mu_A : M \times M \times A \to M \times A$ given by $(m, m', a) \mapsto (mm', a)$.*

(iii) *If A is an object of a category $\mathbf{C}$ with finite products, the functor $A \times (-) : \mathbf{C} \to \mathbf{C}$ carries a comonad structure, with $\varepsilon_B = \pi_2 : A \times B \to B$ and $\delta_B = (\pi_1, \pi_1, \pi_2) : A \times B \to A \times A \times B$.*

Remark 5.4. *One natural question to ask about monads is whether or not it is always possible to find an adjunction inducing a given monad, as in Example 5.3.*(i)*. In 1965, Eilenberg and Moore on the one hand and Kleisli on the other hand provided two different constructions of such an adjunction.*

5.2 The Eilenberg-Moore adjunction

Remark 5.5. *In Example 5.3.*(ii)*, the functor $M \times (-) : \mathbf{Set} \to [M, \mathbf{Set}]$ (with M acting by multiplication on the left factor) is left adjoint to the forgetful functor $[M, \mathbf{Set}] \to \mathbf{Set}$, and this adjunction induces the given monad.*

Definition 5.6 ($\mathbb{T}$-algebra)**.** *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathbf{C}$. By a* $\mathbb{T}$-algebra*, we mean a pair (A, α), where $A \in \operatorname{ob} \mathbf{C}$ and $\alpha : TA \to A$ satisfy the commutative diagram:*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"'] & TTA \arrow[l, "T\alpha"'] \arrow[d, "\mu_A"] \\ & A & TA \arrow[l, "\alpha"] \end{tikzcd}$$

A homomorphism of $\mathbb{T}$-algebras $(A, \alpha) \xrightarrow{f} (B, \beta)$ *is a morphism $A \xrightarrow{f} B$ making the following diagram commute:*

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

We write $\mathbf{C}^{\mathbb{T}}$ for the category of $\mathbb{T}$-algebras and homomorphisms; $\mathbf{C}^{\mathbb{T}}$ is called the Eilenberg-Moore category *of $\mathbb{T}$.*

Lemma 5.7. *The forgetful functor $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ has a left adjoint $F^{\mathbb{T}}$, and the adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ induces the monad $\mathbb{T}$.*

Proof. We define $F^{\mathbb{T}}A$ to be the pair $(TA, \mu_A) \in \operatorname{ob} \mathbf{C}^{\mathbb{T}}$ and $F^{\mathbb{T}}\left(A \xrightarrow{f} B\right)$ to be $(TA, \mu_A) \xrightarrow{Tf} (TB, \mu_B)$. The composite $G^{\mathbb{T}}F^{\mathbb{T}}$ is T, we have a unit $\eta : 1_{\mathbf{C}} \to G^{\mathbb{T}}F^{\mathbb{T}}$ and we take the counit $\varepsilon_{(A,\alpha)} : F^{\mathbb{T}}A = F^{\mathbb{T}}G^{\mathbb{T}}(A, \alpha) \to (A, \alpha)$ to be $\alpha : TA \to A$. Then the composite $F^{\mathbb{T}}A \xrightarrow{F^{\mathbb{T}}\eta_A} F^{\mathbb{T}}G^{\mathbb{T}}F^{\mathbb{T}}A \xrightarrow{\varepsilon_{F^{\mathbb{T}}A}} F^{\mathbb{T}}A$ is $TA \xrightarrow{T\eta_A} TTA \xrightarrow{\mu_A} TA$, which is the identity. Likewise, the composite $G^{\mathbb{T}}(A, \alpha) \xrightarrow{\eta_{G^{\mathbb{T}}(A,\alpha)}} G^{\mathbb{T}}F^{\mathbb{T}}G^{\mathbb{T}}(A, \alpha) \xrightarrow{\varepsilon_{(A,\alpha)}} G^{\mathbb{T}}(A, \alpha)$ is $A \xrightarrow{\eta_A} TA \xrightarrow{\alpha} A$, which is also the identity. Therefore, the two triangular identities are satisfied, which shows that there is an adjunction $\left(F^{\mathbb{T}} \dashv G^{\mathbb{T}}\right)$. It is clear that $G^{\mathbb{T}}F^{\mathbb{T}} = T$ and that the unit of the adjunction is η. We also have $G^{\mathbb{T}}\varepsilon_{F^{\mathbb{T}}} = \mu$ since the algebra structure on $F^{\mathbb{T}}A$ is μ_A; therefore, the monad induced by $\left(F^{\mathbb{T}} \dashv G^{\mathbb{T}}\right)$ is $\mathbb{T}$. □

5.3 The Kleisli adjunction

Remark 5.8. *Kleisli observed that if we have an adjunction $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$ inducing $\mathbb{T}$, then we can replace $\mathbf{D}$ by its full subcategory $\mathbf{D}'$ on objects of the form FA, with $A \in \operatorname{ob} \mathbf{C}$. We may therefore assume that F is surjective (and in fact bijective) on objects. Also, morphisms $FA \to FA'$ in $\mathbf{D}'$ must correspond bijectively to morphisms $A \to GFA' = TA'$ in $\mathbf{C}$.*

Definition 5.9 (Kleisli category)**.** *Given a monad $\mathbb{T}$ on $\mathbf{C}$, we define the* Kleisli category *$\mathbf{C}_{\mathbb{T}}$ by $\operatorname{ob} \mathbf{C}_{\mathbb{T}} = \operatorname{ob} \mathbf{C}$, and morphisms $A \xrightarrow{f} B$ in $\mathbf{C}_{\mathbb{T}}$ are morphisms $A \xrightarrow{f} TB$ in $\mathbf{C}$; the identity $A \to A$ in $\mathbf{C}_{\mathbb{T}}$ is $A \xrightarrow{\eta_A} TA$, and the composite $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$.*

Proof. To check that this is a category, note that the composite $A \xrightarrow{\eta_A} A \xrightarrow{f} B$ is:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & TA \arrow[r, "Tf"] & TTB \arrow[d, "\mu_B"] \\
& TB \arrow[ur, "\eta_{TB}"] \arrow[r, "1_{TB}"'] & TB
\end{tikzcd}$$

and the composite $A \xrightarrow{f} B \xrightarrow{\eta_B} B$ is:

$$\begin{tikzcd}
A \arrow[r, "f"] & TB \arrow[r, "T\eta_B"] \arrow[dr, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] \\
& & TB
\end{tikzcd}$$

For associativity, suppose given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$. Then the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] & TB \arrow[r, "Tg"] & TTC \arrow[r, "\mu_C"] \arrow[d, "TTh"'] & TC \arrow[d, "Th"] \\
& & TTTD \arrow[r, "\mu_{TD}"] \arrow[d, "T\mu_D"'] & TTD \arrow[d, "\mu_D"] \\
& & TTD \arrow[r, "\mu_D"] & TD
\end{tikzcd}$$

□

Lemma 5.10. *There is an adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}} : \mathbf{C} \rightleftarrows \mathbf{C}_{\mathbb{T}}$ inducing the monad $\mathbb{T}$.*

Proof. We define $F_{\mathbb{T}}A = A$ and $F_{\mathbb{T}}\left(A \xrightarrow{f} B\right) = \left(A \xrightarrow{f} B \xrightarrow{\eta_B} TB\right)$. It is clear that $F_{\mathbb{T}}$ preserves identities; functoriality follows from the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "\eta_B"] \arrow[dr, "g"'] & TB \arrow[r, "Tg"] & TC \arrow[r, "T\eta_C"] \arrow[dr, "1_{TC}"'] & TTC \arrow[d, "\mu_C"] \\
& & C \arrow[ur, "\eta_C"'] & & TC
\end{tikzcd}$$

We also define $G_{\mathbb{T}}A = TA$ and $G_{\mathbb{T}}\left(A \xrightarrow{f} B\right) = \left(TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB\right)$. Hence $G_{\mathbb{T}}(\eta_A) = 1_{TA}$, and functoriality follows from the following diagram:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TTB \arrow[r, "TTg"] \arrow[d, "\mu_B"] & TTTC \arrow[r, "T\mu_C"] \arrow[d, "\mu_{TC}"] & TTC \arrow[d, "\mu_C"] \\
& TB \arrow[r, "Tg"] & TTC \arrow[r, "\mu_C"] & TC
\end{tikzcd}$$

We have $G_{\mathbb{T}}F_{\mathbb{T}}A = TA$ and $G_{\mathbb{T}}F_{\mathbb{T}}\left(A \xrightarrow{f} B\right) = Tf$, i.e. $G_{\mathbb{T}}F_{\mathbb{T}} = T$. We take the unit of $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ to be η, and we define the counit $\varepsilon : F_{\mathbb{T}}G_{\mathbb{T}} \to 1_{\mathbf{C}_{\mathbb{T}}}$ by $\varepsilon_A = \left(TA \xrightarrow{1_{TA}} TA\right)$. This is natural by the following diagram:

$$\begin{tikzcd}
TA \arrow[r, "Tf"] & TTB \arrow[r, "\mu_B"] & TB \arrow[r, "\eta_{TB}"] \arrow[dr, "1_{TB}"'] & TTB \arrow[d, "\mu_B"] \\
& & & TB
\end{tikzcd}$$

which shows that $\varepsilon_B(F_{\mathbb{T}}G_{\mathbb{T}}f) = f = f\varepsilon_A$. We verify the triangular identities in the same manner, which proves that there is an adjunction $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$. We have $G_{\mathbb{T}}F_{\mathbb{T}} = T$, η is the unit of the adjunction and $G_{\mathbb{T}}\varepsilon_{F_{\mathbb{T}}A} = \left(TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA\right)$, so the monad induced by $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ is $\mathbb{T}$. ☐

5.4 Adjunction category of a monad

Theorem 5.11. *Given a monad $\mathbb{T}$ on $\mathbf{C}$, let $\mathbf{Adj}(\mathbb{T})$ be the category whose objects are adjunctions $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$ inducing $\mathbb{T}$ and whose morphisms $(F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}) \longrightarrow (F' \dashv G' : \mathbf{C} \rightleftarrows \mathbf{D}')$ are functors $K : \mathbf{D} \to \mathbf{D}'$ satisfying $KF = F'$ and $G'K = G$.*

Then the Kleisli adjunction $(F_{\mathbb{T}} \dashv G_{\mathbb{T}})$ *is initial in* $\mathbf{Adj}(\mathbb{T})$ *and the* Eilenberg-Moore adjunction $\left(F^{\mathbb{T}} \dashv G^{\mathbb{T}}\right)$ *is terminal.*

Proof. The Eilenberg-Moore adjunction is terminal. Given $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$ with counit ε in $\mathbf{Adj}(\mathbb{T})$, we define the *Eilenberg-Moore comparison functor* $K : \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ by $KB = (GB, G\varepsilon_B)$. Using one of the triangular identities for $(F \dashv G)$ and the naturality of ε, we check that this is indeed a $\mathbb{T}$-algebra, and we set $K\left(B \xrightarrow{g} C\right) = Gg$, a homomorphism by naturality of ε. Clearly $G^{\mathbb{T}}K = G$, and $KFA = (GFA, G\varepsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A$ and $KF\left(A \xrightarrow{f} A'\right) = Tf$, which shows that K is a morphism in $\mathbf{Adj}(\mathbb{T})$. We now show that it is unique. Let $K' : (\mathbf{C} \rightleftarrows \mathbf{D}) \longrightarrow \left(\mathbf{C} \rightleftarrows \mathbf{C}^{\mathbb{T}}\right)$ in $\mathbf{Adj}(\mathbb{T})$. We must have $K'B = (GB, \theta_B)$ and $K'g = Gg$ for some natural transformation $\theta : GFG \to G$. Since $K'F = F^{\mathbb{T}}$, we have $\theta_{FA} = \mu_A = G\varepsilon_{FA}$ for all A. Given $B \in \operatorname{ob}\mathbf{D}$, consider the diagram:

$$\begin{tikzcd}
GFGFGB \arrow[r, "GFG\varepsilon_B"] \arrow[d, shift right, "G\varepsilon_{FGB}"'] \arrow[d, shift left, "\theta_{FGB}"] & GFGB \arrow[d, shift right, "G\varepsilon_B"'] \arrow[d, shift left, "\theta_B"] \\
GFGB \arrow[r, "G\varepsilon_B"] & GB
\end{tikzcd}$$

Both squares commute, and the left vertical edges are equal, so $(G\varepsilon_B)(GFG\varepsilon_B) = \theta_B(GFG\varepsilon_B)$; but $GFG\varepsilon_B$ is split epic (by $GF\eta_{GB}$), and hence $G\varepsilon_B = \theta_B$, so $K' = K$.

The Kleisli adjunction is initial. Define the *Kleisli comparison functor* $L : \mathbf{C}_{\mathbb{T}} \to \mathbf{D}$ by $LA = FA$ and $L\left(A \xrightarrow{f} B\right) = FA \xrightarrow{Ff} FGFB \xrightarrow{\varepsilon_{FB}} FB$. (Note that Lf corresponds to f under $(F \dashv G)$, so L is automatically full and faithful.) We check as before that L is a functor. Moreover, $GL = G_{\mathbb{T}}$ and $LF_{\mathbb{T}} = F$, so L defines a morphism in $\mathbf{Adj}(\mathbb{T})$. Finally, if $L' : \mathbf{C}_{\mathbb{T}} \to \mathbf{D}$ also satisfies $L'F_{\mathbb{T}} = F$ and $GL' = G_{\mathbb{T}}$, then $L'A = FA = LA$ for all A. And if $A \xrightarrow{f} B$, then $GL'f$ is $GFA \xrightarrow{GFf} GFGFB \xrightarrow{G\varepsilon_{FB}} GFB$, so $L'f$ is the morphism corresponding to $A \xrightarrow{f} GFB$ under $(F \dashv G)$, so $L' = L$. □

Remark 5.12. *Note that the Kleisli category $\mathbf{C}_{\mathbb{T}}$ has coproducts if $\mathbf{C}$ does, since $F_{\mathbb{T}}$ preserves them. But in general it has very few other limits or colimits.*

Theorem 5.13. (i) *The forgetful functor $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ creates all limits which exist in $\mathbf{C}$.*

(ii) *Suppose $\mathbf{C}$ has colimits of shape $\mathbf{J}$. Then $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ creates them* iff *T preserves them.*

Proof. In the proof, we shall write G instead of $G^{\mathbb{T}}$.

(i) Suppose given a diagram $D : \mathbf{J} \to \mathbf{C}^{\mathbb{T}}$, with $D(j) = (GD(j), \delta_j)$ and suppose given a limit cone $\left(L \xrightarrow{\lambda_j} GD(j)\right)_{j \in \mathrm{ob}\, \mathbf{J}}$ over GD. Then the composites $TL \xrightarrow{T\lambda_j} TGD(j) \xrightarrow{\delta_j} GD(j)$ form a cone over GD since the edges of D are $\mathbb{T}$-algebra homomorphisms, so they induce a unique morphism $\alpha : TL \to L$ making the following diagram commute for all j:

$$\begin{array}{ccc} TL & \xrightarrow{T\lambda_j} & TGD(j) \\ \alpha \big\downarrow & & \big\downarrow \delta_j \\ L & \xrightarrow{\lambda_j} & GD(j) \end{array}$$

We thus have a $\mathbb{T}$-algebra (L, α) by uniqueness of factorisation through limits, and the λ_j are morphisms in $\mathbf{C}^{\mathbb{T}}$. Moreover, given any cone $\left(M \xrightarrow{\beta_j} D(j)\right)_{j \in \mathrm{ob}\, \mathbf{J}}$ over D, we have a unique induced morphism $GM \xrightarrow{\gamma} L$, and γ is a homomorphism $M \to (L, \alpha)$ by uniqueness of factorisations through limits. This proves that $\left((L, \alpha) \xrightarrow{\lambda_j} D(j)\right)_{j \in \mathrm{ob}\, \mathbf{J}}$ is a limit cone in $\mathbf{C}^{\mathbb{T}}$.

(ii) ($\Rightarrow$) Note that $F^{\mathbb{T}}$ preserves any colimits which exist in $\mathbf{C}$, so $T = G^{\mathbb{T}}F^{\mathbb{T}}$ preserves them if $G^{\mathbb{T}}$ does. ($\Leftarrow$) Suppose given a diagram $D : \mathbf{J} \to \mathbf{C}^{\mathbb{T}}$ and a colimit cone $\left(GD(j) \xrightarrow{\lambda_j} L\right)_{j \in \mathrm{ob}\, \mathbf{J}}$ in $\mathbf{C}$. Since $\left(TGD(j) \xrightarrow{T\lambda_j} TL\right)_{j \in \mathrm{ob}\, \mathbf{J}}$ is also a colimit, the composites $TGD(j) \xrightarrow{\delta_j} GD(j) \xrightarrow{\lambda_j} L$ induce a unique $TL \xrightarrow{\alpha} L$. As before, this is a $\mathbb{T}$-algebra structure on L, and it turns the λ_j into a colimit cone in $\mathbf{C}^{\mathbb{T}}$. □

5.5 Monadicity

Definition 5.14 (Monadicity)**.** *We call an adjunction $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$* monadic *if the Eilenberg-Moore comparison functor $K : \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ is part of an equivalence of categories, where $\mathbb{T}$ is the monad induced by the adjunction $F \dashv G$.*

We also say that a functor $G : \mathbf{D} \to \mathbf{C}$ is monadic *if it has a left adjoint and the adjunction is monadic.*

Lemma 5.15. *Suppose an adjunction $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$ induces a monad $\mathbb{T}$. Then the Eilenberg-Moore comparison functor $K : \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ has a left adjoint provided that the pair $F\alpha, \varepsilon_{FA} : FGFA \rightrightarrows FA$ has a coequalizer in $\mathbf{D}$ for each $\mathbb{T}$-algebra (A, α).*

Proof. Suppose $F\alpha, \varepsilon_{FA} : FGFA \rightrightarrows FA$ has a coequalizer $FA \xrightarrow{h} L(A, \alpha)$ for all (A, α). Given a homomorphism $(A, \alpha) \xrightarrow{f} (B, \beta)$, we get a unique $L\,(A, \alpha) \xrightarrow{Lf} L\,(B, \beta)$ making the following diagram commute:

$$\begin{tikzcd}
FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\varepsilon_{FA}"'] \arrow[d, "FGFf"'] & FA \arrow[r] \arrow[d, "Ff"'] & L(A,\alpha) \arrow[d, dashed, "Lf"'] \\
FGFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\varepsilon_{FB}"'] & FB \arrow[r] & L(B,\beta)
\end{tikzcd}$$

Functoriality of $L : \mathbf{C}^{\mathbb{T}} \to \mathbf{D}$ follows from uniqueness.

For any $B \in \operatorname{ob}\mathbf{D}$, morphisms $L(A, \alpha) \to B$ correspond to morphisms $FA \xrightarrow{\ell} B$ such that $\ell\,(F\alpha) = \ell\varepsilon_{FA} = \varepsilon_B\,(FG\ell)$, and hence to morphisms $A \xrightarrow{\overline{\ell}} GB$ satisfying $\overline{\ell}\alpha = 1_B\,(G\ell) = G\left(\varepsilon_B F\overline{\ell}\right) = (G\varepsilon_B)\left(GF\overline{\ell}\right)$, i.e. to homomorphisms $(A, \alpha) \to (GB, G\varepsilon_B) = KB$. This shows that $L \dashv K$. $\square$

Definition 5.16 (Reflexive pair, split pair, etc.). (i) *We say that a parallel pair $f, g : A \rightrightarrows B$ is* reflexive *if there exists $B \xrightarrow{r} A$ such that $fr = gr = 1_B$.*

Note that, with the notations of Lemma 5.15, the pair $F\alpha, \varepsilon_{FA} : FGFA \rightrightarrows FA$ is reflexive, with common splitting $FA \xrightarrow{F\eta_A} FGFA$.

By reflexive coequalizers*, we mean coequalizers of reflexive pairs, i.e colimits of shape:*

$$\begin{tikzcd}
\bullet \arrow[loop above] \arrow[loop below] \arrow[r, shift left=2] \arrow[r, shift right=2] & \bullet \arrow[l]
\end{tikzcd}$$

(ii) *By a* split coequalizer diagram*, we mean a diagram*

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

satisfying $hf = hg$, $hs = 1_C$, $sh = ft$ and $gt = 1_B$.

These equations imply that h is a coequalizer for $f, g : A \rightrightarrows B$.

(iii) *Given a functor $G : \mathbf{D} \to \mathbf{C}$, a pair $f, g : A \rightrightarrows B$ in $\mathbf{D}$ is called G-*split *if there is a split coequalizer diagram*

$$\begin{tikzcd}
GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[r, "h"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

in $\mathbf{C}$.

Note that, with the notations of Lemma 5.15, the pair $F\alpha, \varepsilon_{FA} : FGFA \rightrightarrows FA$ is G-split because their image under G yields the following split coequalizer diagram:

$$\begin{tikzcd}
TTA \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\alpha"'] & TA \arrow[r, "\alpha"] \arrow[l, bend left=50, "\eta_{TA}"] & A \arrow[l, bend left=50, "\eta_A"]
\end{tikzcd}$$

Theorem 5.17 (Precise Monadicity Theorem)**.** *A functor $G : \mathbf{D} \to \mathbf{C}$ is monadic* iff *it has a left adjoint and creates coequalizers of G-split pairs.*

Proof. ($\Rightarrow$) Since equivalence functors create colimits, it is sufficient to show that $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ creates coequalizers of $G^{\mathbb{T}}$-split pairs. But if $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ is a $G^{\mathbb{T}}$-split pair with split coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"] \end{tikzcd}$$

in $\mathbf{C}$, then the diagram

$$\begin{tikzcd} TA \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] & TB \arrow[r, "Th"] & TC \end{tikzcd}$$

is a coequalizer, so as in Theorem 5.13.(ii), we get a unique algebra structure $TC \xrightarrow{\gamma} C$ turning h into a homomorphism, and it makes h a coequalizer in $\mathbf{C}^{\mathbb{T}}$.

($\Leftarrow$) Using Lemma 5.15, we have an adjunction $L \dashv K : \mathbf{C}^{\mathbb{T}} \rightleftarrows \mathbf{D}$, where K is the Eilenberg-Moore comparison functor. Consider the unit $(A, \alpha) \to KL(A, \alpha)$; the underlying morphism $A \to GL(A, \alpha)$ can be read in the following diagram, recalling that $L(A, \alpha)$ was defined in Lemma 5.15 as the coequalizer below:

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\varepsilon_{FA}"'] & GFA \arrow[r, "\alpha"] \arrow[dr] & A \arrow[d, dashed] \\ & & GL(A, \alpha) \end{tikzcd}$$

Since the top row is a coequalizer, the unit $A \to GL(A, \alpha)$ is the unique factorisation of $GFA \to GL(A, \alpha)$ through it. This factorisation is an isomorphism since G preserves the coequalizer defining $L(A, \alpha)$. Similarly, for the counit $LKB \to B$, we have

$$\begin{tikzcd} FGFGB \arrow[r, shift left, "FG\varepsilon_B"] \arrow[r, shift right, "\varepsilon_{FGB}"'] & FGB \arrow[r] \arrow[dr, "\varepsilon_B"'] & LKB \arrow[d, dashed] \\ & & B \end{tikzcd}$$

By assumption, the pair $(FG\varepsilon_B, \varepsilon_{FGB})$ is G-split, with coequalizer

$$\begin{tikzcd} GFGFGB \arrow[r, shift left] \arrow[r, shift right] & GFGB \arrow[r, "G\varepsilon_B"] \arrow[l, bend left=50, "\eta_{GFGB}"] & GB \arrow[l, bend left=50, "\eta_{GB}"] \end{tikzcd}$$

so the counit is also an isomorphism. $\square$

Theorem 5.18 (Reflexive Monadicity Theorem)**.** *A functor $G : \mathbf{D} \to \mathbf{C}$ is monadic provided that the three following conditions are satified:*

(i) *G has a left adjoint.*

(ii) *G reflects isomorphisms.*

(iii) *$\mathbf{D}$ has and G preserves reflexive coequalizers.*

Proof. Same proof as Theorem 5.17. $\square$

5.6 Examples of monadic functors

Lemma 5.19. *Consider two reflexive coequalizer diagrams in* **Set***:*

$$A_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} B_1 \xrightarrow{h_1} C_1 \qquad\qquad A_2 \underset{g_2}{\overset{f_2}{\rightrightarrows}} B_2 \xrightarrow{h_2} C_2$$

Then the following is also a reflexive coequalizer diagram:

$$A_1 \times A_2 \underset{g_1 \times g_2}{\overset{f_1 \times f_2}{\rightrightarrows}} B_1 \times B_2 \xrightarrow{h_1 \times h_2} C_1 \times C_2$$

Proof. Note that reflexivity will be clear once we have proved that the above is a coequalizer diagram. Let $R_i = \{(f_i(a), g_i(a))\,,\ a \in A_i) \subseteq B_i \times B_i$ and let $\overline{R}_i$ be the smallest equivalence relation containing R_i. Then $C_i \cong B_i/\overline{R}_i$. Therefore, it suffices to show that $\overline{R}_1 \times \overline{R}_2 = \overline{R_1 \times R_2}$ as equivalence relations on $B_1 \times B_2$. Note that $(b_i, b_i') \in \overline{R}_i$ iff there exists a chain $b_i = c_0, c_1, \ldots, c_n = b_i'$ with $(c_i, c_{i+1}) \in R_i \cup R_i^\vee$ (where $R_i^\vee = \{(b, b')\,,\ (b', b) \in R_i\}$. Hence, given chains linking b_1 to b_1' and b_2 to b_2', we can link (b_1, b_2) to (b_1', b_2') by way of (b_1', b_2), since both R_1 and R_2 are reflexive relations because f_i, g_i are split epic. □

Example 5.20. (i) *The forgetful functors* **Gp** $\to$ **Set***,* **Rng** $\to$ **Set** *and* $\mathbf{Mod}_R \to$ **Set** *are all monadic (use Lemma 5.19 to show that they create reflexive coequalizers, and apply Theorem 5.18).*

(ii) *Any reflection is monadic (reduce to the case of a reflective subcategory and use Theorem 5.17).*

(iii) *Consider the composite adjunction*

$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{AbGp} \underset{I}{\overset{L}{\rightleftarrows}} \mathbf{tfAbGp}$$

where **tfAbGp** *is the category of torsion-free abelian groups and* $L : \mathbf{AbGp} \to \mathbf{tfAbGp}$ *is defined by* $LA = A/A_t$ *(i.e. the largest torsion-free quotient of* A*). The two factors are monadic by* (i) *and* (ii) *but the composite isn't since* $LF \dashv UI$ *induces the same monad on* **Set** *as* $F \dashv U$*.*

(iv) *The forgetful functor* $U : \mathbf{Top} \to \mathbf{Set}$ *has a left adjoint and preserves coequalizers, but the induced monad is* $(1_{\mathbf{Set}}, 1_{1_{\mathbf{Set}}}, 1_{1_{\mathbf{Set}}})$ *and the corresponding Eilenberg-Moore category is* **Set***, so* U *is not monadic.*

(v) *Consider the composite adjunction*

$$\mathbf{Set} \underset{F}{\overset{D}{\rightleftarrows}} \mathbf{Top} \underset{I}{\overset{\beta}{\rightleftarrows}} \mathbf{KHaus}$$

where $D : \mathbf{Set} \to \mathbf{Top}$ *is the discrete topology and* $\beta : \mathbf{Top} \to \mathbf{KHaus}$ *is the Stone-Čech compactification. The composite is monadic by Theorem 5.17 and the following fact: if* Y *is compact Hausdorff and* $R \subseteq Y \times Y$*, then* Y/R *is Hausdorff* iff R *is closed in* $Y \times Y$*.*

Lemma 5.21. *If*

$$C \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a coreflexive equalizer diagram, then

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "h"'] & B \arrow[d, "f"] \\ B \arrow[r, "g"] & A \end{tikzcd}$$

is a pullback.

Proof. If $x, y : D \rightrightarrows B$ satisfy $fx = gy$, then $x = rfx = rgy = y$ (we have $rf = 1_B$ by coreflexivity), and x, y factor through h because h is an equalizer. $\square$

Lemma 5.22. *If*

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$

is a pullback square in **Set**, *then*

$$\begin{tikzcd} PD \arrow[r, "Ph"] & PB \\ PC \arrow[u, "P^*k"] \arrow[r, "Pg"] & PA \arrow[u, "P^*f"'] \end{tikzcd}$$

commutes.

Example 5.23. (vi) *The functor* $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ *is monadic by Theorem 5.18 (it has a left adjoint* $P^* : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$*, it reflects isomorphisms, and it preserves reflexive coequalizers by Lemmas 5.21 and 5.22).*

5.7 Monadic length

Definition 5.24 (Monadic length)**.** *Suppose given an adjunction* $F \dashv G : \mathbf{C} \rightleftarrows \mathbf{D}$*, where* $\mathbf{D}$ *has reflexive coequalizers. The* monadic tower *associated with* $F \dashv G$ *is the diagram*

$$\begin{tikzcd} \mathbf{D} \arrow[d, shift left] \arrow[dr, shift left] \arrow[drr, shift left] & & & \\ \mathbf{C} \arrow[u, shift left] \arrow[r, shift left] & \mathbf{C}^{\mathbb{T}} \arrow[l, shift left] \arrow[ul, shift left] \arrow[r, shift left] & \left(\mathbf{C}^{\mathbb{T}}\right)^{\mathbb{S}} \arrow[l, shift left] \arrow[ull, shift left] & \cdots \end{tikzcd}$$

where $\mathbb{T}$ *is the monad induced by* $F \dashv G$*,* $L \dashv K : \mathbf{D} \rightleftarrows \mathbf{C}^{\mathbb{T}}$ *is the Eilenberg-Moore comparison functor together with its left adjoint,* $\mathbb{S}$ *is the monad induced by* $L \dashv K$*, etc.*

We say that $F \dashv G$ *has* monadic length n *if we arrive at an equivalence after* n *steps.*

Example 5.25. *An equivalence of categories has monadic length* 0.

6 Filtered colimits

6.1 Filtered categories

Definition 6.1 (Filtered category)**.** *We say that a category* $\mathbf{C}$ *is* filtered *if, for every finite diagram* $D : \mathbf{J} \to \mathbf{C}$*, there exists a cone under* D*.*

Lemma 6.2. *A category* $\mathbf{C}$ *is filtered* iff *the three following conditions are satisfied:*

(i) $\mathbf{C} \neq \varnothing$.

(ii) *Given* $A, B \in \operatorname{ob}\mathbf{C}$*, there exists an object* $C \in \operatorname{ob}\mathbf{C}$ *together with arrows* $A \to C$ *and* $B \to C$*.*

(iii) *Given* $f, g : A \rightrightarrows B$ *in* $\mathbf{C}$*, there exists* $B \xrightarrow{h} C$ *with* $hf = hg$*.*

Proof. ($\Rightarrow$) Clear since each of the three conditions is a particular case of the definition. ($\Leftarrow$) Suppose given $D : \mathbf{J} \to \mathbf{C}$ with $\mathbf{J}$ finite. Condition (i) deals with the case where $\mathbf{J} = \varnothing$, so we may assume that $\mathbf{J} \neq \varnothing$. By (ii), we can therefore find an object A with morphisms $D(j) \xrightarrow{f_j} A$ for all j. Then we use (iii) repeatedly to find an arrow $A \to B$ having equal composites with $f_j, f_{j'}(D\alpha) : D(j) \rightrightarrows A$ for all $j \xrightarrow{\alpha} j'$ in $\mathbf{J}$. □

Remark 6.3. *If* $\mathbf{C}$ *is a preorder, then condition* (iii) *of Lemma 6.2 is redundant. We then use the term* directed *instead of filtered.*

Lemma 6.4. *If* $\mathbf{C}$ *has finite colimits and small filtered colimits, then it has all small colimits.*

Proof. By (the dual of) Theorem 4.5, it suffices to construct small coproducts. Suppose given a set-indexed family $(A_i)_{i \in I}$ of objects of $\mathbf{C}$. For each finite subset $I' \subseteq I$, we can form the coproduct $B_{I'} = \sum_{i \in I'} A_i$. Moreover, if $I' \subseteq I'' \subseteq I$, we get an induced morphism $B_{I'} \to B_{I''}$ given by $f\nu_i = \nu_i$. This makes the assignment $I' \mapsto B_{I'}$ a diagram of shape $P_f I = \{I' \in PI,\ I' \text{ is finite}\}$ in $\mathbf{C}$. But $P_f I$ is directed, so it suffices to take a colimit for the diagram $I' \mapsto B_{I'}$. □

6.2 Commutativity of limits and colimits

Remark 6.5. *Suppose given a diagram* $D : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$*, where* $\mathbf{C}$ *has limits of shape* $\mathbf{I}$ *and colimits of shape* $\mathbf{J}$*.*

We can regard D *as a diagram of shape* $\mathbf{I}$ *in* $[\mathbf{J}, \mathbf{C}]$ *and form its limit* $\lim_{\mathbf{I}} D : \mathbf{J} \to \mathbf{C}$*, and then form* $\operatorname{colim}_{\mathbf{J}} \lim_{\mathbf{I}} D$*; or we can form* $\operatorname{colim}_{\mathbf{J}} D : \mathbf{I} \to \mathbf{C}$ *and then* $\lim_{\mathbf{I}} \operatorname{colim}_{\mathbf{J}} D$*.*

There is a canonical morphism

$$\operatorname*{colim}_{\mathbf{J}} \lim_{\mathbf{I}} D \xrightarrow{\theta} \lim_{\mathbf{I}} \operatorname*{colim}_{\mathbf{J}} D.$$

Indeed, the composites $\lim_{\mathbf{I}} D(j) \to D(i,j) \to \operatorname{colim}_{\mathbf{J}} D(i)$ *form a cone over* $\operatorname{colim}_{\mathbf{J}} D(i)$*, so they induce morphisms* $\lim_{\mathbf{I}} D(j) \to \lim_{\mathbf{I}} \operatorname{colim}_{\mathbf{J}} D$*. When* j *varies, these form a cone under* $\lim_{\mathbf{I}} D$*, so they induce* θ*.*

Definition 6.6 (Commutativity of limits and colimits)**.** *Suppose the category* $\mathbf{C}$ *has limits of shape* $\mathbf{I}$ *and colimits of shape* $\mathbf{J}$*. We say that colimits of shape* $\mathbf{J}$ *commute with limits of shape* $\mathbf{I}$ *in* $\mathbf{C}$ *if for all diagrams* $D : \mathbf{I} \times \mathbf{J} \to \mathbf{C}$*, the canonical morphism* $\operatorname{colim}_{\mathbf{J}} \lim_{\mathbf{I}} D \xrightarrow{\theta} \lim_{\mathbf{I}} \operatorname{colim}_{\mathbf{J}} D$ *is an isomorphism.*

Example 6.7. *In* **Set***, Lemma 5.19 says that reflexive coequalizers commute with binary products.*

Theorem 6.8. *Let* $\mathbf{J}$ *be a small category. Then colimits of shape* $\mathbf{J}$ *commute with all finite limits in* **Set** *if and only if* $\mathbf{J}$ *is filtered.*

Proof. ($\Rightarrow$) Suppose given $D : \mathbf{I} \to \mathbf{J}$ with $\mathbf{I}$ finite. Consider the diagram

$$E : \mathbf{I}^{\mathrm{op}} \times \mathbf{J} \to \mathbf{Set}$$

given by $E(i,j) = \mathbf{J}(Di, j)$. For fixed i, $\operatorname{colim}_{\mathbf{J}} E(i)$ is a singleton since every $Di \xrightarrow{\alpha} j$ gets identified in the colimit with $Di \xrightarrow{1_{Di}} Di$ because the diagram

$$\begin{tikzcd} \mathbf{J}(Di, Di) \arrow[rr, "\mathbf{J}(Di,\alpha)"] \arrow[dr] & & \mathbf{J}(Di, j) \arrow[dl] \\ & \operatorname{colim}_{\mathbf{J}} E(i) & \end{tikzcd}$$

commutes. Thus $\lim_{\mathbf{I}} \operatorname{colim}_{\mathbf{J}} E$ is a singleton. Hence, for some j, $\lim_{\mathbf{I}} E(j)$ is nonempty; but an element of this limit is exactly a cone under D with apex j, so $\mathbf{J}$ is filtered.

($\Leftarrow$) For a general $\mathbf{J}$, the colimit of $D : \mathbf{J} \to \mathbf{Set}$ can be described as $\coprod_{j \in \mathrm{ob}\,\mathbf{J}} D(j)/\sim$, where $\sim$ is the smallest equivalence relation identifying (j,x) with $(j', D(\alpha)(x))$ for all $\alpha : j \to j'$ in $\mathbf{J}$ and all $x \in D(j)$. When $\mathbf{J}$ is filtered, the relation $\sim$ has a simple description:

$$(j,x) \sim (j',x') \iff \exists j \xrightarrow{\alpha} j'',\ \exists j' \xrightarrow{\beta} j'',\ D(\alpha)(x) = D(\beta)(x')\,, \qquad (*)$$

since this relation is transitive and hence is an equivalence relation.

To show that $\operatorname{colim}_{\mathbf{J}} : [\mathbf{J}, \mathbf{Set}] \to \mathbf{Set}$ preserves finite limits, it is sufficient by Theorem 4.5 to show that it preserves the terminal object and pullbacks. Since $\mathbf{J}$ is connected by Lemma 6.2, it is clear that $\operatorname{colim}_{\mathbf{J}}(\Delta 1) \cong 1$, so $\operatorname{colim}_{\mathbf{J}}$ preserves the terminal object. Now consider a diagram $D : \mathbf{I} \times \mathbf{J} \to \mathbf{Set}$, where $\mathbf{I}$ is the category

$$\begin{tikzcd} & i_1 \arrow[d] \\ i_2 \arrow[r] & i_3 \end{tikzcd}$$

We have a diagram

$$\begin{tikzcd} & \operatorname{colim}_{\mathbf{J}} D(i_1, -) \arrow[d] \\ \operatorname{colim}_{\mathbf{J}} D(i_2, -) \arrow[r] & \operatorname{colim}_{\mathbf{J}} D(i_3, -) \end{tikzcd}$$

Given an element (x_1, x_2) of the pullback, we can represent x_1 and x_2 by elements $y_1 \in D(i_1, j_1)$ and $y_2 \in D(i_2, j_2)$ and we can assume without loss of generality that, say, $j_1 = j_2 = j$ (because $\mathbf{J}$ is filtered). The images of y_1 and y_2 in $D(i_3, j)$ represent the same element of $\operatorname{colim}_{\mathbf{J}} D(i_3, -)$, so by $(*)$ and Lemma 6.2, we can find $j \xrightarrow{\alpha} j'$ such that $D(\alpha)(y_1) = D(\alpha)(y_2)$. So there exists $y_0 \in \lim_{\mathbf{I}} D(-, j')$ representing the element (x_1, x_2) of $\lim_{\mathbf{I}} \operatorname{colim}_{\mathbf{J}} D$, i.e. the map

$$\operatorname*{colim}_{\mathbf{J}} \lim_{\mathbf{I}} D \xrightarrow{\theta} \lim_{\mathbf{I}} \operatorname*{colim}_{\mathbf{J}} D$$

is surjective. Similarly, if we are given two elements x_1, x_2 of $\operatorname{colim}_{\mathbf{J}} \lim_{\mathbf{I}} D$ with the same image under θ, we can represent them by elements (y_1, y_2) and (y_1', y_2') of $\lim_{\mathbf{I}} D(-, j)$ for some j; and the pairs (y_1, y_1') and (y_2, y_2') each represent the same element of $\operatorname{colim}_{\mathbf{J}} D(i_1, -)$ or $\operatorname{colim}_{\mathbf{J}} D(i_2, -)$. So we can find $j \xrightarrow{\alpha} j'$ such that $D(i_1, \alpha)(y_1) = D(i_1, \alpha)(y_1')$ and $D(i_2, \alpha)(y_2) = D(i_2, \alpha)(y_2')$; hence (y_1, y_2) and (y_1', y_2') represent the same element of $\operatorname{colim}_{\mathbf{J}} \lim_{\mathbf{I}} D$. $\square$

Corollary 6.9. *Let* $\mathbf{A}$ *be a category whose objects are sets* A *equipped with finitary operations* $A^n \to A$ *satisfying some equations (and whose arrows are homomorphisms), for instance* $\mathbf{Gp}, \mathbf{Rng}, \mathbf{Mod}_R$*. Then:*

(i) *The forgetful functor $U : \mathbf{A} \to \mathbf{Set}$ creates filtered colimits.*

(ii) *Filtered colimits commute with finite limits in $\mathbf{A}$.*

Proof. (i) Same argument as for reflexive coequalizers in Example 5.20.(i). (ii) This follows from Theorem 6.8 since U reflects both filtered colimits and finite limits. □

Remark 6.10. (i) *Filtered colimits and finite limits don't commute in $\mathbf{Set}^{\mathrm{op}}$. Indeed, consider the following diagram of shape $\mathbb{N}^{\mathrm{op}} \times \mathbf{2}$ in $\mathbf{Set}$:*

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{N} \arrow[r, "\cdot + 1"] \arrow[d] & \mathbb{N} \arrow[r, "\cdot + 1"] \arrow[d] & \mathbb{N} \arrow[d] \\ \cdots \arrow[r] & 1 \arrow[r] & 1 \arrow[r] & 1 \end{tikzcd}$$

Applying $\lim_{\mathbb{N}^{\mathrm{op}}}$ to this diagram yields

$$\begin{tikzcd} \varnothing \arrow[d] \\ 1 \end{tikzcd}$$

so $\lim_{\mathbb{N}^{\mathrm{op}}}$ does not preserve epimorphisms, and hence it does not preserve pushouts.

(ii) *The argument of Corollary 6.9 extends to categories such as $\mathbf{Cat}$, where we have an operation whose domain is a pullback:*

$$\begin{tikzcd} & \operatorname{mor}\mathbf{C} \arrow[d, "\mathrm{dom}"] \\ \operatorname{mor}\mathbf{C} \arrow[r, "\mathrm{cod}"] & \operatorname{ob}\mathbf{C} \end{tikzcd}$$

(iii) *Given an infinite cardinal κ, we say that $\mathbf{J}$ is κ-filtered if every diagram $D : \mathbf{I} \to \mathbf{J}$ where $\operatorname{card}(\operatorname{mor}\mathbf{I}) < \kappa$ has a cone under it. Then the argument of Theorem 6.8 extends to show that κ-filtered colimits commute with κ-small limits (i.e. limits of shape $\mathbf{I}$ where $\operatorname{card}(\operatorname{mor}\mathbf{I}) < \kappa$) in $\mathbf{Set}$.*

6.3 Finitary functors and Lawvere theories

Definition 6.11 (Finitary functor)**.** *Suppose $\mathbf{C}$ has filtered colimits. We say that a functor $F : \mathbf{C} \to \mathbf{D}$ is* finitary *if it preserves filtered colimits. If $\mathbf{C}$ is also locally small, we say that an object $A \in \operatorname{ob}\mathbf{C}$ is* finitely presentable *if $\mathbf{C}(A, -) : \mathbf{C} \to \mathbf{Set}$ is finitary.*

Example 6.12. (i) *If $\mathbf{I}$ is a finite category, then $\lim_{\mathbf{I}} : [\mathbf{I}, \mathbf{Set}] \to \mathbf{Set}$ is finitary.*

(ii) *In $\mathbf{Set}$, any object A can be represented as a filtered colimit of finite sets, for instance as $A = \bigcup_{A' \in P_f A} A'$ or as the colimit of $D : (I \downarrow A) \to \mathbf{Set}$, where $I : \mathbf{Set}_f \to \mathbf{Set}$ is the inclusion functor and $D\left(B \xrightarrow{f} A\right) = B$. Note that $(I \downarrow A)$ has finite colimits since $\mathbf{Set}_f$ does and I preserves them, so it is filtered. Hence any finitary functor $F : \mathbf{Set} \to \mathbf{Set}$ is determined by its restriction to $\mathbf{Set}_f$: in fact, $F \cong \operatorname{Lan}_I I^* F$, where $I^* : [\mathbf{Set}, \mathbf{Set}] \to [\mathbf{Set}_f, \mathbf{Set}]$ is the restriction functor and Lan_I is its left adjoint.*

(iii) *In* **Gp**, *an object* G *is finitely presentable* iff *it is finitely presented, i.e. a coequalizer* $F(r) \rightrightarrows F(n) \to G$. *Note that* $F(n)$ *represents* $U^n : \mathbf{Gp} \to \mathbf{Set}$, *and is thus finitary, so* $\mathbf{Gp}(G, -)$ *is an equalizer of* $U^n \rightrightarrows U^r$ *and is thus also finitary.*

Conversely, we can represent G *as a filtered colimit of finitely presented groups, so if it is finitely presentable, then* $G \xrightarrow{1_G} G$ *factors through some* $H \to G$ *where* H *is finitely presented, i.e.* G *is a coequalizer of* $1_H, e : H \rightrightarrows H$, *so it is obtained from* H *by adding finitely many more relations, and is thus finitely presented.*

Definition 6.13 (Finitary monad)**.** *We say that a monad* $\mathbb{T} = (T, \eta, \mu)$ *on* **Set** *is* finitary *if* $T : \mathbf{Set} \to \mathbf{Set}$ *is finitary (or equivalently by Theorem 5.13, the forgetful functor* $\mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$ *creates filtered colimits).*

Lemma 6.14. *Let* $\mathbb{T} = (T, \eta, \mu)$ *be a monad on* **Set**. *Then for any* n, *elements of* Tn *correspond bijectively to natural transformations* $\left(G^{\mathbb{T}}\right)^n \to G^{\mathbb{T}}$, *where* $G^{\mathbb{T}}$ *is the forgetful functor* $\mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$.

Proof. The free algebra $F^{\mathbb{T}}(n)$ represents the functor $\left(G^{\mathbb{T}}\right)^n$, since homomorphisms $F^{\mathbb{T}}(n) \to (A, \alpha)$ correspond to n-tuples of elements of A. So by the Yoneda Lemma, natural transformations $\left(G^{\mathbb{T}}\right)^n \to G^{\mathbb{T}}$ correspond to elements of $G^{\mathbb{T}} F^{\mathbb{T}}(n) = Tn$. $\square$

Definition 6.15 (Lawvere theory)**.** *A* Lawvere theory *is a (small) category* **L** *equipped with a functor* $F : \mathbf{Set}_f \to \mathbf{L}$ *which is bijective on objects and preserves finite coproducts, where* $\mathbf{Set}_f$ *is the skeleton of the category of finite sets whose objects are the sets* $\{1, 2, \ldots, n\}$ *for* $n \in \mathbb{N}$.

Given a monad $\mathbb{T}$ *on* **Set**, *the full subcategory of* $\mathbf{Set}_{\mathbb{T}}$ *on the objects* $\{1, 2, \ldots, n\}$ *is a Lawvere category.*

Given a Lawvere theory **L**, *an* algebra *for* **L** *is a functor* $A : \mathbf{L}^{\mathrm{op}} \to \mathbf{Set}$ *preserving finite products.*

Remark 6.16. *An algebra* A *for a Lawvere category* **L** *is determined by* $A1$ *since we have* $An = A1^n$ *for all* n, *together with* n*-ary operations* $\omega_A : A1^n \to A1$ *for each* $\omega : 1 \to n$ *in* **L** *satisfying:*

(i) *Given* $\nu_i : 1 \to n$ *in* $\mathbf{Set}_f$, *we have* $(F\nu_i)_A = \pi_i : A1^n \to A1$.

(ii) *For any* $\theta_1, \ldots, \theta_n : 1 \to n$ *in* **L** *and* $\omega : 1 \to n$, *the composite*

$$(A1)^n \xrightarrow{(\theta_1)_A, \ldots, (\theta_n)_A} (A1)^n \xrightarrow{\omega_A} A1$$

must equal $((\theta_1 \amalg \cdots \amalg \theta_n)\, \omega)_A : (A1)^n \to A1$.

Theorem 6.17. *The following concepts are equivalent:*

(i) *Finitary algebraic categories, i.e. categories whose objects are sets* A *equipped with operations* $A^n \to A$ *satisfying some equations,*

(ii) *Finitary monads on* **Set**,

(iii) *Lawvere theories.*

Sketch of proof. (i) $\Rightarrow$ (ii) Use Example 5.20.(i) and Corollary 6.9. (ii) $\Rightarrow$ (iii) Take the finite part of the Kleisli category. (iii) $\Rightarrow$ (i) Take the category of algebras.

Showing that we get back to where we started amounts to showing that every derived operation on Lawvere algebras is equal to some ω_A, i.e. that $\mathbf{L}(1, n)$ is the set of natural transformations $G^n \to G$, where G is the forgetful functor from **L**-algebras to **Set**. $\square$

7 Additive and abelian categories

7.1 Pointed, semi-additive and additive categories

Definition 7.1 (Enriched category)**.** *Let* $\mathbf{C}$ *be a category with a forgetful functor* $G : \mathbf{C} \to \mathbf{Set}$*. We say that a locally small category* $\mathbf{A}$ *is* enriched *over* $\mathbf{C}$ *if the functor* $\mathbf{A}(-,-) : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$ *has a factorisation through* G*.*

(i) *If* $\mathbf{C} = \mathbf{Set}_*$ *is the category of pointed sets, we call* $\mathbf{A}$ *a* pointed category*. This means that for every* $A, B \in \mathrm{ob}\, \mathbf{A}$*, we have a distinguished morphism* $A \xrightarrow{0} B$ *satisfying* $f0 = 0 = 0g$ *whenever the composites are defined.*

(ii) *If* $\mathbf{C} = \mathbf{CMon}$ *is the category of commutative monoids, we call* $\mathbf{A}$ *a* semi-additive category*. This means that we have a binary operation* $+$ *on each set* $\mathbf{A}(A, B)$ *which is associative and commutative, has* 0 *as an identity, and satisfies* $f(g+h) = fg + fh$ *and* $(g+h)k = gk + hk$ *whenever the composites are defined.*

(iii) *If* $\mathbf{C} = \mathbf{AbGp}$ *is the category of abelian groups, we call* $\mathbf{A}$ *an* additive category*. This means that* $\mathbf{A}$ *is a semi-additive category and also has an operation* $f \mapsto (-f)$ *on* $\mathbf{A}(A, B)$ *satisfying* $f + (-f) = 0$*.*

7.2 Zero objects and biproducts

Lemma 7.2. (i) *If* A *is an object of a pointed category, the following are equivalent:*

- (a) A *is initial,*
- (b) A *is terminal,*
- (c) $1_A = 0 : A \to A$*.*

(ii) *If* A, B, C *are objects of a semi-additive category, the following are equivalent:*

- (a) *There exist* $A \xrightarrow{\nu_1} C \xleftarrow{\nu_2} B$ *making* C *a coproduct,*
- (b) *There exist* $A \xleftarrow{\pi_1} C \xrightarrow{\pi_2} B$ *making* C *a product,*
- (c) *There exist* $\nu_1, \nu_2, \pi_1, \pi_2$ *satisfying* $\pi_1\nu_1 = 1_A$*,* $\pi_2\nu_2 = 1_B$*,* $\pi_2\nu_1 = 0 = \pi_1\nu_2$ *and* $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$*.*

Proof. (i)(a) $\Rightarrow$ (i)(c) There is only one morphism $A \to A$. (i)(c) $\Rightarrow$ (i)(a) Any $A \xrightarrow{f} B$ satisfies $f = f1_A = f0 = 0$. (i)(b) $\Leftrightarrow$ (i)(c) Dual of (i)(a) $\Leftrightarrow$ (i)(c).

(ii)(a) $\Rightarrow$ (ii)(c) We define $\pi_1 : C \to A$ by $\pi_1\nu_1 = 1_A$, $\pi_1\nu_2 = 0$, and $\pi_2 : C \to B$ similarly. Thus

$$(\nu_1\pi_1 + \nu_2\pi_2)\,\nu_1 = \nu_1 \qquad \text{and} \qquad (\nu_1\pi_1 + \nu_2\pi_2)\,\nu_2 = \nu_2,$$

so $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$. (ii)(c) $\Rightarrow$ (ii)(a) Given $A \xrightarrow{f} D \xleftarrow{g} B$, the morphism $h = f\pi_1 + g\pi_2 : C \to D$ satisfies $h\nu_1 = f$ and $h\nu_2 = g$. But any k satisfying $k\nu_1 = f$ and $k\nu_2 = g$ must also satisfy

$$k = k\,(\nu_1\pi_1 + \nu_2\pi_2) = f\pi_1 + g\pi_2 = h,$$

so C has the universal property of coproducts. (ii)(b) $\Leftrightarrow$ (ii)(c) Dual of (ii)(a) $\Leftrightarrow$ (ii)(c). □

Definition 7.3 (Zero objects, biproducts)**.** *In a category* $\mathbf{C}$*, an object which is both initial and terminal is called a* zero object *and denoted* 0*. An object which is both a product and a coproduct of* (A, B) *is called a* biproduct *and denoted* $A \oplus B$*.*

Notation 7.4. *We denote morphisms $A \to B \times C$ by column vectors $\begin{pmatrix} f \\ g \end{pmatrix}$, and morphisms $B+C \to D$ by row vectors $\begin{pmatrix} h & k \end{pmatrix}$. Hence a morphism $A + B \to C \times D$ can be represented by a matrix $\begin{pmatrix} f & g \\ h & k \end{pmatrix}$.*

Lemma 7.5. (i) *If a category* $\mathbf{C}$ *has a zero object, then it has a unique pointed structure.*

(ii) *If* $\mathbf{C}$ *is a pointed category with finite products and coproducts such that, for each pair of objects (A, B), the morphism $A + B \xrightarrow{c} A \times B$ with matrix $\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$ is an isomorphism, then* $\mathbf{C}$ *has a unique semi-additive structure.*

Proof. (i) The morphism $0 \xrightarrow{0} 0$ has to be 1_0, and $A \xrightarrow{0} B$ has to be the unique composite $A \to 0 \to B$.

(ii) Given $f, g : A \rightrightarrows B$, we define $f +_L g$ to be the composite

$$A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{\begin{pmatrix} 1_B & 1_B \end{pmatrix}} B,$$

and $f +_R g$ to be

$$A \xrightarrow{\begin{pmatrix} 1_A \\ 1_A \end{pmatrix}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} B.$$

Note that $(f +_L g)\, h = fh +_L gh$ when the composites are defined, and dually $k\,(f +_R g) = kf +_R kg$. Next, $f +_L 0 = f$ since

$$\begin{tikzcd}
A \arrow[rr, "{\begin{pmatrix} f \\ 0 \end{pmatrix}}"] \arrow[drrr, "f"'] & & B \times B \arrow[rr, "c^{-1}"] & & B + B \arrow[rr, "{\begin{pmatrix} 1_B & 1_B \end{pmatrix}}"] & & B \\
& & & B \arrow[ul, "{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}"] \arrow[ur, "\nu_1"'] \arrow[urrr, "1_B"'] & & &
\end{tikzcd}$$

commutes. Similarly, $0 +_L f = f$, and dually $f +_R 0 = 0 +_R f = f$.

Given $f, g, h, k : A \to B$, consider the composite

$$A \xrightarrow{\begin{pmatrix} 1_A \\ 1_A \end{pmatrix}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{\begin{pmatrix} 1_B & 1_B \end{pmatrix}} B.$$

This equals

$$A \xrightarrow{\begin{pmatrix} f +_R g \\ h +_R k \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} B,$$

i.e. $(f +_R g) +_L (h +_R k)$. It also equals $(f +_L h) +_R (g +_L k)$. This gives the *interchange law*:

$$(f +_R g) +_L (h +_R k) = (f +_L h) +_R (g +_L k)\,.$$

Putting $g = h = 0$ gives $f +_L k = f +_R k$, i.e. $+_L = +_R = +$. Putting $f = k = 0$ gives $g + h = h + g$, so $+$ is commutative. Putting $h = 0$ gives $(f + g) + k = f + (g + k)$, so $+$ is associative. Therefore, $+$ is a semi-additive structure on $\mathbf{C}$.

For uniqueness, note that any semi-additive structure $+$ must satisfy

$$\left(B \xrightarrow{\nu_1 + \nu_2} B + B \xrightarrow{c} B \times B\right) = (c\nu_1 + c\nu_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the composite

$$A \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix}f & g\end{pmatrix}} B$$

must equal $f + g$, which implies that $+ = +_R$. $\square$

Corollary 7.6. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be semi-additive categories with finite biproducts. If* $F : \mathbf{C} \to \mathbf{D}$ *is a functor, then the following assertions are equivalent:*

(i) *F is semi-additive, i.e.* $F(0) = 0$ *and* $F(f + g) = Ff + Fg$.

(ii) *F preserves finite products.*

(iii) *F preserves finite coproducts.*

In particular, any functor with a left or right adjoint is semi-additive.

Example 7.7. *Consider the multiplicative monoid* $(\mathbb{N}, \times)$. *It has a semi-additive structure given by* $+$, *but it has many others: any permutation* π *of the set of prime numbers extends uniquely to an automorphism* $\overline{\pi}$ *of* $(\mathbb{N}, \times)$, *so if we define* $+^{\pi}$ *by*

$$m +^{\pi} n = \overline{\pi}\left(\overline{\pi}^{-1}(m) + \overline{\pi}^{-1}(n)\right),$$

then $+^{\pi}$ *is a semi-additive structure on* $(\mathbb{N}, \times)$.

7.3 Kernels and cokernels

Definition 7.8 (Kernel, cokernel)**.** *In a pointed category, by a* kernel *of* $A \xrightarrow{f} B$ *we mean an equalizer of* $f, 0 : A \rightrightarrows B$, *i.e. a universal morphism* $C \xrightarrow{k} A$ *satisfying* $fk = 0$. *Dually, we have a notion of* cokernel.

We say that a monomorphism is normal *if it occurs as a kernel. We say that* $A \xrightarrow{f} B$ *is a* pseudo-monomorphism *if its kernel is a zero morphism, i.e. if* $fk = 0$ *implies* $k = 0$.

In an additive category, every regular monomorphism is normal since an equalizer of $f, g : A \rightrightarrows B$ *is also a kernel of* $A \xrightarrow{f-g} B$; *similarly, any pseudo-monomorphism is monic.*

Example 7.9. (i) *In* $\mathbf{Gp}$, *every injective homomorphism is regular monic, but* $f : G \rightarrowtail H$ *is normal monic* iff $f(G)$ *is a normal subgroup of* H. *However, every surjective homomorphism* $f : G \twoheadrightarrow H$ *is normal epic.*

(ii) *In* $\mathbf{Set}_*$, *every monomorphism is normal: if* $f : (A, a) \to (B, b)$ *is injective, then it is the kernel of* $(B, b) \to (B/\sim, [b])$, *where* $\sim$ *identifies every element of* $f(A)$ *with* b. *However, not every epimorphism is normal: for instance,* $(\{1, 2, 3\}, 1) \xrightarrow{f} (\{1, 2\}, 1)$ *defined by* $f(2) = f(3) = 2$ *is not a cokernel.*

Lemma 7.10. *If* $\mathbf{C}$ *is pointed and has cokernels, then a morphism* f *is normal monic* iff

$$f \cong \ker(\operatorname{coker} f).$$

In particular, if $\mathbf{C}$ *has both kernels and cokernels, there is a bijection between isomorphism classes of normal subobjects and normal quotients of any object.*

Proof. ($\Leftarrow$) Obvious. ($\Rightarrow$) Suppose $A \xrightarrow{f} B$ is the kernel of $B \xrightarrow{g} C$. Then g factors as $B \xrightarrow{\operatorname{coker} f} D \xrightarrow{h} C$, so $(\operatorname{coker} f)\, k = 0$ implies $gk = 0$, hence k factors through f. Therefore, $f \cong \ker(\operatorname{coker} f)$. $\square$

Notation 7.11. *In a pointed category with kernels and cokernels, we write* $\operatorname{im} f$ *for* $\ker(\operatorname{coker} f)$ *and* $\operatorname{coim} f$ *for* $\operatorname{coker}(\ker f)$.

Lemma 7.12. *Suppose* $\mathbf{C}$ *is pointed and has kernels and cokernels, and suppose that every monomorphism in* $\mathbf{C}$ *is normal. Then every morphism of* $\mathbf{C}$ *factors uniquely as a pseudo-epimorphism followed by a monomorphism.*

Proof. Given $A \xrightarrow{f} B$, form the diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g"'] \arrow[dd, "h"'] & & B \arrow[rr, "\operatorname{coker} f"] \arrow[drr, "t"'] & & C \arrow[d, "s"] \\
& I \arrow[ur, "\operatorname{im} f"'] \arrow[dr, "\operatorname{coker} g"] & & & T \\
K \arrow[ur, "\operatorname{im} g"'] & & J & &
\end{tikzcd}$$

It suffices to show that $\operatorname{coker} g = 0$. To do this, consider the cokernel $B \xrightarrow{t} T$ of $(\operatorname{im} f)(\operatorname{im} g)$. Then we have $tf = t(\operatorname{im} f)(\operatorname{im} g)h = 0$, so there is a factorisation $C \xrightarrow{s} T$ of t through $\operatorname{coker} f$. Hence $t(\operatorname{im} f) = s(\operatorname{coker} f)(\operatorname{im} f) = 0$. But recall that $t = \operatorname{coker}((\operatorname{im} f)(\operatorname{im} g))$, and $(\operatorname{im} f)(\operatorname{im} g)$ is normal monic, so $\ker t = (\operatorname{im} f)(\operatorname{im} g)$ by Lemma 7.10. Therefore, there is a factorisation $I \xrightarrow{u} K$ of $\operatorname{im} f$ through $(\operatorname{im} f)(\operatorname{im} g)$: we have $\operatorname{im} f = (\operatorname{im} f)(\operatorname{im} g)u$. Since $\operatorname{im} f$ is monic, $1 = (\operatorname{im} g)u$, so $\operatorname{im} g$ is epic and therefore $\operatorname{coker} g = \operatorname{coker}(\operatorname{im} g) = 0$.

For uniqueness, if f factors as $A \xrightarrow{u} M \xrightarrow{v} B$ with v monic and u pseudo-epic, then $\operatorname{coker} f \cong \operatorname{coker} v$, so $v \cong \ker(\operatorname{coker} v) \cong \ker(\operatorname{coker} f)$. □

7.4 Abelian categories

Definition 7.13 (Abelian category)**.** *By an* abelian category *we mean an additive category with all finite limits and colimits (equivalently, finite biproducts, kernels and cokernels) in which every monomorphism and every epimorphism is normal (equivalently, regular).*

Example 7.14. (i) *The categories* $\mathbf{AbGp}$ *and* $\mathbf{Mod}_R$ *(for any ring* R*) are abelian.*

(ii) *If* $\mathbf{C}$ *is small and* $\mathbf{A}$ *is abelian, then* $[\mathbf{C}, \mathbf{A}]$ *is abelian, with structure defined pointwise.*

(iii) *If* $\mathbf{C}$ *is small and additive and* $\mathbf{A}$ *is abelian, then the full subcategory* $\mathbf{Add}(\mathbf{C}, \mathbf{A}) \subseteq [\mathbf{C}, \mathbf{A}]$ *of additive functors is closed under finite limits and colimits in* $[\mathbf{C}, \mathbf{A}]$ *and hence abelian.*

Note in particular that $\mathbf{Mod}_R \cong \mathbf{Add}(R, \mathbf{AbGp})$ *for any ring* R*.*

Lemma 7.15. *Suppose* $\mathbf{A}$ *is additive with finite biproducts. Given a square*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"'] \\
C \arrow[r, "k"] & D
\end{tikzcd}$$

in $\mathbf{A}$*, its* flattening *is the diagram*

$$A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} h & -k \end{pmatrix}} D.$$

Then:

(i) *The square commutes* iff $\begin{pmatrix} h & -k \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0$.

(ii) *The square is a pullback* iff $\begin{pmatrix} f \\ g \end{pmatrix} = \ker \begin{pmatrix} h & -k \end{pmatrix}$.

(iii) *The square is a pushout* iff $\begin{pmatrix} h & -k \end{pmatrix} = \operatorname{coker} \begin{pmatrix} f \\ g \end{pmatrix}$.

Proof. (i) The composite is zero iff $hf - kg = 0$. (ii) The pair of arrows $A \xrightarrow{f} B$, $A \xrightarrow{g} C$ is universal among cones over $B \xrightarrow{h} D$, $C \xrightarrow{k} D$ iff $\begin{pmatrix} f \\ g \end{pmatrix}$ is universal among morphisms having zero composite with $\begin{pmatrix} h & -k \end{pmatrix}$. (iii) This is almost dual to (ii). □

Corollary 7.16. *In an abelian category* $\mathbf{A}$*,*

(i) *Epimorphisms are stable under pullback, i.e. if*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

is a pullback and h is epic, then g is epic;

(ii) *Image factorisations are stable under pullbacks.*

Proof. (i) Since the square is a pullback, we have $\begin{pmatrix} f \\ g \end{pmatrix} = \ker \begin{pmatrix} h & -k \end{pmatrix}$ by Lemma 7.15. But $\begin{pmatrix} h & -k \end{pmatrix}$ is epic since h is, so $\begin{pmatrix} h & -k \end{pmatrix} = \operatorname{coker} \begin{pmatrix} f \\ g \end{pmatrix}$ by Lemma 7.10, and hence the square is a pushout by Lemma 7.15. So if we are given $C \xrightarrow{x} E$ with $xg = 0$, then the pair $C \xrightarrow{x} E$, $B \xrightarrow{0} E$ forms a cone under $A \xrightarrow{f} B$, $A \xrightarrow{g} C$. Thus, there exists $D \xrightarrow{y} E$ with $yk = x$ and $yh = 0$. But h is epic, so $y = 0$ and hence $x = 0$.

(ii) This is immediate from (i) and the fact (Lemma 4.17) that monomorphisms are stable under pullbacks in any category. □

7.5 Exact sequences

Definition 7.17 (Exact sequence)**.** *Given a (finite or infinite) sequence*

$$\cdots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \cdots$$

in an abelian category, we say that the sequence is exact at A_n *if*

$$\ker f_n \cong \operatorname{im} f_{n-1},$$

or equivalently $\operatorname{coker} f_{n-1} \cong \operatorname{coim} f_n$*. We say that the sequence is* exact *if it is exact at every interior vertex.*

Example 7.18. *In an abelian category,*

(i) *A sequence* $0 \to A \xrightarrow{f} B$ *is exact* iff f *is monic,*

(ii) *A sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ *is exact* iff $f \cong \ker g$.

Definition 7.19 (Exact functor)**.** *We say that a functor* $F : \mathbf{A} \to \mathbf{B}$ *between abelian categories is*

(i) Exact *if it preserves all exact sequences,*

(ii) Left exact *if it preserves exact sequences of the form* $0 \to A \to B \to C$*,*

(iii) Right exact *if it preserves exact sequences of the form* $A \to B \to C \to 0$*.*

Remark 7.20. *Note that a left exact functor preserves exact sequences of the form* $0 \to A \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} B \to 0$ *since* $\begin{pmatrix}0 & 1\end{pmatrix}$ *is split epic. Therefore it preserves finite biproducts, and hence is additive. Hence:*

(i) F *is left exact* iff *it preserves all finite limits,*

(ii) F *is exact* iff *it preserves kernels and cokernels,* iff *it preserves all finite limits and colimits.*

Lemma 7.21 (Five Lemma)**.** *Suppose given a commutative diagram with exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "a_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \arrow[r, "b_4"] & B_5 \end{tikzcd}$$

in an abelian category. Then

(i) *If* f_2, f_4 *are monic and* f_1 *is epic, then* f_3 *is monic.*

(ii) *If* f_2, f_4 *are epic and* f_5 *is monic, then* f_3 *is epic.*

Proof. Note that (ii) is the dual of (i), so it suffices to prove (i). Suppose given $X \xrightarrow{x} A_3$ such that $f_3 x = 0$. Then $f_4 a_3 x = b_3 f_3 x = 0$. Since f_4 is monic, $a_3 x = 0$. Therefore x factors through $\ker a_3 \cong \operatorname{im} a_2$. Forming the pullback

$$\begin{tikzcd} Y \arrow[r, two heads, "e"] \arrow[d, "y"'] & X \arrow[d] \arrow[dr, "x"] & \\ A_2 \arrow[r, two heads] \arrow[rr, bend right, "a_2"'] & I_2 \arrow[r, tail, "\operatorname{im} a_2"] & A_3 \end{tikzcd}$$

we get morphisms e, y with $a_2 y = xe$ and e epic (by Corollary 7.16). Now $b_2 f_2 y = f_3 a_2 y = f_3 x e = 0$, so $f_2 y$ factors through $\ker b_2 = \operatorname{im} b_1$. Again, forming the pullback

$$\begin{tikzcd} Z \arrow[rr, two heads, "d"] \arrow[d, "z"'] & & Y \arrow[d] \arrow[dr, "f_2 y"] & \\ A_1 \arrow[r, two heads, "f_1"] & B_1 \arrow[r, two heads] \arrow[rr, bend right, "b_1"'] & I_1 \arrow[r, tail, "\operatorname{im} b_1"] & B_2 \end{tikzcd}$$

we get d and z satisfying $b_1 f_1 z = f_2 y d$. Thus $f_2 y d = f_2 a_1 z$, and f_2 is monic, so $yd = a_1 z$. Now $xed = a_2 yd = a_2 a_1 z = 0$, and ed is epic, so $x = 0$. □

Lemma 7.22 (Snake Lemma)**.** *Suppose given a black commutative diagram with exact rows and columns in an abelian category:*

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& A_1 \arrow[r] \arrow[d, "u_1"'] & A_2 \arrow[r] \arrow[d, "u_2"'] & A_3 \arrow[d, "u_3"'] \arrow[ddll, rounded corners, "\delta"] & \\
& B_1 \arrow[r] \arrow[d, "v_1"'] & B_2 \arrow[r] \arrow[d, "v_2"'] & B_3 \arrow[r] \arrow[d, "v_3"'] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d, "w_1"'] & C_2 \arrow[r] \arrow[d, "w_2"'] & C_3 \arrow[d, "w_3"'] & \\
& D_1 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & D_3 \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

Then there exist red morphisms making the red sequence exact.

7.6 Homological algebra

Definition 7.23 (Chain complex)**.** *By a* complex *in an abelian category* $\mathbf{A}$*, we mean a sequence*

$$
C_\bullet = \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \right)
$$

indexed by $\mathbb{Z}$ *and satisfying* $d_n d_{n+1} = 0$ *for all* n*.*

A morphism of complexes $f_\bullet : C_\bullet \to D_\bullet$ *is a sequence of morphisms* $f_n : C_n \to D_n$ *satisfying* $f_n d_{n+1} = d_{n+1} f_{n+1}$ *for all* n*.*

We write $C\mathbf{A}$ *for the category of complexes in* $\mathbf{A}$*. Note that* $C\mathbf{A}$ *is abelian; in fact,* $C\mathbf{A} \cong \mathbf{Add}(\mathbf{Z}, \mathbf{A})$*, where* $\mathbf{Z}$ *has* $\operatorname{ob}\mathbf{Z} = \mathbb{Z}$ *and* $\mathbf{Z}(m,n) = \mathbb{Z}$ *if* $n \in \{m, m-1\}$*,* $\mathbf{Z}(m,n) = 0$ *otherwise.*

Definition 7.24 (Homology of a complex)**.** *The* homology objects $H_n(C_\bullet)$ *of a complex* $C_\bullet$ *are usually defined as follows: let* $Z_n(C_\bullet) \rightarrowtail C_n$ *be* $\ker d_n$*, let* $I_n(C_\bullet) \rightarrowtail C_n$ *be* $\operatorname{im} d_{n+1}$ *and define* $Z_n(C_\bullet) \twoheadrightarrow H_n(C_\bullet)$ *to be the cokernel of* $I_n(C_\bullet) \rightarrowtail Z_n(C_\bullet)$*.*

In fact, the definition is self-dual. Consider the diagram:

$$
\begin{tikzcd}
C_{n+1} \arrow[rrrr, "d_{n+1}"] \arrow[dr, two heads] & & & & C_n \arrow[rrrr, "d_n"] \arrow[dr, two heads] & & & & C_{n-1} \\
& Q_{n+1} \arrow[dr, two heads] & & Z_n \arrow[ur, tail] \arrow[dr, two heads] & & Q_n \arrow[dr, two heads] & & Z_{n-1} \arrow[ur, tail] & \\
& & I_n \arrow[ur, tail] & & H_n \arrow[ur, tail] & & I_{n-1} \arrow[ur, tail] & &
\end{tikzcd}
$$

Define $(C_n \twoheadrightarrow Q_n) = \operatorname{coker} d_{n+1}$*, then* $(I_n \rightarrowtail C_n) = \ker(C_n \to Q_n)$*, and since* $Z_n \to C_n$ *is monic, we have* $(I_n \to Z_n) = \ker(Z_n \to C_n \to Q_n)$*. Therefore* $(Z_n \to H_n) = \operatorname{coim}(Z_n \to Q_n)$ *and dually* $(H_n \to Q_n) = \operatorname{im}(Z_n \to Q_n)$*.*

Note that Z_n, Q_n, I_n, H_n *are all (additive) functor* $C\mathbf{A} \to \mathbf{A}$*.*

Theorem 7.25 (Mayer-Vietoris)**.** *Suppose given an exact sequence*

$$
0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0
$$

in $C\mathbf{A}$*. Then there is an exact sequence*

$$
\cdots \to H_n(A_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(B_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(C_\bullet) \xrightarrow{\delta_n} H_{n-1}(A_\bullet) \to \cdots .
$$

Proof. First consider:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_n(A_\bullet) \arrow[r] \arrow[d] & Z_n(B_\bullet) \arrow[r] \arrow[d] & Z_n(C_\bullet) \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d_n"'] & B_n \arrow[r] \arrow[d, "d_n"'] & C_n \arrow[r] \arrow[d, "d_n"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & Q_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Q_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Q_{n-1}(C_\bullet) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The top and bottom rows are exact by the Snake Lemma (Lemma 7.22), plus the fact that $Z_n(A_\bullet) \to A_n \to B_n$ is monic. Now consider:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n(A_\bullet) \arrow[r] \arrow[d] & H_n(B_\bullet) \arrow[r] \arrow[d] & H_n(C_\bullet) \arrow[d] \arrow[dddll, red, "\delta_n"] & \\
 & Q_n(A_\bullet) \arrow[r] \arrow[d] & Q_n(B_\bullet) \arrow[r] \arrow[d] & Q_n(C_\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(C_\bullet) \arrow[d] & \\
 & H_{n-1}(A_\bullet) \arrow[r] \arrow[d] & H_{n-1}(B_\bullet) \arrow[r] \arrow[d] & H_{n-1}(C_\bullet) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This gives the required long exact sequence. $\square$

Definition 7.26 (Homotopy)**.** *Suppose given $f_\bullet, g_\bullet : C_\bullet \rightrightarrows D_\bullet$ in $C\mathbf{A}$. By a* homotopy *from $f_\bullet$ to $g_\bullet$ we mean a sequence of morphisms $h_n : C_n \to D_{n+1}$ satisfying $f_n - g_n = h_{n-1}d_n + d_{n+1}h_n$ for all n.*

$$\begin{tikzcd}
 & C_n \arrow[r, "d_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"] \\
D_{n+1} \arrow[r, "d_{n+1}"] & D_n &
\end{tikzcd}$$

We write $f_\bullet \simeq g_\bullet$ if there exists such an $h_\bullet$.

Note that $\simeq$ is an equivalence relation on $\operatorname{mor} C\mathbf{A}$*, and in fact a congruence in the sense of Example 1.3.*(iv)*. We write $H\mathbf{A}$ for the corresponding quotient category of $C\mathbf{A}$. The category $H\mathbf{A}$ is additive since we have*

$$(f_\bullet \simeq g_\bullet \text{ and } k_\bullet \simeq \ell_\bullet) \Longrightarrow f_\bullet + k_\bullet \simeq g_\bullet + \ell_\bullet,$$

but it does not necessarily have kernels and cokernels.

Lemma 7.27. *Homotopic morphisms in $C\mathbf{A}$ induce the same morphisms on homology, i.e. the functors $H_n : C\mathbf{A} \to \mathbf{A}$ factor through $C\mathbf{A} \to H\mathbf{A}$.*

Proof. Suppose that $f_\bullet \simeq g_\bullet$ with a homotopy $h_\bullet$. The composites

$$Z_n(C_\bullet) \rightarrowtail C_n \underset{g_n}{\overset{f_n}{\rightrightarrows}} D_n \twoheadrightarrow Q_n(D_\bullet)$$

differ by the sum of

$$Z_n(C_\bullet) \rightarrowtail C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{h_{n-1}} D_n \twoheadrightarrow Q_n(D_\bullet),$$

and

$$Z_n(C_\bullet) \rightarrowtail C_n \xrightarrow{h_n} D_{n+1} \xrightarrow{d_{n+1}} D_n \twoheadrightarrow Q_n(D_\bullet),$$

but both of these are zero. Since $H_n(f_\bullet)$ and $H_n(g_\bullet)$ are the factorisations of the first two composites through $Z_n(C_\bullet) \twoheadrightarrow H_n(C_\bullet)$ and $H_n(D_\bullet) \rightarrowtail Q_n(D_\bullet)$, they are equal. $\square$

7.7 Projective resolutions

Definition 7.28 (Projective objects)**.** *An object P in $\mathbf{A}$ is called* projective *if $\mathbf{A}(P,-)$ preserves epimorphisms (if $\mathbf{A}$ is abelian, this is equivalent to saying that $\mathbf{A}(P,-)$ is an exact functor $\mathbf{A} \to$ $\mathbf{AbGp}$).*

We say that $\mathbf{A}$ has enough projectives *if, for every $A \in \operatorname{ob}\mathbf{A}$, there exists $P \twoheadrightarrow A$ with P projective.*

Example 7.29. $\mathbf{Mod}_R$ *has enough projectives since free modules are projective.*

Definition 7.30 (Projective resolution)**.** *By a* projective resolution *of an object A of $\mathbf{A}$, we mean a complex $P_\bullet$ such that P_n is projective for all n, $P_n = 0$ if $n < 0$, and*

$$H_n(P_\bullet) = \begin{cases} A & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Equivalently, a projective resolution is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0,$$

with P_n projective for all n.

Remark 7.31. *If $\mathbf{A}$ has enough projectives, then every object has a projective resolution: given A, choose $P_0 \twoheadrightarrow A$ with P_0 projective, let $(Z_0 \rightarrowtail P_0) = \ker(P_0 \to A)$, choose $P_1 \twoheadrightarrow Z_0$ with P_1 projective, let $(Z_1 \rightarrowtail P_1) = \ker(P_1 \to Z_0)$ and so on.*

Lemma 7.32. *Let $P_\bullet, Q_\bullet$ be projective resolutions of A, B respectively. Then*

(i) *Any $A \xrightarrow{f} B$ induces $P_\bullet \xrightarrow{f_\bullet} Q_\bullet$ s.t. $H_0(f_\bullet) = f$.*

(ii) *Any two such morphisms $f_\bullet, g_\bullet : P_\bullet \rightrightarrows Q_\bullet$ are homotopic.*

Hence, if $\mathbf{A}$ is an abelian category with enough projectives, then the construction of projective resolutions defines a functor $\mathrm{PR} : \mathbf{A} \to H\mathbf{A}$.

Proof. (i) The morphism $P_0 \xrightarrow{f_0} Q_0$ exists since P_0 is projective (because the map $Q_0 \twoheadrightarrow B$ is an epimorphism and therefore so is $\mathbf{A}(P_0, Q_0) \twoheadrightarrow \mathbf{A}(P_0, B)$).

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "d_2"] \arrow[d, "f_2"'] \arrow[dr] & & P_1 \arrow[rr, "d_1"] \arrow[d, "f_1"'] \arrow[dr] & & P_0 \arrow[r, two heads] \arrow[d, "f_0"'] & A \arrow[d, "f"'] \\
\cdots \arrow[r] & Q_2 \arrow[r, two heads] & Z_1(Q_\bullet) \arrow[r, tail] & Q_1 \arrow[r, two heads] & Z_0(Q_\bullet) \arrow[r, tail] & Q_0 \arrow[r, two heads] & B
\end{tikzcd}$$

Then the composite $P_1 \to P_0 \to Q_0 \to B$ is zero since $P_1 \to P_0 \to A$ is zero, so we can find $P_1 \to Z_0(Q_\bullet)$ making the above diagram commute. Then f_1 exists by projectivity of P_1, and so on.

(ii) Since $(f_0 - g_0)$ factors through $Z_0(Q_\bullet) \rightarrowtail Q_0$, there exists $P_0 \xrightarrow{h_0} Q_1$ such that $d_1 h_0 = f_0 - g_0$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "d_2"] \arrow[d] & & P_1 \arrow[rr, "d_1"] \arrow[d] \arrow[dll] \arrow[dl] & & P_0 \arrow[r, two heads] \arrow[d, "f_0 - g_0"] \arrow[dll, "h_0"'] \arrow[dl] & A \arrow[d, "0"] \\
\cdots \arrow[r] & Q_2 \arrow[r, two heads] & Z_1(Q_\bullet) \arrow[r, tail] & Q_1 \arrow[r, two heads] & Z_0(Q_\bullet) \arrow[r, tail] & Q_0 \arrow[r, two heads] & B
\end{tikzcd}$$

Now $d_1(f_1 - g_1 - h_0 d_1) = (f_0 - g_0 - d_1 h_0) d_1$, so $(f_1 - g_1 - h_0 d_1)$ factors through $Z_1(Q_\bullet) \to Q_1$, so there exists $P_1 \xrightarrow{h_1} Q_2$ with $d_2 h_1 = f_1 - g_1 - h_0 d_1$, and so on. $\square$

Remark 7.33. *Since the proof of Lemma 7.32 does not use the projectivity of $Q_\bullet$, it shows that* PR *is left adjoint to $H_{0|\mathbf{C}}$, where $\mathbf{C} \subseteq H\mathbf{A}$ is the full subcategory on complexes $C_\bullet$ with $H_n(C_\bullet) = 0$ for $n > 0$.*

7.8 Derived functors

Definition 7.34 (Derived functors)**.** *Let $F : \mathbf{A} \to \mathbf{B}$ be an additive functor between abelian categories, where $\mathbf{A}$ has enough projectives. Note that F induces functors $CF : C\mathbf{A} \to C\mathbf{B}$ and $HF : H\mathbf{A} \to H\mathbf{B}$. We define the n-*th left derived functor $L^n F : \mathbf{A} \to \mathbf{B}$ *to be the composite*

$$\mathbf{A} \xrightarrow{\mathrm{PR}} H\mathbf{A} \xrightarrow{HF} H\mathbf{B} \xrightarrow{H_n} \mathbf{B}$$

for all $n \geqslant 0$.

Remark 7.35. *Note that $L^n F$ is additive; if F is exact, then $L^0 F \cong F$ and $L^n F = 0$ for all $n > 0$. If F is merely right exact, we still have $L^0 F \cong F$ since F preserves the exactness of $P_1 \to P_0 \to A \to 0$.*

Lemma 7.36. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in an abelian category $\mathbf{A}$ with enough projectives. Then we can choose projective resolutions $P_\bullet \to Q_\bullet \to R_\bullet$ so that $Q_n = P_n \oplus R_n$ for all n, and the morphisms $P_n \to Q_n \to R_n$ are*

$$P_n \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} P_n \oplus R_n \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} R_n.$$

Proof. Construct $P_\bullet, R_\bullet$ arbitrarily. Then $P_0 \oplus R_0$ is projective because coproducts of projectives are projective, and $\begin{pmatrix}fu_0 & w_0\end{pmatrix}$ makes the squares in the following diagram commute.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1 \arrow[r, two heads] \arrow[d] & K_0 \arrow[r] \arrow[d] & P_0 \arrow[r, two heads, "u_0"] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & P_1 \oplus R_1 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & P_0 \oplus R_0 \arrow[r, "\begin{pmatrix}fu_0 & w_0\end{pmatrix}"] \arrow[d, "\begin{pmatrix}0 & 1\end{pmatrix}"'] & B \arrow[r] \arrow[d, "g"'] & 0 \\
\cdots \arrow[r] & R_1 \arrow[r, two heads] \arrow[d] \arrow[ur] & M_0 \arrow[r] \arrow[d] & R_0 \arrow[r, two heads, "v_0"'] \arrow[d] \arrow[ur, "w_0"] & C \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 & 0 &
\end{tikzcd}$$

To show it is epic, suppose given $B \xrightarrow{x} X$ such that $x\begin{pmatrix}fu_0 & w_0\end{pmatrix} = 0$. Then $xfu_0 = 0$, and u_0 is epic, so $xf = 0$. It follows that $x = yg$ for some y; but $0 = xw_0 = ygw_0 = yv_0$, so $y = 0$ because v_0 is epic, and hence $x = 0$. Therefore, $\begin{pmatrix}fu_0 & w_0\end{pmatrix}$ is epic. Now form kernels K_0, L_0, M_0 as in the diagram. Then the sequence $0 \to K_0 \to L_0 \to M_0 \to 0$ is exact by the Snake Lemma (Lemma 7.22). Now we can define $P_1 \oplus R_1 \twoheadrightarrow L_0$ as before, and so on. $\square$

Remark 7.37. *Lemma 7.36 does not say that* $Q_\bullet = P_\bullet \oplus Q_\bullet$ *in* $C\mathbf{A}$. *If it were, then the transition maps* $Q_n \to Q_{n-1}$ *would have matrices* $\begin{pmatrix} d_n & 0 \\ 0 & d_n \end{pmatrix}$, *but in fact they have the form* $\begin{pmatrix} d_n & 0 \\ x & d_n \end{pmatrix}$ *with* x *not necessarily* 0.

Theorem 7.38. *Suppose given an additive functor* $F : \mathbf{A} \to \mathbf{B}$ *where* $\mathbf{A}, \mathbf{B}$ *are abelian and* $\mathbf{A}$ *has enough projectives. Then for any short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathbf{A}$, *we get an exact sequence*

$$\cdots \to L^1FA \to L^1FB \to L^1FC \to L^0FA \to L^0FB \to L^0FC \to 0.$$

In particular, L^0F *is right exact.*

Proof. Form projective resolutions of A, B, C as in Lemma 7.36. Since F is additive, it preserves the exactness of the columns $0 \to P_n \to P_n \oplus R_n \to R_n \to 0$ in the diagram of the proof of Lemma 7.36. Therefore, $0 \to FP_\bullet \to FQ_\bullet \to FR_\bullet \to 0$ is a short exact sequence in $C\mathbf{B}$. The result follows by applying the Mayer-Vietoris Theorem (Theorem 7.25) to this sequence. □

References

[1] S. Awodey. *Category Theory.*

[2] F. Borceux. *Handbook of Categorical Algebra.*

[3] T. Leinster. *Basic Category Theory.*

[4] S Mac Lane. *Categories for the Working Mathematician.*

[5] E. Riehl. *Category Theory in Context.*