Riemann Surfaces

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1 Elliptic functions and complex tori

1.1 Lattices and complex tori

Definition 1.1.1 (Lattice). *A* lattice *in $\mathbb{C}$ is an additive subgroup of $\mathbb{C}$ of the form:*

$$\Lambda = \{n_1\omega_1 + n_2\omega_2,\ n_1, n_2 \in \mathbb{Z}\},$$

where (ω_1, ω_2) is a $\mathbb{R}$-basis of $\mathbb{C}$. We then say that (ω_1, ω_2) is a basis *of Λ. We also say that $D = \{t_1\omega_1 + t_2\omega_2,\ t_1, t_2 \in [0, 1]\}$ is a* fundamental domain *for the action of Λ on $\mathbb{C}$ by translation.*

Remark 1.1.2. *A lattice can have several different bases. If (ω_1, ω_2) is a basis of Λ, then the bases of Λ are the couples $(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$.*

Proposition 1.1.3. *An additive subgroup Λ of $\mathbb{C}$ is a lattice* iff *Λ is discrete and the quotient $\mathbb{C}/\Lambda$ is compact.*

Corollary 1.1.4. *If Λ is a lattice in $\mathbb{C}$ and K is a compact subset of $\mathbb{C}$, then $\Lambda \cap K$ is finite.*

Definition 1.1.5 (Complex torus). *A* complex torus *is a space of the form $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$. Topologically, a complex torus is homeomorphic to $(\mathbb{R}/\mathbb{Z})^2$.*

1.2 Elliptic functions

Notation 1.2.1. *In this section, Λ is a lattice in $\mathbb{C}$.*

Definition 1.2.2 (Elliptic function). *An* elliptic function *for Λ is a meromorphic function $f \in \mathcal{M}(\mathbb{C})$ which is Λ-periodic, i.e. for every $z \in \mathbb{C}$ and $\lambda \in \Lambda$, if z and $z + \lambda$ are not poles of f, then $f(z) = f(z + \lambda)$. We write $\mathbb{C}(\Lambda)$ for the set of elliptic functions for Λ.*

Remark 1.2.3. *Let $f \in \mathbb{C}(\Lambda)$. Then the set S of poles of f is stable by translation by Λ, so it defines a subset $\overline{S} \subseteq \mathbb{C}/\Lambda$. Moreover, $\overline{S}$ is discrete and closed in $\mathbb{C}/\Lambda$, which is compact, so $\overline{S}$ is finite.*

Lemma 1.2.4. *$\mathbb{C}(\Lambda)$ is a field.*

Proof. We know that $\mathcal{M}(\mathbb{C})$ is a field, and it is clear that if $f \in \mathbb{C}(\Lambda)$, then $\frac{1}{f}$ is Λ-periodic. □

Definition 1.2.5 (Order of vanishing and residue of an elliptic function at a point). *Let $f \in \mathbb{C}(\Lambda)^\times$ and $p \in \mathbb{C}/\Lambda$. The* order of vanishing *of f at p and the* residue *of f at p are well-defined; we denote them by $\operatorname{ord}_p(f)$ and $\operatorname{Res}_p(f)$ respectively.*

Proposition 1.2.6. *Let $f \in \mathbb{C}(\Lambda)^\times$. We have the following equalities:*

(i) $\sum_{p\in\mathbb{C}/\Lambda} \operatorname{Res}_p(f) = 0$,

(ii) $\sum_{p\in\mathbb{C}/\Lambda} \operatorname{ord}_p(f) = 0$,

(iii) $\sum_{p\in\mathbb{C}/\Lambda} \operatorname{ord}_p(f) \cdot p = 0$.

Proof. (i) Let (ω_1, ω_2) be a basis of Λ and let D be the associated fundamental domain. We choose $a \in \mathbb{C}$ s.t. $D' = D + a$ contains no pole or zero of f on its boundary $\partial D'$. Applying the Residue Theorem to f, we obtain:

$$\sum_{p \in \mathbb{C}/\Lambda} \mathrm{Res}_p(f) = \int_{\partial D'} f(z) \, \mathrm{d}z.$$

But using the Λ-periodicity of f, we have $\int_{\partial D'} f(z) \, \mathrm{d}z = 0$. (ii) Apply (i) to $g = \frac{f'}{f}$. (iii) Set $h(z) = z\frac{f'(z)}{f(z)}$ (note that h is not an elliptic function). Apply the Residue Theorem to h on $\partial D'$, and note that:

$$\int_{a+\omega_1}^{a+\omega_1+\omega_2} h(z) \, \mathrm{d}z = \int_a^{a+\omega_2} h(z) \, \mathrm{d}z + \omega_1 \int_a^{a+\omega_2} \frac{f'(u)}{f(u)} \, \mathrm{d}u.$$

Now, use the change of variable $v = f(u)$ and note that $\gamma : u \in [a, a+\omega_2] \longmapsto f(u) \in \mathbb{C}^\times$ is a closed $\mathcal{C}^1$ path. By the Residue Theorem, $\int_a^{a+\omega_2} \frac{f'(u)}{f(u)} \, \mathrm{d}u = \int_\gamma \frac{\mathrm{d}v}{v} \in 2i\pi\mathbb{Z}$. Therefore, $\int_{a+\omega_1}^{a+\omega_1+\omega_2} h(z) \, \mathrm{d}z - \int_a^{a+\omega_2} h(z) \, \mathrm{d}z \in 2i\pi\Lambda$. We obtain $\int_{\partial D'} h(z) \, \mathrm{d}z \in 2i\pi\Lambda$, which gives the desired result. □

Remark 1.2.7. *Note that, in Proposition 1.2.6,* (iii) *is an equality in the additive group* $\mathbb{C}/\Lambda$.

Corollary 1.2.8. *A nonconstant elliptic function has at least two poles (counted with multiplicity).*

1.3 The Weierstraß ℘-function

Lemma 1.3.1. *The sum* $\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3}$ *converges.*

Proof. Use the fact that there exists $c_\Lambda > 0$ s.t.

$$\forall N \in \mathbb{N}^*, \ |\{\lambda \in \Lambda, \ N \leq |\lambda| < N+1\}| \leq c_\Lambda N.$$

□

Definition 1.3.2 (Weierstraß ℘-function)**.** *The* Weierstraß ℘-function *is defined on* $\mathbb{C}\backslash\Lambda$ *by:*

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Proposition 1.3.3. $\wp$ *is an even elliptic function.*

Proof. Show that $\wp$ is defined by a series of holomorphic functions which converges uniformly on every compact subset of $\mathbb{C}\backslash\Lambda$. □

Remark 1.3.4. *In* $\mathbb{C}/\Lambda$, $\wp$ *has only one pole (at* 0*); its order of vanishing is* (-2).

Lemma 1.3.5. *For every* $z_0 \in \mathbb{C}\backslash\Lambda$, *the elliptic function* $(\wp - \wp(z_0))$ *has a double pole at* $z = 0$ *and simple zeroes at* $z = \pm z_0$.

Proof. By Remark 1.3.4, $(\wp - \wp(z_0))$ has a double pole at $z = 0$ and is holomorphic everywhere else. It is also clear that $\wp$ has zeroes at $\pm z_0$. Now, Proposition 1.2.6 guarantees that $\wp$ has no other zero. □

1.4 Principal divisors

Definition 1.4.1 (Divisors)**.** *We define* $\mathbb{Z}[\mathbb{C}/\Lambda]$ *to be the free* $\mathbb{Z}$*-module with basis* $\mathbb{C}/\Lambda$. *Its elements will be denoted by* $\sum_{i=1}^r n_i [p_i]$, *with* $n_i \in \mathbb{Z}$ *and* $p_i \in \mathbb{C}/\Lambda$; *they will be called* divisors. *Note that divisors are formal sums, not elements of* $\mathbb{C}/\Lambda$.

Definition 1.4.2 (Divisor of an elliptic function)**.** *Given* $f \in \mathbb{C}(\Lambda)^\times$, *we define the* divisor of f *by:*

$$\mathrm{div}(f) = \sum_{p \in \mathbb{C}/\Lambda} \mathrm{ord}_p(f) \cdot [p] \in \mathbb{Z}[\mathbb{C}/\Lambda].$$

Example 1.4.3. *Lemma 1.3.5 can be restated more concisely in the following way:*

$$\operatorname{div}\left(\wp - \wp\left(z_0\right)\right) = \left[z_0\right] + \left[-z_0\right] - 2\left[0\right].$$

Proposition 1.4.4. *The map* $\operatorname{div} : \mathbb{C}\left(\Lambda\right)^{\times} \to \mathbb{Z}\left[\mathbb{C}/\Lambda\right]$ *is a group homomorphism. Elements of* $\operatorname{Im}\operatorname{div}$ *will be called* principal divisors*.*

Definition 1.4.5 (Degree of a divisor)**.** *We define a group homomorphism* $\deg : \mathbb{Z}\left[\mathbb{C}/\Lambda\right] \to \mathbb{Z}$ *by* $\deg\left(\sum_{p\in\mathbb{C}/\Lambda} n_p[p]\right) = \sum_{p\in\mathbb{C}/\Lambda} n_p$*. Moreover, we define the* group of degree 0 divisors *by:*

$$I_\Lambda = \operatorname{Ker}\deg \subseteq \mathbb{Z}\left[\mathbb{C}/\Lambda\right].$$

Remark 1.4.6. $\mathbb{Z}\left[\mathbb{C}/\Lambda\right]$ *can be equipped with the structure of a commutative ring by seeing it as the group algebra of* $\mathbb{C}/\Lambda$*. In other words, we set:*

$$\left(\sum_{i=1}^{m} a_i\left[p_i\right]\right)\left(\sum_{j=1}^{n} b_j\left[q_j\right]\right) = \sum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} a_i b_j\left[p_i + q_j\right].$$

With this ring structure, I_Λ *is in fact an ideal of* $\mathbb{Z}\left[\mathbb{C}/\Lambda\right]$*, called the* augmentation ideal*.*

Theorem 1.4.7. *Let* $D = \sum_{p\in\mathbb{C}/\Lambda} n_p[p] \in \mathbb{Z}\left[\mathbb{C}/\Lambda\right]$*. Then* $D \in \operatorname{Im}\operatorname{div}$ *if and only if* $\deg D = 0$ *and* $\sum_{p\in\mathbb{C}/\Lambda} n_p \cdot p = 0$*.*

Proof. Note that ($\Rightarrow$) was proved in Proposition 1.2.6, so it suffices to prove ($\Leftarrow$). *First step.* Let $D \in \mathbb{Z}\left[\mathbb{C}/\Lambda\right]$ s.t. $\deg D = 0$ and $\sum_{p\in\mathbb{C}/\Lambda} n_p \cdot p = 0$. We shall prove that $D \in I_\Lambda^2$. Consider the map:

$$\varphi : p \in \mathbb{C}/\Lambda \longmapsto [[p] - [0]] \in I_\Lambda/I_\Lambda^2.$$

Then φ is a group homomorphism. And:

$$D = \sum_{p\in\mathbb{C}/\Lambda} n_p \cdot [p] = \sum_{p\in\mathbb{C}/\Lambda} n_p\left([p] - [0]\right) = \sum_{p\in\mathbb{C}/\Lambda} n_p\varphi(p) = \varphi\left(\sum_{p\in\mathbb{C}/\Lambda} n_p \cdot p\right) \equiv 0 \mod I_\Lambda^2.$$

This shows that $D \in I_\Lambda^2$. *Second step.* We shall show that $I_\Lambda^2 \subseteq \operatorname{Im}\operatorname{div}$. Note that I_Λ is generated by the divisors $([p] - [0])$ with $p \in \mathbb{C}/\Lambda$; therefore, I_Λ^2 is generated (as an abelian group) by the divisors $D_{p,q} = \left([p] - [0]\right)\left([q] - [0]\right)$ for $p, q \in \mathbb{C}/\Lambda$. Hence, it is enough to show that each $D_{p,q}$ is in $\operatorname{Im}\operatorname{div}$. This is obvious if $p = 0$ or $q = 0$. Otherwise, we choose $r \in \mathbb{C}/\Lambda$ s.t. $2r = q$ and $p + r \neq 0$, and we set:

$$f(z) = \frac{\wp\left(z - r\right) - \wp\left(p + r\right)}{\left(\wp(z) - \wp(p)\right)\left(\wp\left(z - r\right) - \wp(r)\right)}.$$

We check that $f \in \mathbb{C}(\Lambda)^{\times}$ and that $D_{p,q} = \operatorname{div}(f)$. □

1.5 Abel-Jacobi Theorem

Definition 1.5.1 (Picard group)**.** *The* Picard group $\operatorname{Pic}\left(\mathbb{C}/\Lambda\right)$ *is defined by:*

$$\operatorname{Pic}\left(\mathbb{C}/\Lambda\right) = \mathbb{Z}\left[\mathbb{C}/\Lambda\right] / \operatorname{Im}\operatorname{div}.$$

Moreover, we define $\operatorname{Pic}^0\left(\mathbb{C}/\Lambda\right) = I_\Lambda/\operatorname{Im}\operatorname{div}$*. This is the analogue of the ideal class group of a number field.*

Theorem 1.5.2 (Abel-Jacobi)**.** *The map:*

$$\left|\begin{aligned} \mathbb{C}/\Lambda &\longrightarrow \operatorname{Pic}^0\left(\mathbb{C}/\Lambda\right) \\ p &\longmapsto [[p] - [0]] \end{aligned}\right.$$

is a group isomorphism.

Proof. Define $\psi : I_\Lambda/I_\Lambda^2 \to \mathbb{C}/\Lambda$ by $\psi\left(\left[\sum_{p\in\mathbb{C}/\Lambda} n_p \cdot [p]\right]\right) = \sum_{p\in\mathbb{C}/\Lambda} n_p \cdot p$ and check that ψ is the inverse of the map φ defined in the proof of Theorem 1.4.7. □

1.6 Structure of Riemann surface on the complex torus

Definition 1.6.1 (Holomorphic functions on the complex torus)**.** *Let $\mathcal{U}$ be an open subset of $\mathbb{C}/\Lambda$ and denote by $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ the canonical projection. Consider the open subset $\widetilde{\mathcal{U}} = \pi^{-1}(\mathcal{U})$ of $\mathbb{C}$; it is stable by translation by Λ. Now define:*

$$\mathcal{O}(\mathcal{U}) = \left\{ f : \mathcal{U} \to \mathbb{C},\ (f \circ \pi) \text{ is holomorphic on } \widetilde{\mathcal{U}} \right\}.$$

$\mathcal{O}(\mathcal{U})$ is a $\mathbb{C}$-algebra that is isomorphic to the algebra of Λ-periodic holomorphic functions on $\widetilde{\mathcal{U}}$. Elements of $\mathcal{O}(\mathcal{U})$ are called holomorphic functions *on $\mathcal{U}$.*

Lemma 1.6.2. *If G is a discrete group acting by homeomorphisms on a topological space X, then the projection map $\pi : X \to X/G$ is continuous (by definition of the quotient topology) and open.*

Proposition 1.6.3. *Every point $p \in \mathbb{C}/\Lambda$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{C}$.*

Proof. Using the discreteness of Λ, choose an open neighbourhood W of 0 in $\mathbb{C}$ s.t. $W \cap \Lambda = \{0\}$. Write $p_0 = [z_0]$, with $z_0 \in \mathbb{C}$ and consider $\widetilde{V} = z_0 + W$ and $V = \pi\left(\widetilde{V}\right)$, where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the canonical projection. By Lemma 1.6.2, V is an open subset of $\mathbb{C}/\Lambda$. Now the map $\pi_{|\widetilde{V}} : \widetilde{V} \to V$ is continuous, bijective and open, so it is a homeomorphism. □

Proposition 1.6.4. *Let $p \in \mathbb{C}/\Lambda$. Let $\phi : \widetilde{V} \subseteq \mathbb{C} \longrightarrow V \subseteq \mathbb{C}/\Lambda$ be the homeomorphism between an open subset of $\mathbb{C}$ and an open neighbourhood of p in $\mathbb{C}/\Lambda$ given by Proposition 1.6.3. Then ϕ sends the holomorphic functions on $\widetilde{V}$ to the holomorphic functions on V.*

2 Riemann surfaces and holomorphic maps

2.1 Motivation

Example 2.1.1.

(i) Algebraic curves. *Let $P \in \mathbb{C}[X,Y]$ be an irreducible polynomial. Consider the set $C_P = \{(x,y) \in \mathbb{C}^2,\ P(x,y) = 0\}$; C_P is called the* algebraic curve *defined by P. Assume that C_P is nonsingular, i.e. $\forall (x,y) \in C_P$, $\left(\frac{\partial P}{\partial x}(x,y), \frac{\partial P}{\partial y}(x,y)\right) \neq (0,0)$. Then C_P is a Riemann surface. Furthermore, there exists an integer $g \in \mathbb{N}$, called the* genus *of C_P s.t. C_P is isomorphic (as a Riemann surface) to the surface of genus g (i.e. with g holes), minus a finite set of points.*

(ii) Hyperbolic geometry. *Consider the half-plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$. Then we have an action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. Moreover, the Riemann surface $\mathbb{H}/SL_2(\mathbb{Z})$ is isomorphic to $\mathbb{C}$. More generally, if Γ is a subgroup of $SL_2(\mathbb{Z})$ with finite index, then $\mathbb{H}/\Gamma$ is a Riemann surface called a* modular curve.

(iii) Power series. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with positive radius of convergence. Then f defines a holomorphic function on an open ball $B(0,R)$, with $R > 0$. And Riemann proved that there exists a largest Riemann surface X_f s.t. f extends to a holomorphic function on X_f. For instance, if $f = \log : B(1,1) \to \mathbb{C}$, then X_f is a covering of $\mathbb{C}^\times$, that is isomorphic to $\mathbb{C}$, and the isomorphism is given by the exponential map.*

2.2 Definition of Riemann surfaces

Definition 2.2.1 (Chart)**.** *Let X be a topological space. A* (holomorphic) chart *on X is the data of an open subset $\mathcal{U} \subseteq X$ and of a homeomorphism $\phi : \mathcal{U} \to V$, where V is an open subset of $\mathbb{C}$. We say that two holomorphic charts $\phi : \mathcal{U} \to V$ and $\phi' : \mathcal{U}' \to V'$ are* compatible *if the map ψ defined by the following commutative diagram is holomorphic:*

$$\begin{array}{ccc} & \mathcal{U}\cap\mathcal{U}' & \\ {}^{\phi}\swarrow & & \searrow^{\phi'} \\ \phi\left(\mathcal{U}\cap\mathcal{U}'\right) & \xrightarrow{\ \psi\ } & \phi'\left(\mathcal{U}\cap\mathcal{U}'\right) \end{array}$$

Definition 2.2.2 (Atlas)**.** *Let X be a topological space. A* (holomorphic) atlas *on X is a collection $\left(\phi_i : \mathcal{U}_i \to V_i\right)_{i\in I}$ of holomorphic charts s.t. $X = \bigcup_{i\in I} \mathcal{U}_i$ and the charts are pairwise compatible. Two atlases $\mathcal{A} = \left(\phi_i\right)_{i\in I}$ and $\mathcal{A}' = \left(\phi'_j\right)_{j\in J}$ on X are said to be* equivalent *if ϕ_i is compatible with ϕ'_j for all $(i,j) \in I \times J$. This defines an equivalence relation on the set of atlases on X.*

Definition 2.2.3 (Riemann surface)**.** *A* Riemann surface *is a nonempty Hausdorff topological space equipped with an atlas (or with an equivalence class of atlases).*

Definition 2.2.4 (Holomorphic functions on a Riemann surface)**.** *Let X be a Riemann surface equipped with an atlas $\mathcal{A} = \left(\phi_i : \mathcal{U}_i \to V_i\right)_{i\in I}$. Let $\mathcal{U}$ be an open subset of X. A function $f : \mathcal{U} \to \mathbb{C}$ is said to be* holomorphic *if for every $i \in I$, the map f_i defined by the following commutative diagram is holomorphic:*

$$\begin{array}{ccc} \mathcal{U}\cap\mathcal{U}_i & \overset{f}{\searrow} & \\ {\scriptstyle\phi_i}\big\downarrow & & \mathbb{C} \\ \phi_i\left(\mathcal{U}\cap\mathcal{U}_i\right) & \underset{f_i}{\nearrow} & \end{array}$$

This notion of holomorphic functions does not change when $\mathcal{A}$ is replaced by an equivalent atlas $\mathcal{A}'$. We denote by $\mathcal{O}_X(\mathcal{U})$ the $\mathbb{C}$-algebra of holomorphic functions on $\mathcal{U}$.

Remark 2.2.5. *Let X be a Riemann surface. Then the $\mathbb{C}$-algebras $\mathcal{O}_X(\mathcal{U})$, for $\mathcal{U} \subseteq X$ open, have the following properties:*

(i) *If $\mathcal{U}' \subseteq \mathcal{U} \subseteq X$ are open subsets, then there is a restriction map $\mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_X(\mathcal{U}')$.*

(ii) *The restriction maps satisfy the* gluing condition*: if $\mathcal{U} \subseteq X$ is an open subset and $\left(\mathcal{U}_i\right)_{i\in I}$ is an open covering of $\mathcal{U}$, and $\left(f_i\right)_{i\in I} \in \prod_{i\in I} \mathcal{O}_X(\mathcal{U}_i)$ is a collection of holomorphic functions s.t. $f_{i|\mathcal{U}_i\cap\mathcal{U}_j} = f_{j|\mathcal{U}_i\cap\mathcal{U}_j}$ for all $i,j \in I$, then there exists a unique $f \in \mathcal{O}_X(\mathcal{U})$ s.t. $\forall i \in I,\ f_i = f_{|\mathcal{U}_i}$.*

This point of view gives rise to an alternative definition of Riemann surfaces: they are topological spaces equipped with a collection $\left(\mathcal{O}_X(\mathcal{U})\right)_{\substack{\mathcal{U}\subseteq X \\ \mathcal{U}\ open}}$ satisfying some conditions. $\mathcal{O}_X$ is called a sheaf.

Example 2.2.6. *Every open subset of $\mathbb{C}$ is a Riemann surface (with a single chart).*

Example 2.2.7 (Riemann sphere)**.** *The* Riemann sphere *is the space $\mathbb{P}^1(\mathbb{C}) = \left(\mathbb{C}^2\backslash\{0\}\right)/\mathbb{C}^\times$, endowed with the quotient topology. It is a compact topological space (it is actually the one-point compactification of $\mathbb{C}$). We define two holomorphic charts on $\mathbb{P}^1(\mathbb{C})$ by:*

$$\psi_0 : z \in \mathbb{C} \longmapsto (1:z) \in \mathbb{P}^1(\mathbb{C})\backslash\{(0:1)\},$$
$$\psi_1 : z \in \mathbb{C} \longmapsto (z:1) \in \mathbb{P}^1(\mathbb{C})\backslash\{(1:0)\}.$$

These charts cover $\mathbb{P}^1(\mathbb{C})$ and are compatible, so they give $\mathbb{P}^1(\mathbb{C})$ the structure of a Riemann sphere.

Proposition 2.2.8. *Every holomorphic function on $\mathbb{P}^1(\mathbb{C})$ is constant. Hence, $\mathcal{O}\left(\mathbb{P}^1(\mathbb{C})\right) = \mathbb{C}$.*

Proof. Let $f \in \mathcal{O}\left(\mathbb{P}^1(\mathbb{C})\right)$. If ψ_0 is the chart defined above, then $f \circ \psi_0$ is a holomorphic function defined on $\mathbb{C}$. Moreover, f is continuous (because f is holomorphic) on the compact space $\mathbb{P}^1(\mathbb{C})$, so f is bounded. As a consequence, $f \circ \psi_0$ is an entire function that is bounded, so $f \circ \psi_0$ is constant. Hence, f is constant on $\mathbb{P}^1(\mathbb{C})\backslash\{(0:1)\}$, so f is constant on $\mathbb{P}^1(\mathbb{C})$. $\square$

Remark 2.2.9. *From now on, the map ψ_0 will be used to identify $\mathbb{C}$ with the corresponding subset of $\mathbb{P}^1(\mathbb{C})$, and the point $(0:1)$ will be denoted by ∞. Hence, $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, $\psi_0 : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ is the inclusion and $\psi_1 : \mathbb{C} \to \mathbb{C}^\times \cup \{\infty\}$ is the map given by $z \mapsto \frac{1}{z}$.*

Example 2.2.10 (Complex tori)**.** *If Λ is a lattice in $\mathbb{C}$, then the* complex torus $\mathbb{C}/\Lambda$ *is a Riemann surface, with the structure defined in Section 1.6.*

2.3 Holomorphic maps between Riemann surfaces

Definition 2.3.1 (Holomorphic map). *Let X and Y be two Riemann surfaces. A continuous map $f : X \to Y$ is said to be* holomorphic *if for every open subset $V \subseteq Y$ and for every holomorphic map $h : V \to \mathbb{C}$, $h \circ f : f^{-1}(V) \to \mathbb{C}$ is holomorphic. In other words:*

$$\forall V \subseteq Y \text{ open}, \forall h \in \mathcal{O}_Y(V),\ (h \circ f) \in \mathcal{O}_X\left(f^{-1}(V)\right).$$

Remark 2.3.2. *If $Y = \mathbb{C}$, then a holomorphic map $X \to Y$ is simply a holomorphic function $X \to \mathbb{C}$.*

Proposition 2.3.3. *Let $f : X \to Y$ be a continuous map between Riemann surfaces.*

(i) *The property of being holomorphic is local on the source: for any open cover $X = \bigcup_{i\in I} \mathcal{U}_i$, $f : X \to Y$ is holomorphic* iff *$f_{|\mathcal{U}_i} : \mathcal{U}_i \to Y$ is holomorphic for all $i \in I$.*

(ii) *The property of being holomorphic is local on the target: for any open cover $Y = \bigcup_{j\in J} \mathcal{V}_j$, $f : X \to Y$ is holomorphic* iff *$f_{|f^{-1}(\mathcal{V}_j)} : f^{-1}(\mathcal{V}_j) \to \mathcal{V}_j$ is holomorphic for all $j \in J$.*

Proposition 2.3.4. *If $f : X \to Y$ and $g : Y \to Z$ are holomorphic maps between Riemann surfaces, then $g \circ f : X \to Z$ is also holomorphic.*

Example 2.3.5. *Let Λ be a lattice in $\mathbb{C}$. Then the projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is holomorphic.*

Definition 2.3.6 (Biholomorphism). *Let X and Y be two Riemann surfaces. A map $f : X \to Y$ is said to be an* isomorphism (of Riemann surfaces) *or a* biholomorphism *if f is holomorphic, bijective, and f^{-1} is holomorphic.*

2.4 Generalisation to Riemann surfaces of standard results of complex analysis

Theorem 2.4.1 (Identity Theorem). *Let X and Y be two Riemann surfaces. Assume that X is connected and consider two holomorphic maps $f, g : X \to Y$.*

(i) *If $f \neq g$, then the set $\{p \in X,\ f(p) = g(p)\}$ is a closed discrete subset of X.*

(ii) *If the set $\{p \in X,\ f(p) = g(p)\}$ has a limit point, then $f = g$.*

In particular, if f and g coincide on some nonempty open subset of X, then $f = g$.

Proof. Assume that $f \neq g$. It is clear that $A_{f,g} = \{p \in X,\ f(p) = g(p)\}$ is closed. Let us show that $A_{f,g}$ is discrete. Let $x_0 \in A_{f,g}$ and let $y_0 = f(x_0) = g(x_0)$. Consider charts $\mathcal{U}_0$ of X around x_0, $\mathcal{V}_0$ of Y around y_0. We may assume that $f(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and $g(\mathcal{U}_0) \subseteq \mathcal{V}_0$ by shrinking $\mathcal{U}_0$ if necessary. Thus, we have two maps $f, g : \mathcal{U}_0 \to \mathcal{V}_0$ with $f(x_0) = g(x_0)$. By reading these maps in the charts, we may assume that $\mathcal{U}_0$ and $\mathcal{V}_0$ are subsets of $\mathbb{C}$. We now set $h = (f - g) : \mathcal{U}_0 \to \mathbb{C}$. Since $h(x_0) = 0$ and h is holomorphic, either h does not vanish in a punctured neighbourhood of x_0, or $h = 0$ in a neighbourhood of x_0. In the first case, x_0 is isolated in $A_{f,g}$ and we are done. In the second case, consider the set:

$$\Omega = \left\{p \in X,\ \exists W \text{ open neighbourhood of } p \text{ in } X,\ f_{|W} = g_{|W}\right\}.$$

As $x_0 \in \Omega$, $\Omega \neq \varnothing$. It is clear that Ω is open in X. Moreover, if there existed $p \in \overline{\Omega}\backslash\Omega$, then $p \in \overline{A}_{f,g} = A_{f,g}$, so $f(p) = g(p)$. And p is not isolated in $A_{f,g}$ because $p \in \overline{\Omega}\backslash\Omega$. By the same reasoning as before, we deduce that $f = g$ in a neighbourhood of p, so $p \in \Omega$, which is a contradiction. Therefore, Ω is open and closed in the connected space X, so $X = \Omega$, a contradiction. □

Corollary 2.4.2 (Discreteness of the fibres)**.** *Let $f : X \to Y$ be a holomorphic map between two Riemann surfaces, with X connected. If f is not constant, then for every $q \in Y$, the fibre $f^{-1}(\{q\})$ is a closed discrete subset of X. In particular, if X is compact, then the fibre $f^{-1}(\{q\})$ is finite.*

Theorem 2.4.3 (Oppen Mapping Theorem)**.** *Let $f : X \to Y$ be a holomorphic map between two Riemann surfaces, with X connected. If f is not constant, then f is open.*

Corollary 2.4.4. *Let $f : X \to Y$ be a holomorphic map between two Riemann surfaces, with X compact and connected and Y connected. If f is not constant, then f is surjective (and Y is compact).*

Corollary 2.4.5. *If X is a compact connected Riemann surface, then every holomorphic function on X is constant: $\mathcal{O}(X) = \mathbb{C}$.*

3 Meromorphic functions

3.1 Meromorphic functions

Definition 3.1.1 (Meromorphic function at a point)**.** *Let X be a Riemann surface. Let Ω be an open subset of X containing a point p. Let $f : \Omega\backslash\{p\} \to \mathbb{C}$ be a holomorphic function. We say that f is* meromorphic at p *(resp. has an* essential singularity at p*) if for every holomorphic chart $(\mathcal{U}, \phi)$ of X around p, the map $g = f \circ \phi^{-1} : \phi(\mathcal{U} \cap \Omega) \setminus \{\phi(p)\} \to \mathbb{C}$ is meromorphic at $\phi(p)$ (resp. has an essential singularity at $\phi(p)$). Moreover, f extends to a holomorphic function on Ω* iff *g extends to a holomorphic function on a neighbourhood of $\phi(p)$ for every holomorphic chart $(\mathcal{U}, \phi)$.*

Definition 3.1.2 (Meromorphic function)**.** *Let X be a Riemann surface. A* meromorphic function *on X is the data of a closed discrete subset S of X and of a holomorphic function $f : X\backslash S \to \mathbb{C}$, s.t. f is meromorphic at each point of S. In this case, we write $f : X \dashrightarrow \mathbb{C}$. We identify two meromorphic functions $f, g : X \dashrightarrow \mathbb{C}$ if there exists a closed discrete subset S of X s.t. f and g are defined and coincide on $X\backslash S$; in this case, we write $f \sim g$. We define $\mathcal{M}(X)$ to be the set of meromorphic functions on X, quotiented by the equivalence relation $\sim$.*

Proposition 3.1.3. *Let X be a Riemann surface. Let Ω be an open subset of X containing a point p. Let $f : \Omega\backslash\{p\} \to \mathbb{C}$ be a holomorphic function. Then f is meromorphic at p* iff *f extends to a holomorphic map $\hat{f} : \Omega \to \mathbb{P}^1(\mathbb{C})$, i.e. such that the following diagram commutes, with the notations of Example 2.2.7:*

$$\begin{array}{ccc} \Omega\backslash\{p\} & \xrightarrow{\quad f \quad} & \mathbb{C} \\ \subseteq\big\downarrow & & \big\downarrow \psi_0 \\ \Omega & \overset{\hat{f}}{\dashrightarrow} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

Corollary 3.1.4. *If X is a Riemann surface, then any meromorphic function $f : X \dashrightarrow \mathbb{C}$ can be extended to a holomorphic map $\hat{f} : X \to \mathbb{P}^1(\mathbb{C})$.*

Definition 3.1.5 (Order of vanishing)**.** *Let X be a Riemann surface. Let Ω be an open subset of X containing a point p. Let $f : \Omega\backslash\{p\} \to \mathbb{C}$ be a holomorphic function that is meromorphic at p. If $\phi : \mathcal{U} \to V$ is a holomorphic chart of X containing p, with $\phi(p) = z_0$, then $f \circ \phi^{-1}$ has a Laurent expansion at z_0: $f \circ \phi^{-1}(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n$ around z_0, for some $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$. We define the* order of vanishing *of f at p by:*

$$\mathrm{ord}_p(f) = \min\{n \in \mathbb{Z},\ a_n \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

The order of vanishing does not depend on the choice of ϕ, because the transition maps are biholomorphic. Moreover:

(i) *We say that* f *has a* zero *at* p *if* $\mathrm{ord}_p(f) > 0$*; in this case, the* order of the zero *is* $\mathrm{ord}_p(f)$*.*

(ii) *We say that* f *has a* pole *at* p *if* $\mathrm{ord}_p(f) < 0$*; in this case, the* order of the pole *is* $|\mathrm{ord}_p(f)|$*.*

Example 3.1.6.

(i) *Let* Λ *be a lattice in* $\mathbb{C}$*. Then the Weierstraß* $\wp$*-function is a meromorphic function on* $\mathbb{C}/\Lambda$ *with a pole of order* 2 *at* 0*, i.e.* $\mathrm{ord}_0(\wp) = -2$*.*

(ii) *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire function. Viewing* $\mathbb{C}$ *as a subset of* $\mathbb{P}^1(\mathbb{C})$*,* f *is meromorphic at* ∞ iff f *is a polynomial.*

Proposition 3.1.7. *If* X *is a connected Riemann surface, then* $\mathcal{M}(X)$ *is a field.*

Proposition 3.1.8. *Let* X *be a connected Riemann surface,* $p \in X$*. We have a function* $\mathrm{ord}_p : \mathcal{M}(X) \to \mathbb{Z} \cup \{\infty\}$*, which has the following properties:*

(i) *For* $f \in \mathcal{M}(X)$*,* $\mathrm{ord}_p(f) = +\infty \iff f = 0$*.*

(ii) *For* $f, g \in \mathcal{M}(X)$*,* $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$*.*

(iii) *For* $f, g \in \mathcal{M}(X)$*,* $\mathrm{ord}_p(f+g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$*.*

We say that ord_p *is a* discrete valuation *on* $\mathcal{M}(X)$*.*

Theorem 3.1.9. *If* X *is a compact connected Riemann surface, then there exists* $f \in \mathcal{M}(X)\backslash\mathbb{C}$ *s.t.* $\mathcal{M}(X)$ *is a finite extension of* $\mathbb{C}(f)$*. We say that the field extension* $\mathcal{M}(X)/\mathbb{C}$ *has transcendance degree* 1*.*

Example 3.1.10.

(i) $\mathcal{M}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}(z)$*.*

(ii) *If* Λ *is a lattice in* $\mathbb{C}$*, then* $\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp')$*, and* $\wp'$ *is algebraic over* $\mathbb{C}(\wp)$*.*

Proposition 3.1.11. *Let* X *be a connected Riemann surface. For* $f \in \mathcal{M}(X)$*, denote by* $\hat{f}$ *the induced holomorphic map* $X \to \mathbb{P}^1(\mathbb{C})$ *(c.f. Corollary 3.1.4). Then the map* $f \mapsto \hat{f}$ *induces a bijection between* $\mathcal{M}(X)$ *and* $\{g : X \to \mathbb{P}^1(\mathbb{C})$ *holomorphic,* $g \neq \infty\}$*.*

Remark 3.1.12. *Let* X *be a Riemann surface. Consider a meromorphic function* $f : X \dashrightarrow \mathbb{C}$ *and denote by* $\hat{f}$ *the induced holomorphic map* $X \to \mathbb{P}^1(\mathbb{C})$*. Then, for* $p \in X$*,* f *has a pole at* p iff $\hat{f}(p) = \infty$*.*

3.2 Ramification theory

Definition 3.2.1 (Ramification index of a holomorphic map)**.** *Let* $f : X \to Y$ *be a holomorphic map between two Riemann surfaces, let* $p \in X$ *and set* $q = f(p) \in Y$*. Assume that* f *is not constant near* p*. Consider local coordinates* $(\mathcal{U}, \phi)$ *of* X *near* p *(i.e. a holomorphic chart with* $\phi(p) = 0$*) and* $(\mathcal{V}, \psi)$ *of* Y *near* q*. In a neighbourhood of* 0*, we can write* $\psi \circ f \circ \phi^{-1}(z) = \sum_{n \in \mathbb{N}^*} a_n z^n$*. We define the* ramification index *of* f *at* p *by:*

$$e_f(p) = \min\{n \in \mathbb{N}^*,\ a_n \neq 0\} \in \mathbb{N}^*.$$

This definition does not depend on the choice of local coordinates on X *and* Y*. Moreover, using the local normal form of a holomorphic function, one can show that for every choice of* $(\mathcal{V}, \psi)$*, there exists a choice of* $(\mathcal{U}, \phi)$ *s.t.* $\psi \circ f \circ \phi^{-1}(z) = z^e$ *in a neighbourhood of* 0*, where* $e = e_f(p)$*.*

Example 3.2.2.

(i) *Consider the map* $f : z \in \mathbb{P}^1(\mathbb{C}) \longmapsto z^2 \in \mathbb{P}^1(\mathbb{C})$. *Then* $e_f(z) = 1$ *if* $z \in \mathbb{C}^\times$ *and* $e_f(0) = e_f(\infty) = 2$.

(ii) *Consider* $\cos : \mathbb{C} \to \mathbb{C}$. *Then* $e_{\cos}(z_0) = 1$ *if* $z_0 \notin \pi\mathbb{Z}$ *and* $e_{\cos}(z_0) = 2$ *if* $z_0 \in \pi\mathbb{Z}$.

Proposition 3.2.3. *Let* $f : X \dashrightarrow \mathbb{C}$ *be a meromorphic function on a Riemann surface* X *and let* $\hat{f} : X \to \mathbb{P}^1(\mathbb{C})$ *be the induced holomorphic map. For* $p \in X$, *we have:*

(i) *If* f *is holomorphic at* p, *then* $e_{\hat{f}}(p) = \operatorname{ord}_p(f - f(p))$.

(ii) *If* f *has a pole at* p, *then* $e_{\hat{f}}(p) = |\operatorname{ord}_p(f)|$.

Corollary 3.2.4. *If* Λ *is a lattice in* $\mathbb{C}$, *then the complex torus* $\mathbb{C}/\Lambda$ *is not isomorphic to the Riemann sphere* $\mathbb{P}^1(\mathbb{C})$.

Proof. Assume for contradiction that $f : \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ is an isomorphism. We may view f as a meromorphic function on $\mathbb{C}/\Lambda$ with only one pole at some point $p \in \mathbb{C}/\Lambda$. By Proposition 3.2.3, the pole of f at p is simple. Therefore, $\operatorname{div}(f)$ must be of the form $[q] - [p]$ for some $q \in \mathbb{C}/\Lambda$. But by Proposition 1.2.6, we must have $q = p$, which is a contradiction. □

Remark 3.2.5. *In fact,* $\mathbb{C}/\Lambda$ *and* $\mathbb{P}^1(\mathbb{C})$ *are not even homeomorphic because* $\Pi_1(\mathbb{P}^1(\mathbb{C})) = 0$ *and* $\Pi_1(\mathbb{C}/\Lambda) \simeq \mathbb{Z}^2$.

Definition 3.2.6 (Ramification points and branch points)**.** *Let* $f : X \to Y$ *be a holomorphic map between two Riemann surfaces, let* $p \in X$. *We say that* f *is* unramified *at* p *if* $e_f(p) = 1$ *(equivalently,* f *is a local isomorphism, or homeomorphism, at* p*). Otherwise, we say that* f *is* ramified *at* p, *or that* p *is a* ramification point *of* f. *The set* $R(f) \subseteq X$ *of ramification points of* f *is called the* ramification locus *of* f. *The set* $B(f) = f(R(f)) \subseteq Y$ *is called the* branch locus *of* f *and its elements are called* branch points.

Remark 3.2.7. *In differential geometry, ramification points are called critical points and branch points are called critical values.*

Proposition 3.2.8. *Let* $f : X \to Y$ *be a holomorphic map between two Riemann surfaces. If* X *is connected and* f *is not constant, then* $R(f)$ *is closed and discrete in* X. *If in addition* X *is compact (which implies that* Y *is compact), then* $R(f)$ *and* $B(f)$ *are both finite.*

Proof. Let $p \in X \backslash R(f)$. Taking local coordinates $(\mathcal{U}, \phi)$ at p and $(\mathcal{V}, \psi)$ at $f(p)$, we have $\psi \circ f \circ \phi^{-1}(z) \underset{0}{\sim} \lambda z$ for some $\lambda \in \mathbb{C}^\times$. Therefore, f is a local isomorphism around p, so f is unramified in a neighbourhood of p. This shows that $X \backslash R(f)$ is open, i.e. $R(f)$ is closed. Now, let $p \in R(f)$. We can find charts $(\mathcal{U}, \phi)$ at p and $(\mathcal{V}, \psi)$ at $f(p)$ s.t. $\psi \circ f \circ \phi^{-1}(z) = z^e$ in a neighbourhood of 0, with $e = e_f(p) \geq 2$. Hence, $(\psi \circ f \circ \phi^{-1})'(z) = ez^{e-1}$ in a neighbourhood of 0, so $(\psi \circ f \circ \phi^{-1})'(z) \neq 0$ if $z \neq 0$ in a neighbourhood of 0. This shows that p is isolated in $R(f)$, so $R(f)$ is discrete. □

Proposition 3.2.9. *Let* $f : X \to Y$ *be a nonconstant holomorphic map between two compact connected Riemann surfaces.*

(i) *If* $h \in \mathcal{M}(Y)$, *then* $h \circ f \in \mathcal{M}(X)$.

(ii) *For every* $p \in X$, *we have:*

$$\operatorname{ord}_p(h \circ f) = e_f(p) \cdot \operatorname{ord}_{f(p)}(h).$$

Remark 3.2.10. *If* $f : X \to Y$ *is a nonconstant holomorphic map between two compact connected Riemann surfaces, then* f *induces a map:*

$$f^* : h \in \mathcal{M}(Y) \longmapsto h \circ f \in \mathcal{M}(X),$$

which is a morphism of $\mathbb{C}$*-algebras, and thus a morphism of fields. Therefore, we may view* $\mathcal{M}(X)$ *as a field extension of* $\mathcal{M}(Y)$. *Moreover, we have a discrete valuation* ord_p *on* $\mathcal{M}(X)$, *and the restriction to* $\mathcal{M}(Y)$ *is* $e_f(p) \cdot \operatorname{ord}_{f(p)}$.

3.3 Degree of a holomorphic map

Theorem 3.3.1. *Let $f : X \to Y$ be a nonconstant holomorphic map between two compact connected Riemann surfaces. For $q \in Y$, define:*

$$d_q = \sum_{p \in f^{-1}(\{q\})} e_f(p) \in \mathbb{N}^*.$$

Then d_q does not depend on q; it is called the topological degree *of f and denoted by $\deg f$.*

Remark 3.3.2. *Let $f : X \to Y$ be a nonconstant holomorphic map between two compact connected Riemann surfaces. Write $R'(f) = f^{-1}(B(f))$. Then f induces a local isomorphism $X \backslash R'(f) \to Y \backslash B(f)$, which is a topological covering of degree $\deg f$.*

Example 3.3.3. *If $f : X \to Y$ is a nonconstant holomorphic map between two compact connected Riemann surfaces, we have seen (in Remark 3.2.10) that f induces a field extension $\mathcal{M}(X)/\mathcal{M}(Y)$, and we have $[\mathcal{M}(X) : \mathcal{M}(Y)] = \deg f$.*

Corollary 3.3.4. *Let X be a compact connected Riemann surface. If there exists a meromorphic function on X with only a simple pole, then X is isomorphic to $\mathbb{P}^1(\mathbb{C})$.*

Theorem 3.3.5. *Let X be a compact connected Riemann surface. For every $f \in \mathcal{M}(X)^\times$, we have:*

$$\sum_{p \in X} \operatorname{ord}_p(f) = 0.$$

Proof. We may assume that f is nonconstant, and we view it as a holomorphic map $\hat{f} : X \to \mathbb{P}^1(\mathbb{C})$. We have:

$$\sum_{p \in \hat{f}^{-1}(\{0\})} \operatorname{ord}_p(f) = \sum_{p \in \hat{f}^{-1}(\{0\})} e_{\hat{f}}(p) = \deg \hat{f} = \sum_{p \in \hat{f}^{-1}(\{\infty\})} e_{\hat{f}}(p) = -\sum_{p \in \hat{f}^{-1}(\{\infty\})} \operatorname{ord}_p(f).$$

Therefore $\sum_{p \in X} \operatorname{ord}_p(f) = \sum_{p \in \hat{f}^{-1}(\{0\})} \operatorname{ord}_p(f) + \sum_{p \in \hat{f}^{-1}(\{\infty\})} \operatorname{ord}_p(f) = 0.$ ☐

3.4 Divisors

Remark 3.4.1. *Let X be a compact connected Riemann surface. Given $p_1, \dots, p_r, q_1, \dots, q_s \in X$, $m_1, \dots, m_r, n_1, \dots, n_s \in \mathbb{N}^*$, a fundamental problem is to tell whether or not there exists a meromorphic function $f \in \mathcal{M}(X)$ with a zero of order m_i at each point p_i and a pole of order n_j at each point q_j, and which is holomorphic and nonvanishing on $X \backslash \{p_1, \dots, p_r, q_1, \dots, q_s\}$. To state this problem in a more concise way, we shall introduce the language of divisors.*

Definition 3.4.2 (Divisors)**.** *If X is a Riemann surface, we denote by* $\operatorname{Div}(X)$ *the free $\mathbb{Z}$-module with basis X. Its elements are called* divisors*; they are formal linear combinations of points of X.*

- *Given $D = \sum_{p \in X} n_p[p] \in \operatorname{Div}(X)$, we define the* order *of D at p by $\operatorname{ord}_p(D) = n_p \in \mathbb{Z}$, and the* degree *of D by $\deg D = \sum_{p \in X} n_p \in \mathbb{Z}$.*
- *We say that a divisor $D = \sum_{p \in X} n_p[p] \in \operatorname{Div}(X)$ is* effective*, and we write $D \geq 0$, if $\forall p \in X$, $n_p \geq 0$. If $D_1, D_2 \in \operatorname{Div}(X)$, we say that $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.*
- *Given $D = \sum_{p \in X} n_p[p] \in \operatorname{Div}(X)$, the* support *of D is the finite set $\{p \in X,\ \operatorname{ord}_p(D) \neq 0\}$.*

Definition 3.4.3 (Divisor of a meromorphic function)**.** *Let X be a compact connected Riemann surface and let $f \in \mathcal{M}(X)^\times$. The* divisor *of f is defined by:*

$$\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_p(f) \cdot [p] \in \operatorname{Div}(X).$$

Proposition 3.4.4. *If X is a compact connected Riemann surface, then* $\operatorname{div} : \mathcal{M}(X)^\times \to \operatorname{Div}(X)$ *is a group homomorphism. Its image is called the* group of principal divisors, *and denoted by* $\Pr(X)$.

Lemma 3.4.5. *Let X be a compact connected Riemann surface. Then the following sequence is exact:*

$$0 \to \mathbb{C}^\times \overset{\subseteq}{\to} \mathcal{M}(X)^\times \xrightarrow{\operatorname{div}} \Pr(X) \to 0.$$

Definition 3.4.6 (Degree zero divisors and Picard group)**.** *Let X be a compact connected Riemann surface. Then we have a $\mathbb{Z}$-linear map* $\deg : \operatorname{Div}(X) \to \mathbb{Z}$*; we define the group of* degree zero divisors *by:*

$$\operatorname{Div}^0(X) = \operatorname{Ker}\deg \subseteq \operatorname{Div}(X).$$

Theorem 3.3.5 implies that $\Pr(X) \subseteq \operatorname{Div}^0(X) \subseteq \operatorname{Div}(X)$*. The quotient:*

$$\operatorname{Pic}^0(X) = \operatorname{Div}^0(X)/\Pr(X),$$

is called the Picard group *of X; it can be interpreted as* $H^1\left(X, \mathcal{O}_X^\times\right)$.

Example 3.4.7.

(i) $\operatorname{Pic}^0\left(\mathbb{P}^1(\mathbb{C})\right) = 0$.

(ii) *If Λ is a lattice in $\mathbb{C}$, then* $\operatorname{Pic}^0(\mathbb{C}/\Lambda) \simeq \mathbb{C}/\Lambda$.

In general, according to the Abel-Jacobi Theorem, if X is a compact connected Riemann surface, then there exists an integer $g \in \mathbb{N}$, called the genus *of X, s.t.* $\operatorname{Pic}^0(X) \simeq \mathbb{C}^g/\Lambda$*, where Λ is a $\mathbb{Z}$-lattice of rank $2g$ in $\mathbb{C}^g$.*

3.5 Riemann-Roch spaces

Definition 3.5.1 (Riemann-Roch space associated to a divisor)**.** *Let X be a compact connected Rieman surface. Given $D \in \operatorname{Div}(X)$, we define:*

$$\mathcal{L}(D) = \left\{f \in \mathcal{M}(X)^\times,\ \operatorname{div}(f) \geq -D\right\} \cup \{0\} \subseteq \mathcal{M}(X).$$

$\mathcal{L}(D)$ is called the Riemann-Roch space *associated to D. Then $\mathcal{L}(D)$ is a sub-$\mathbb{C}$-vector space of $\mathcal{M}(X)$.*

Proposition 3.5.2. *Let X be a compact connected Riemann surface. If $D \in \operatorname{Div}(X)$, then the $\mathbb{C}$-vector space $\mathcal{L}(D)$ is finite-dimensional.*

Proof. Firstly, note that $\mathcal{L}(D_1) \subseteq \mathcal{L}(D_2)$ if $D_1 \leq D_2$, so it suffices to prove the proposition for effective divisors (i.e. $D \geq 0$). Thus, we can proceed by induction on $\deg D$. If $\deg D = 0$ (i.e. $D = 0$), then $\mathcal{L}(D) = \mathcal{O}(X) = \mathbb{C}$. Now, assume that the result is true for all effective divisors of degree $d \in \mathbb{N}$ and let D be an effective divisor with $\deg D = d+1$. Write $D = D' + [p]$, with $p \in X$, D' effective and $\deg D' = d$. We have $\mathcal{L}(D') \subseteq \mathcal{L}(D)$, and $\mathcal{L}(D')$ is finite-dimensional. We will construct a linear form on $\mathcal{L}(D)$ whose kernel is $\mathcal{L}(D')$. To do this, consider a holomorphic system of coordinates $\phi : \mathcal{U} \to V$ around p in X. If $f \in \mathcal{L}(D)$, then $f \circ \phi^{-1}(z)$ has a pole of order at most $n+1 = \operatorname{ord}_p(D)$ at 0, so we can write:

$$f \circ \phi^{-1}(z) = \frac{\alpha}{z^{n+1}} + \mathcal{O}_0\left(\frac{1}{z^n}\right),$$

for some unique $\alpha \in \mathbb{C}$. Denote by $\lambda_\phi : \mathcal{L}(D) \to \mathbb{C}$ the linear map given by $f \mapsto \alpha$. By construction, $\operatorname{Ker}\lambda_\phi = \mathcal{L}(D')$, so $\dim \mathcal{L}(D) \leq \dim \mathcal{L}(D') + 1 < +\infty$. □

Remark 3.5.3. *Let X be a compact connected Riemann surface. If $D \in \mathrm{Div}(X)$ is an effective divisor, then the proof of Proposition 3.5.2 gives a bound on the dimension of $\mathcal{L}(D)$:*

$$\dim \mathcal{L}(D) \leq \deg D + 1.$$

Theorem 3.5.4 (Riemann-Roch)**.** *Let X be a compact connected Riemann surface of genus g. If $D \in \mathrm{Div}(X)$ is an effective divisor with $\deg D > 2g - 2$, then:*

$$\dim \mathcal{L}(D) = \deg D + 1 - g.$$

Corollary 3.5.5. *If X is a compact connected Riemann surface, then for every $p \in X$, there exists a nonconstant meromorphic function on X whose only pole is at p.*

Example 3.5.6. *If $X = \mathbb{P}^1(\mathbb{C})$, then $\mathcal{L}(n[\infty])$ is the set of polynomials of degree at most n.*

Remark 3.5.7. *The* genus g *of a compact connected Riemann surface X is a topological invariant. The integer $\chi = 2 - 2g$ is called the* Euler-Poincaré characteristic *of X. To compute it, consider any triangulation of X, with V vertices, E edges and F faces. Then:*

$$\chi = V - E + F.$$

4 Differential forms

4.1 Complex and holomorphic differential forms

Notation 4.1.1. *Let X be a Riemann surface. In particular, X is a differentiable manifold of real dimension 2, and we denote by $\mathcal{A}^k_{\mathbb{R}}(X)$ the real vector space of smooth differential k-forms on X, for $0 \leq k \leq 2$.*

Definition 4.1.2 (Complex differential forms)**.** *Let X be a Riemann surface. For $0 \leq k \leq 2$, we define:*

$$\mathcal{A}^k_{\mathbb{C}}(X) = \mathcal{A}^k_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

Elements of $\mathcal{A}^k_{\mathbb{C}}(X)$ are called complex k-forms *on X.*

Remark 4.1.3. *Let X be a Riemann surface.*

- *Any form $\omega \in \mathcal{A}^k_{\mathbb{C}}(X)$ can be written uniquely as $\omega = \alpha + i\beta$, with $\alpha, \beta \in \mathcal{A}^k_{\mathbb{R}}(X)$.*
- *For $k = 0$, we have $\mathcal{A}^0_{\mathbb{R}}(X) = \mathcal{C}^\infty(X, \mathbb{R})$ and $\mathcal{A}^0_{\mathbb{C}}(X) = \mathcal{C}^\infty(X, \mathbb{C})$.*
- *The operator $\mathrm{d} : \mathcal{A}^k_{\mathbb{R}}(X) \to \mathcal{A}^{k+1}_{\mathbb{R}}(X)$ extends to $\mathrm{d} : \mathcal{A}^k_{\mathbb{C}}(X) \to \mathcal{A}^{k+1}_{\mathbb{C}}(X)$.*
- *We have a complex conjugation operator $\overline{\cdot} : \mathcal{A}^k_{\mathbb{C}}(X) \to \mathcal{A}^k_{\mathbb{C}}(X)$ defined by $\overline{\omega \otimes \lambda} = \omega \otimes \overline{\lambda}$. The fixed points of this involution are the elements of $\mathcal{A}^k_{\mathbb{R}}(X)$.*
- *For $\omega \in \mathcal{A}^k_{\mathbb{C}}(X)$, we have $\overline{\mathrm{d}\omega} = \mathrm{d}\overline{\omega}$.*

Notation 4.1.4. *If V is an open subset of $\mathbb{C}$, we have the standard coordinate $z = x + iy$ on V; we denote by $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y \in \mathcal{A}^1_{\mathbb{C}}(V)$ and $\mathrm{d}\overline{z} = \mathrm{d}x - i\,\mathrm{d}y \in \mathcal{A}^1_{\mathbb{C}}(V)$.*

Definition 4.1.5 (Holomorphic differential form in a chart)**.** *Let X be a Riemann surface and let $\phi : \mathcal{U} \to V$ be a holomorphic chart of X. We say that a differential form $\omega \in \mathcal{A}^1_{\mathbb{C}}(\mathcal{U})$ is* holomorphic *(resp.* anti-holomorphic*) on $\mathcal{U}$ if $\phi_*\omega = f(z)\,\mathrm{d}z$ (resp. $\phi_*\omega = \overline{f(z)}\,\mathrm{d}\overline{z}$), with $f \in \mathcal{O}(V)$.*

Remark 4.1.6. *A complex form ω is holomorphic* iff *$\overline{\omega}$ is anti-holomorphic.*

Definition 4.1.7 (Holomorphic differential form)**.** *Let X be a Riemann surface. A differential form $\omega \in \mathcal{A}^1_{\mathbb{C}}(X)$ is said to be* holomorphic *(resp.* anti-holomorphic*) if it is holomorphic (resp. anti-holomorphic) on every holomorphic chart of X. We denote by $\Omega^1(X)$ (resp. $\overline{\Omega}^1(X)$) the set of holomorphic (resp. anti-holomorphic) differential forms on X.*

Remark 4.1.8. *Let X be a Riemann surface. Let $\phi_1 : \mathcal{U}_1 \to V_1$ and $\phi_2 : \mathcal{U}_2 \to V_2$ be two charts of X. Then a complex differential form $\omega \in \mathcal{A}^1_{\mathbb{C}}(X)$ is holomorphic on $\mathcal{U}_1 \cap \mathcal{U}_2$ for ϕ_1* iff *it is holomorphic on $\mathcal{U}_1 \cap \mathcal{U}_2$ for ϕ_2.*

Example 4.1.9.

(i) *If $\mathcal{U}$ is an open subset of $\mathbb{C}$, then $\Omega^1(\mathcal{U}) = \mathcal{O}(\mathcal{U})\,\mathrm{d}z$.*

(ii) $\Omega^1(\mathbb{P}^1(\mathbb{C})) = 0$.

(iii) *If Λ is a lattice in $\mathbb{C}$, then $\Omega^1(\mathbb{C}/\Lambda) = \mathbb{C}\,\mathrm{d}z$.*

Proposition 4.1.10. *Let $f : X \to Y$ be a holomorphic map between Riemann surfaces. Then f induces a $\mathbb{C}$-linear map $f^* : \mathcal{A}^1_{\mathbb{C}}(Y) \to \mathcal{A}^1_{\mathbb{C}}(X)$, and f^* sends $\Omega^1(Y)$ to $\Omega^1(X)$.*

4.2 Integration of differential forms

Remark 4.2.1. *Let X be a Riemann surface. Consider a $\mathcal{C}^1$ path $\gamma : [0,1] \to X$. For any $\omega \in \mathcal{A}^1_{\mathbb{C}}(X)$, we can define $\int_\gamma \omega$.*

(i) *If ω is* closed*, i.e. $\mathrm{d}\omega = 0$, then for any $\gamma' : [0,1] \to X$ that is homotopic to γ (with fixed endpoints), we have $\int_{\gamma'} \omega = \int_\gamma \omega$.*

(ii) *If ω is* exact*, i.e. $\omega = \mathrm{d}F$, with $F \in \mathcal{C}^\infty(X, \mathbb{C})$, then $\int_\gamma \omega = F(\gamma(1)) - F(\gamma(0))$.*

Proposition 4.2.2. *All holomorphic and anti-holomorphic forms are closed.*

Proof. It suffices to prove the result for holomorphic forms. Moreover, the result being local, it suffices to prove it for holomorphic forms ω on an open subset $\mathcal{U} \subseteq \mathbb{C}$. Hence, we can write $\omega = f(z)\,\mathrm{d}z$ with $f \in \mathcal{O}(\mathcal{U})$, so that $\mathrm{d}\omega = f'(z)\,\mathrm{d}z \wedge \mathrm{d}z = 0$. ☐

Remark 4.2.3. *Every Rieman surface has a canonical orientation induced by the orientation of $\mathbb{C}$.*

Definition 4.2.4 (Hermitian scalar product on Ω^1)**.** *Let X be a Riemann surface. We define a Hermitian scalar product $\langle \cdot, \cdot \rangle$ on $\Omega^1(X)$ by:*

$$\langle \omega, \nu \rangle = \frac{i}{2} \int_X \omega \wedge \overline{\nu}.$$

Proof. It is clear that $\overline{\langle \omega, \nu \rangle} = \langle \nu, \omega \rangle$ and that $\langle \cdot, \nu \rangle$ is $\mathbb{C}$-linear. Let $\omega \in \Omega^1(X)$. We shall show that $\langle \omega, \omega \rangle \in \mathbb{R}_+$. In a holomorphic chart $\phi : \mathcal{U} \to V$, we have $\phi_* \omega = f(z)\,\mathrm{d}z$ with $f \in \mathcal{O}(V)$, so $\phi_*\left(\frac{i}{2}\omega \wedge \overline{\omega}\right) = |f(z)|^2\,\mathrm{d}x \wedge \mathrm{d}y$. Therefore, the restriction of the integral to any chart is nonnegative, so $\langle \omega, \omega \rangle \in \mathbb{R}_+$. Moreover, if $\langle \omega, \omega \rangle = 0$, then the restriction of ω to any chart is zero, so $\omega = 0$. ☐

Example 4.2.5. *Let Λ be a lattice in $\mathbb{C}$. If $\omega = \mathrm{d}z \in \Omega^1(\mathbb{C}/\Lambda)$, then $\langle \omega, \omega \rangle = \int_{\mathbb{C}/\Lambda} \mathrm{d}x \wedge \mathrm{d}y = \mathcal{A}(D_1)$, where D_1 is any fundamental domain of Λ.*

4.3 Forms of type $(1,0)$ or $(0,1)$

Definition 4.3.1 (Forms of type $(1,0)$ or $(0,1)$)**.** *Let X be a Riemann surface. A differential form $\omega \in \mathcal{A}^1_{\mathbb{C}}(X)$ is said to be* of type $(1,0)$ *(resp.* of type $(0,1)$*) if for every holomorphic chart $\phi : \mathcal{U} \to V$, there exists $\alpha \in \mathcal{C}^\infty(V, \mathbb{C})$ s.t. $\phi_* \omega = \alpha(z)\ \mathrm{d}z$ (resp. $\phi_*\omega = \alpha(z)\ \mathrm{d}\overline{z}$). We denote by $\mathcal{A}^{1,0}(X)$ (resp. $\mathcal{A}^{0,1}(X)$) the set of differential forms on X of type $(1,0)$ (resp. $(0,1)$).*

Lemma 4.3.2. *Let X be a Riemann surface. Then:*

$$\mathcal{A}^1_{\mathbb{C}}(X) = \mathcal{A}^{1,0}(X) \oplus \mathcal{A}^{0,1}(X).$$

For $\omega \in \mathcal{A}^1_{\mathbb{C}}(X)$, we shall write $\omega = \omega^{1,0} + \omega^{0,1}$, with $\omega^{1,0} \in \mathcal{A}^{1,0}(X)$ and $\omega^{0,1} \in \mathcal{A}^{0,1}(X)$.

Remark 4.3.3. *For a Riemann surface X, we have $\Omega^1(X) \subseteq \mathcal{A}^{1,0}(X) \subseteq \mathcal{A}^1_{\mathbb{C}}(X)$ and $\overline{\Omega}^1(X) \subseteq \mathcal{A}^{0,1}(X) \subseteq \mathcal{A}^1_{\mathbb{C}}(X)$.*

Notation 4.3.4. *Let X be a Riemann surface. We define two $\mathbb{C}$-linear operators ∂ and $\overline{\partial}$ by:*

$$\partial : \left| \begin{array}{rcl} \mathcal{C}^\infty(X,\mathbb{C}) & \longrightarrow & \mathcal{A}^{1,0}(X) \\ f & \longmapsto & (\mathrm{d}f)^{1,0} \end{array} \right. \qquad \text{and} \qquad \overline{\partial} : \left| \begin{array}{rcl} \mathcal{C}^\infty(X,\mathbb{C}) & \longrightarrow & \mathcal{A}^{0,1}(X) \\ f & \longmapsto & (\mathrm{d}f)^{0,1} \end{array} \right. .$$

For $f \in \mathcal{C}^\infty(X, \mathbb{C})$, we have $\mathrm{d}f = \partial f + \overline{\partial} f$.

Proposition 4.3.5. *Let X be a Riemann surface. Then ∂ and $\overline{\partial}$ are $\mathbb{C}$-linear and we have:*

$$\forall f, g \in \mathcal{C}^\infty(X, \mathbb{C}),\ \partial(fg) = f\partial g + g \partial f,$$

and similarly for $\overline{\partial}$.

Example 4.3.6. *Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and let $f \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{C})$. Then:*

$$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\mathrm{d}z \qquad \text{and} \qquad \overline{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\mathrm{d}\overline{z}$$

Hence, by Cauchy-Riemann, f is holomorphic iff $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$ iff $\mathrm{d}f \in \mathcal{A}^{1,0}(X)$.

Lemma 4.3.7. *Let X be a Riemann surface and let $f \in \mathcal{C}^\infty(X, \mathbb{C})$. Then the following three assertions are equivalent:*

(i) $f \in \mathcal{O}(X)$.

(ii) $\mathrm{d}f \in \mathcal{A}^{1,0}(X)$.

(iii) $\overline{\partial} f = 0$.

Proof. All the assertions are local, so we may assume that X is an open subset of $\mathbb{C}$ and use Example 4.3.6. □

Lemma 4.3.8. *Let X be a Riemann surface and let $\omega \in \mathcal{A}^{1,0}(X)$. Then:*

$$\omega \in \Omega^1(X) \iff \mathrm{d}\omega = 0.$$

In other words, $\Omega^1(X) = \mathcal{A}^{1,0}(X) \cap \operatorname{Ker} \mathrm{d}$.

Proof. We may assume that X is an open subset of $\mathbb{C}$ and write $\omega = \alpha(z)\ \mathrm{d}z$, with $\alpha \in \mathcal{C}^\infty(X, \mathbb{C})$. Thus:

$$\mathrm{d}\omega = \mathrm{d}\alpha \wedge \mathrm{d}z = \underbrace{\partial\alpha \wedge \mathrm{d}z}_{0} + \underbrace{\overline{\partial}\alpha \wedge \mathrm{d}z}_{\beta(z)\ \mathrm{d}\overline{z} \wedge \mathrm{d}z},$$

with $\beta \in \mathcal{C}^\infty(X, \mathbb{C})$. Thus, $\mathrm{d}\omega = 0$ iff $\overline{\partial}\alpha = 0$ iff $\alpha \in \mathcal{O}(X)$ iff $\omega \in \Omega^1(X)$. □

Notation 4.3.9. *Let X be a Riemann surface. We define a $\mathbb{C}$-linear operator $\partial\overline{\partial}$ by:*

$$\partial\overline{\partial} : \left| \begin{array}{rcl} \mathcal{C}^\infty(X, \mathbb{C}) & \longrightarrow & \mathcal{A}^2_{\mathbb{C}}(X) \\ f & \longmapsto & \mathrm{d}\left(\overline{\partial} f\right) \end{array} \right. .$$

Example 4.3.10. *Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and let $f \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{C})$. Then:*

$$\partial\overline{\partial} f = -\frac{i}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \mathrm{d}x \wedge \mathrm{d}y = \frac{i}{2}\Delta f \ \mathrm{d}x \wedge \mathrm{d}y.$$

Definition 4.3.11 (Harmonic functions)**.** *If X is a Riemann surface, we define $\mathcal{H}(X) = \mathrm{Ker}\left(\partial\overline{\partial}\right) \subseteq \mathcal{C}^\infty(X, \mathbb{C})$. Elements of $\mathcal{H}(X)$ are called* harmonic functions *on X. We have $\mathcal{O}(X) \subseteq \mathcal{H}(X)$ and $\overline{\mathcal{O}}(X) \subseteq \mathcal{H}(X)$.*

Proposition 4.3.12. *If X is a compact connected Riemann surface, then $\mathcal{H}(X) = \mathbb{C}$.*

Proof. Let $f \in \mathcal{H}(X)$ and let $\omega = \partial\overline{f} \in \mathcal{A}^{1,0}(X)$. Note that $\omega \wedge \overline{\omega} = \mathrm{d}\overline{f} \wedge \overline{\omega}$, so:

$$\int_X \omega \wedge \overline{\omega} = \int_X \mathrm{d}\overline{f} \wedge \overline{\omega} = \int_X \mathrm{d}\left(\overline{f\omega}\right) - \int_X \overline{f}\,\mathrm{d}\overline{\omega} = \int_{\partial X} \overline{f\omega} - \int_X \overline{f}\,\mathrm{d}\overline{\omega} = -\int_X \overline{f}\,\mathrm{d}\overline{\omega} = \int_X \overline{f}\left(\partial\overline{\partial} f\right) = 0.$$

Therefore, $\omega = 0$ so $f \in \mathcal{O}(X)$ by Lemma 4.3.7. Thus, $f \in \mathbb{C}$ because X is compact and connected. □

Theorem 4.3.13. *Let X be a compact connected Riemann surface. Then:*

$$\mathrm{Im}\left(\partial\overline{\partial}\right) = \left\{\alpha \in \mathcal{A}^2_{\mathbb{C}}(X),\ \int_X \alpha = 0\right\}.$$

4.4 Meromorphic differential forms

Definition 4.4.1 (Meromorphic form at a point)**.** *Let X be a Riemann surface and $p \in X$. Let $\phi : \mathcal{U} \to V$ be a holomorphic chart containing p, with $\phi(p) = z_0$. If $\omega \in \Omega^1(\mathcal{U}\backslash\{p\})$, we can write $\phi_*\omega = f(z)\,\mathrm{d}z$ for some $f \in \mathcal{O}(V\backslash\{z_0\})$. We say that ω is* meromorphic at p *if f is meromorphic at z_0. In this case, we define:*

$$\mathrm{ord}_p(\omega) = \mathrm{ord}_{z_0}(f) \in \mathbb{Z} \cup \{\infty\}.$$

Remark 4.4.2. *If X is a Riemann surface equipped with a chart $\phi : \mathcal{U} \to V$, $p \in \mathcal{U}$ and $\omega \in \Omega^1(\mathcal{U}\backslash\{p\})$ is meromorphic at p, then:*

(i) $\mathrm{ord}_p(\omega) \geq 0 \Longleftrightarrow \omega$ *extends to a holomorphic form at p.*

(ii) $\mathrm{ord}_p(\omega) = \infty \Longleftrightarrow \omega = 0$ *on a neighbourhood of p.*

(iii) $\mathrm{ord}_p(\omega) \geq 1 \Longleftrightarrow \omega_p = 0$.

Definition 4.4.3 (Meromorphic form)**.** *A* meromorphic form *on X is the data of a closed discrete subset S of X and of $\omega \in \Omega^1(X\backslash S)$ s.t. ω is meromorphic at each point of S. We write $\Omega^1(\mathcal{M}(X))$ for the set of meromorphic forms on X. It is a $\mathcal{M}(X)$-vector space if X is connected.*

Example 4.4.4. *On $\mathbb{P}^1(\mathbb{C})$, $\mathrm{d}z \in \Omega^1(\mathbb{C})$ is meromorphic at ∞ and* $\mathrm{ord}_\infty(\mathrm{d}z) = -2$.

Definition 4.4.5 (Divisor of a meromorphic form)**.** *If X is a compact connected Riemann surface and $\omega \in \Omega^1(\mathcal{M}(X))\backslash\{0\}$, we define the* divisor *of ω by:*

$$\mathrm{div}(\omega) = \sum_{p \in X} \mathrm{ord}_p(\omega) \cdot [p] \in \mathrm{Div}(X).$$

Remark 4.4.6. *Let X be a Riemann surface.*

(i) $\Omega^1(X) = \{\omega \in \Omega^1(\mathcal{M}(X)),\ \mathrm{div}(\omega) \geq 0\}$.

(ii) *For* $f \in \mathcal{M}(X)$ *and* $\omega \in \Omega^1(\mathcal{M}(X))$, *we have* $\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega)$.

Definition 4.4.7 (Genus)**.** *If X is a compact connected Riemann surface, then $\Omega^1(X)$ is a finite-dimensional $\mathbb{C}$-vector space. The* genus *of X is defined by:*

$$g = \dim_{\mathbb{C}} \Omega^1(X).$$

Proof. If $\Omega^1(X) = 0$, then it is clearly finite-dimensional. Otherwise, let $\omega_0 \in \Omega^1(X)\backslash\{0\}$. For $\omega \in \Omega^1(\mathcal{M}(X))$, there exists a unique $f \in \mathcal{M}(X)$ s.t. $\omega = f\omega_0$. Now:

$$\omega \in \Omega^1(X) \iff \mathrm{div}(\omega) \geq 0 \iff f \in \mathcal{L}(\mathrm{div}(\omega_0)).$$

Hence, there is a surjection $\mathcal{L}(\mathrm{div}(\omega_0)) \to \Omega^1(X)$, which concludes the proof because $\mathcal{L}(\mathrm{div}(\omega_0))$ is finite-dimensional by Proposition 3.5.2. ☐

Remark 4.4.8. *The above argument actually shows that $\Omega^1(\mathcal{M}(X))$ is a $\mathcal{M}(X)$-vector space of dimension ≤ 1. In fact, one can prove that $\dim_{\mathcal{M}(X)} \Omega^1(\mathcal{M}(X)) = 1$.*

4.5 Residues of differential forms

Remark 4.5.1. *On Riemann surfaces, it is not possible to define the residue of a meromorphic function at a point, because such a residue would depend on the chart in which we read the function. However, we will be able to define the residue of a meromorphic differential form at a point.*

Definition 4.5.2 (Residue of a meromorphic differential form)**.** *Let X be a Riemann surface. Let Ω be an open subset of X containing a point p. Let $\omega \in \Omega^1(\Omega\backslash\{p\})$ be a holomorphic differential form that is meromorphic at p. If $\phi : \mathcal{U} \to V$ is a holomorphic chart of X containing p, with $\phi(p) = z_0$, then we can write $\varphi_*\omega = f(z)\,\mathrm{d}z$, with f meromorphic at z_0. We define $\mathrm{Res}_p(\omega) = \mathrm{Res}_{z_0}(f)$. The following proposition will show that $\mathrm{Res}_p(\omega)$ is independent of the choice of the chart.*

Proposition 4.5.3. *Let X be a Riemann surface. Let Ω be an open subset of X containing a point p. Let $\omega \in \Omega^1(\Omega\backslash\{p\})$ be a holomorphic differential form that is meromorphic at p. Consider a small loop γ in X around p, s.t. in a holomorphic chart $\phi : \mathcal{U} \to V$ containing p, γ is the boundary of a disk containing no pole of ω except p, oriented counter-clockwise. Then:*

$$\int_\gamma \omega = 2i\pi \cdot \mathrm{Res}_p(\omega).$$

Corollary 4.5.4. *The residue of a meromorphic differential form at a point does not depend on the chart in which we read it.*

Example 4.5.5. *Consider $\omega = \frac{\mathrm{d}z}{z} \in \Omega^1(\mathcal{M}(\mathbb{P}^1(\mathbb{C})))$. Then ω is holomorphic on $\mathbb{C}^\times$ and has poles at 0 and ∞, with $\mathrm{Res}_0(\omega) = 1$ and $\mathrm{Res}_\infty(\omega) = -1$.*

Theorem 4.5.6 (Residue Theorem)**.** *Let X be a Riemann surface and let D be a compact domain in X whose boundary γ can be parametrised by paths $[0,1] \to X$ which are $\mathcal{C}^0$ and piecewise $\mathcal{C}^1$. We endow the paths γ with a canonical orientation as follows: at a point $p \in \gamma$, we denote by $\vec{n}$ a vector in T_pX that is normal to γ and pointing towards the exterior of D, and we orient γ with a tangent vector $\vec{t}$ s.t. $\left(\vec{n}, \vec{t}\right)$ is a direct basis of T_pX (which is canonically oriented because it is a 1-dimensional $\mathbb{C}$-vector space). Now, consider a meromorphic differential form ω defined on a neighbourhood of D and s.t. the only poles of ω are at $p_1, \dots, p_n \in \mathring{D}$. Then:*

$$\int_{\partial D} \omega = 2i\pi \sum_{j=1}^{n} \mathrm{Res}_{p_j}(\omega).$$

Proof. Let $D_1, \ldots, D_n \subseteq \mathring{D}$ be small non-overlapping open disks around $p_1, \ldots, p_n$ respectively. Let $D' = D \backslash (D_1 \cup \cdots \cup D_n)$. Write $\gamma_j = \partial D_j$ for $1 \leq j \leq n$, and orient γ_j clockwise. Then ω is holomorphic on a neighbourhood of D' in X. And by Stokes' Theorem:

$$\int_{D'} \mathrm{d}\omega = \int_{\partial D'} \omega = \int_{\partial D} \omega + \sum_{j=1}^{n} \int_{\gamma_j} \omega = \int_{\partial D} \omega - 2i\pi \sum_{j=1}^{n} \operatorname{Res}_{p_j}(\omega).$$

But ω is holomorphic and thus closed on D', so $\mathrm{d}\omega = 0$, which gives the result. □

Corollary 4.5.7. *If X is a compact connected Riemann surface and $\omega \in \Omega^1(\mathcal{M}(X))$, then:*

$$\sum_{p \in X} \operatorname{Res}_p(\omega) = 0.$$

Remark 4.5.8. *Using the Residue Theorem, we can give a new proof of Theorem 3.3.5 by considering the differential form $\omega = \frac{\mathrm{d}f}{f}$.*

4.6 Riemann-Roch Theorem

Definition 4.6.1 (Canonical divisor)**.** *A* canonical divisor *on a compact connected Riemann surface X is a divisor of the form* $\operatorname{div}(\omega) \in \operatorname{Div}(X)$ *for some $\omega \in \Omega^1(\mathcal{M}(X)) \backslash \{0\}$.*

Remark 4.6.2. *If D and D' are two canonical divisors on a compact connected Riemann surface X, then $D - D'$ is a principal divisor (by the same argument as in the proof of Definition 4.4.7). In other words, X has a unique canonical divisor up to addition of a principal divisor. In particular, all the canonical divisors on X have the same degree.*

Notation 4.6.3. *Let X be a compact connected Riemann surface. For $D \in \operatorname{Div}(X)$, we define:*

$$\ell(D) = \dim_{\mathbb{C}} \mathcal{L}(D) \in \mathbb{N}.$$

Remark 4.6.4. *Let X be a compact connected Riemann surface. If $D, D' \in \operatorname{Div}(X)$ with $D - D'$ principal, then we have an isomorphism $\mathcal{L}(D') \to \mathcal{L}(D)$ (given by $g \mapsto fg$ if $D' - D = \operatorname{div}(f)$). In particular, $\ell(D') = \ell(D)$.*

Theorem 4.6.5 (Riemann-Roch)**.** *Let X be a compact connected Riemann surface of genus g, let K_X be a canonical divisor on X. Then, for every $D \in \operatorname{Div}(X)$, we have:*

$$\ell(D) - \ell(K_X - D) = \deg D + 1 - g.$$

In particular, $\ell(D) \geq \deg D + 1 - g$.

Corollary 4.6.6. *Let X be a compact connected Riemann surface of genus g. If $D \in \operatorname{Div}(X)$ is s.t. $\deg D \geq g + 1$, then $\mathcal{L}(D)$ contains a nonconstant function.*

Lemma 4.6.7. *Let X be a compact connected Riemann surface of genus g. Then for any canonical divisor K_X on X, we have $\deg(K_X) = 2g - 2$.*

Proof. Apply Theorem 4.6.5 to $D = K_X$. Thus $\ell(K_X) - \ell(0) = \deg(K_X) + 1 - g$. But $\mathcal{L}(0) = \mathcal{O}(X) = \mathbb{C}$, so $\ell(0) = 1$. Moreover, we have an isomorphism $\mathcal{L}(K_X) \to \Omega^1(X)$ given by $f \mapsto f\omega$ if $K_X = \operatorname{div}(\omega)$; thus $\ell(K_X) = \dim_{\mathbb{C}} \Omega^1(X) = g$. This yields $\deg(K_X) = 2g - 2$. □

Corollary 4.6.8. *Let X be a compact connected Riemann surface of genus g. If $D \in \operatorname{Div}(X)$ is s.t. $\deg D > 2g - 2$, then:*

$$\ell(D) = \deg D + 1 - g.$$

Proof. If $\deg D > 2g - 2 = \deg(K_X)$, then $\deg(K_X - D) < 0$, so $\ell(K_X - D) = 0$. □

Corollary 4.6.9. *Every compact connected Riemann surface of genus* 0 *is isomorphic to* $\mathbb{P}^1(\mathbb{C})$*.*

Proof. Let X be a compact connected Riemann surface of genus 0 and let $p \in X$. Consider $D = [p] \in \mathrm{Div}(X)$. Then $\deg D = 1 > 2g - 2$, so $\ell(D) = 1 + 1 - g = 2$ by Corollary 4.6.8. Thus, there exists $f \in \mathcal{L}(D) \backslash \mathbb{C}$. Now, f has only a simple pole at p, so X is isomorphic to $\mathbb{P}^1(\mathbb{C})$ by Corollary 3.3.4. □

Theorem 4.6.10. *Every compact connected Rieman surface is algebraic, i.e. can be defined by some polynomial equations in a projective space.*

Proof. Let $D \in \mathrm{Div}(X)$ s.t. $\ell(D) \geq 2$. Let $(f_1, \dots, f_n)$ be a $\mathbb{C}$-basis of $\mathcal{L}(D)$. Consider the map:

$$\phi_D : p \in X \backslash S \longmapsto (f_1(p) : \cdots : f_n(p)) \in \mathbb{P}^{n-1}(\mathbb{C}),$$

where S is the (finite) set consisting in poles of some f_i and common zeros of $f_1, \dots, f_n$. Then one can show that ϕ_D extends to a holomorphic map $\phi_D : X \to \mathbb{P}^{n-1}(\mathbb{C})$, and that, if $\deg D \geq 2g + 1$, then ϕ_D is a holomorphic embedding. Now, one uses Chow's Theorem, which states that every closed complex analytic submanifold of $\mathbb{P}^N(\mathbb{C})$ is algebraic. □

4.7 Hodge decomposition

Theorem 4.7.1. *Let X be a compact connected Riemann surface. Then the canonical map:*

$$\psi : \left| \begin{array}{rcl} \Omega^1(X) \oplus \overline{\Omega}^1(X) & \longrightarrow & H^1_{\mathrm{dR}}(X, \mathbb{C}) \\ (\omega, \omega') & \longmapsto & [\omega + \omega'] \end{array} \right. ,$$

is an isomorphism of $\mathbb{C}$-vector spaces. In particular:

$$\dim_{\mathbb{C}} H^1_{\mathrm{dR}}(X, \mathbb{C}) = 2g.$$

Therefore, the genus is a topological invariant: two homeomorphic Riemann surfaces have the same genus.

Proof. ψ is well-defined because holomorphic and anti-holomorphic differential forms are closed. *Injectivity.* Let $(\omega, \omega') \in \Omega^1(X) \oplus \overline{\Omega}^1(X)$ s.t. $[\omega + \omega'] = 0$, i.e. there exists $f \in \mathcal{C}^\infty(X, \mathbb{C})$ s.t. $\omega + \omega' = \mathrm{d}f$. Then $\overline{\partial} f = \omega'$, so $\partial\overline{\partial} f = \mathrm{d}\omega' = 0$, so f is harmonic, and so f is constant because X is compact and connected (c.f. Proposition 4.3.12). Hence, $\omega = -\omega' = 0$. *Surjectivity.* Let $\alpha \in \mathcal{A}^1_{\mathbb{C}}(X)$ be a closed 1-form. Write $\alpha = \alpha^{1,0} + \alpha^{0,1}$. By Stokes' Theorem:

$$\int_X \mathrm{d}\alpha^{0,1} = \int_{\partial X} \alpha^{0,1} = 0.$$

By Theorem 4.3.13, there exists $f \in \mathcal{C}^\infty(X, \mathbb{C})$ s.t. $\mathrm{d}\alpha^{0,1} = \partial\overline{\partial} f$. In other words, $\mathrm{d}\left(\alpha^{0,1} - \overline{\partial} f\right) = 0$. Now, consider $\alpha' = \alpha - \mathrm{d}f$; α' is a closed form that is cohomologous to α. Thus, $(\alpha')^{0,1} = \alpha^{0,1} - \overline{\partial} f$ is closed and of type $(0,1)$, so it is anti-holomorphic (c.f. Lemma 4.3.8). Likewise, $(\alpha')^{1,0}$ is holomorphic, so $\alpha' = (\alpha')^{1,0} + (\alpha')^{0,1} \in \Omega^1(X) \oplus \overline{\Omega}^1(X)$. □

Theorem 4.7.2 (Hodge Theorem)**.** *Let $X \subseteq \mathbb{P}^N(\mathbb{C})$ be a compact complex analytic submanifold of any dimension. Then, for any $k \in \mathbb{N}$, there is a decomposition:*

$$H^k_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{\substack{p,q \in \mathbb{N} \\ p+q=k}} H^{p,q}(X),$$

where $H^{p,q}(X)$ is defined by the closed forms of type (p, q).

Conjecture 4.7.3 (Hodge Conjecture)**.** *$H^{k,k}(X) \cap H^{2k}(X, \mathbb{Q})$ is spanned by algebraic classes, i.e. classes associated to algebraic subvarieties of X of codimension k.*

5 Quotients of Riemann surfaces

5.1 Groups acting on topological spaces

Definition 5.1.1 (Faithful or free group action)**.** *Let G be a group acting on a set X.*

(i) *We say that the action $G \curvearrowright X$ is* faithful *if $\forall g \in G \backslash \{1\}$, $\exists x \in X$, $gx \neq x$.*

(ii) *We say that the action $G \curvearrowright X$ is* free *if $\forall g \in G \backslash \{1\}$, $\forall x \in X$, $gx \neq x$.*

A free group action is faithful.

Proposition 5.1.2 (Universal property of the quotient)**.** *Let G be a group acting on a topological space X. Then, for every continuous map $f : X \to Y$ s.t. $\forall g \in G$, $\forall x \in X$, $f(gx) = f(x)$, there exists a unique continuous map $\overline{f} : X/G \to Y$ s.t. the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \pi}\big\downarrow & \overset{\overline{f}}{\nearrow} & \\ X/G & & \end{array}$$

Definition 5.1.3 (Continuous group action)**.** *Let G be a topological group acting on a topological space X. The group action $G \curvearrowright X$ is said to be* continuous *if the map $\left| \begin{array}{rcl} G \times X & \longrightarrow & X \\ (g, x) & \longmapsto & gx \end{array} \right.$ is continuous.*

Remark 5.1.4. *Let G be a topological group acting on a topological space X.*

(i) *If the action $G \curvearrowright X$ is continuous, then it is by homeomorphisms.*

(ii) *If G is discrete, then the action $G \curvearrowright X$ is continuous* iff *it is by homeomorphisms.*

Proposition 5.1.5. *Let G be a topological group acting by homeomorphisms on a topological space X. Then the projection map $\pi : X \to X/G$ is continuous and open.*

Definition 5.1.6 (Proper group action)**.** *Let G be a topological group acting continuously on a topological space X. The group action $G \curvearrowright X$ is said to be* proper *if one of the following two equivalent conditions is satisfied:*

(i) *The map $\left| \begin{array}{rcl} G \times X & \longrightarrow & X \\ (g, x) & \longmapsto & gx \end{array} \right.$ is proper (i.e. the preimage of every compact subset is a compact subset).*

(ii) *For every compact subset $K \subseteq X$, the set $\{g \in G,\ gK \cap K \neq \varnothing\}$ is compact.*

If G is discrete, then the action $G \curvearrowright X$ is proper iff *for every compact subset $K \subseteq X$, the set $\{g \in G,\ gK \cap K \neq \varnothing\}$ is finite.*

Proposition 5.1.7. *Let G be a discrete group acting properly and continuously on a Hausdorff and locally compact space X. Then X/G is Hausdorff and for all $x \in X$, the stabiliser $\mathrm{Stab}(x) = \{g \in G,\ gx = x\}$ is finite.*

5.2 Examples of quotients of Riemann surfaces

Remark 5.2.1. *Given an action of a discrete group on a Riemann surface which is proper and holomorphic (i.e. by biholomorphisms), we will show (in Theorem 5.3.8) that the quotient can be endowed with a unique structure of Riemann surface s.t. it satisfies the universal property of quotients (Proposition 5.1.2), where continuous maps are replaced by holomorphic maps.*

Example 5.2.2.

(i) *If Λ is a lattice in $\mathbb{C}$, then we have an action $\Lambda \curvearrowright \mathbb{C}$ by translation, which is proper and holomorphic. The quotient $\mathbb{C}/\Lambda$ is the complex torus that we already know.*

(ii) *We have an action $\mathbb{Z} \curvearrowright \mathbb{C}$ by translation, which is proper and holomorphic. The quotient $\mathbb{C}/\mathbb{Z}$ is isomorphic to $\mathbb{C}^\times$ via the map $e : z \in \mathbb{C} \longmapsto \exp(2i\pi z) \in \mathbb{C}^\times$.*

(iii) *If Λ is a lattice in $\mathbb{C}$ and $\sigma : z \in \mathbb{C}/\Lambda \longmapsto -z \in \mathbb{C}/\Lambda$, then σ is an involution so it induces an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}/\Lambda$, which is proper and holomorphic. The quotient $\frac{\mathbb{C}/\Lambda}{\mathbb{Z}/2\mathbb{Z}}$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ via the Weierstraß function $\wp : \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$.*

5.3 Structure of Riemann surface on a quotient

Proposition 5.3.1. *Let G be a discrete group acting on a connected Riemann surface X. We assume that the action is faithful, proper and holomorphic.*

(i) *For every $p \in X$, the stabiliser $G_p = \mathrm{Stab}(p)$ is a finite cyclic group.*

(ii) *The set $\{p \in X,\ G_p \neq 1\}$ is closed and discrete in X.*

Proof. (i) By properness, G_p must be finite. Now, choose a local coordinate z at p. For $g \in G_p$, we can write locally at p:

$$g(z) = \sum_{n\in\mathbb{N}} a_n(g) z^n.$$

Since $gp = p$, we have $a_0 = 0$. Moreover, g is biholomorphic so it is unramified at p, and therefore $a_1(g) \in \mathbb{C}^\times$. This defines a map $a_1 : G_p \to \mathbb{C}^\times$, which is a group homomorphism. Let us show that a_1 is injective. Let $g \in \operatorname{Ker} a_1$. If $g \neq 1$, then g does not act as the identity (because the action is faithful), so the Taylor expansion at p of $g(z) - z$ is nonzero (by connectedness). Thus, we can write:

$$g(z) = z + \alpha z^n + \mathcal{O}_0\left(z^{n+1}\right),$$

with $\alpha \in \mathbb{C}^\times$ and $n \geq 2$. Therefore, for $k \in \mathbb{N}$, we have $g^k(z) = z + k\alpha z^n + \mathcal{O}_0\left(z^{n+1}\right)$. Taking $k = |G_p|$, we get a contradiction. Therefore, $a_1 : G_p \to \mathbb{C}^\times$ is an injective group homomorphism, so G_p is isomorphic to a finite subgroup of $\mathbb{C}^\times$, so it is cyclic. (ii) Let $S = \{p \in X,\ G_p \neq 1\}$. Let $p \in S$ and let K be a compact neighbourhood of p in X. Then the set $E = \{g \in G,\ gK \cap K \neq \varnothing\}$ is finite by properness, and it contains G_p. Moreover, for $g \in E\backslash\{1\}$, the set $\mathrm{Fix}(g) = \{q \in X,\ gq = q\}$ is closed and discrete in X by the Identity Theorem (Theorem 2.4.1), so $\mathrm{Fix}(g) \cap K$ is finite. Now:

$$S \cap K \subseteq \bigcup_{g\in E\backslash\{1\}} \mathrm{Fix}(g) \cap K,$$

so $S \cap K$ is finite, and S is closed and discrete. □

Remark 5.3.2. *The map $a_1 : G_p \to \mathbb{C}^\times$ defined in the proof of Proposition 5.3.1 does not depend on the choice of the local coordinate around p. Indeed, it can be defined intrinsically by noting that, for $g \in G_p$, the linear map $\mathrm{d}g : T_pX \to T_pX$ is the multiplication by $a_1(g)$. Therefore, locally around p, elements of G_p act as rotations.*

Proposition 5.3.3. *Let G be a discrete group acting on a connected Riemann surface X. We assume that the action is faithful, proper and holomorphic. If $p \in X$ and $G_p = \mathrm{Stab}(p)$, then there exists an open neighbourhood $\mathcal{U}$ of p in X such that:*

(i) *$\mathcal{U}$ is stable by G_p.*

(ii) $\forall g \in G \backslash G_p$, $g\mathcal{U} \cap \mathcal{U} = \varnothing$.

(iii) *The natural map $\alpha : \mathcal{U}/G_p \to X/G$ is a homeomorphism onto an open subset of X/G.*

(iv) $\forall p' \in \mathcal{U}\backslash\{p\}$, $G_{p'} = 1$.

Moreover, we may assume that $\mathcal{U}$ is contained in some fixed neighbourhood of p.

Proof. Let K be a compact neighbourhood of p in X. The set $E = \{g \in G,\ gK \cap K \neq \varnothing\}$ is finite and contains G_p. Write $E\backslash G_p = \{g_1, \ldots, g_n\}$. Since X is Hausdorff and $g_i p \neq p$, there exist open neighbourhoods V_i of p and V_i' of $g_i p$ s.t. $V_i \cap V_i' = \varnothing$. Now, define:

$$V = \mathring{K} \cap \bigcap_{i=1}^{n} \left(V_i \cap g_i^{-1} V_i'\right).$$

V is an open neighbourhood of p that is contained in K. Now, set $\mathcal{U} = \bigcap_{g \in G_p} gV$; this is an open neighbourhood of p that is stable by G_p. If $g \in G\backslash G_p$, then either $g \notin E$ and $g\mathcal{U} \cap \mathcal{U} \subseteq gK \cap K = \varnothing$, or $g = g_i$ for some i, and so $g\mathcal{U} \cap \mathcal{U} \subseteq g_i V \cap V \subseteq V_i' \cap V_i = \varnothing$. Furthermore, the natural map $\alpha : \mathcal{U}/G_p \to X/G$ is injective by (ii), and we show that it is open by using the openness of $\pi : X \to X/G$ (c.f. Proposition 5.1.5). Finally, if $p' \in \mathcal{U}\backslash\{p\}$, then $G_{p'} = 1$ because $S = \{p' \in X,\ G_{p'} \neq 1\}$ is closed and discrete so we may assume that $S \cap K = \{p\}$. □

Example 5.3.4. *Consider the action of $\mu_n = \{z \in \mathbb{C},\ z^n = 1\}$ on $\mathbb{C}$ by multiplication. This action is faithful, proper and holomorphic. We have a holomorphic map $h : z \in \mathbb{C} \longmapsto z^n \in \mathbb{C}$, which is μ_n-invariant and therefore induces a bijection $\overline{h} : \mathbb{C}/\mu_n \to \mathbb{C}$. This map $\overline{h}$ can be taken as a chart on $\mathbb{C}/\mu_n$.*

Proposition 5.3.5. *Let G be a discrete group acting on a connected Riemann surface X. We assume that the action is faithful, proper and holomorphic. Then there exists a structure of Riemann surface on X/G. This structure satisfies the following properties:*

(i) *The quotient map $\pi : X \to X/G$ is holomorphic.*

(ii) *For every $p \in X$, $e_\pi(p) = |G_p|$, with $G_p = \mathrm{Stab}(p)$.*

(iii) *The map π has degree $|G|$ (possibly ∞).*

Proof. *Definition of the charts.* Let $p \in X$. Let $\mathcal{U}$ be a neighbourhood of p as in Proposition 5.3.3, let $m = |G_p|$. We may assume that $\mathcal{U}$ is contained in some chart of X, so that we have a chart $\varphi : \mathcal{U} \to V \subseteq \mathbb{C}$ with $\varphi(p) = 0$. *First case*: $m = 1$. Then we have a map $\alpha : \mathcal{U} \to X/G$ which is a homeomorphism onto its image $\mathcal{W}$; this gives a chart $\varphi \circ \alpha^{-1} : \mathcal{W} \to V$ of X/G. *Second case*: $m \geq 2$. Now, consider:

$$h : q \in \mathcal{U} \longmapsto \prod_{g \in G_p} \varphi(gq) \in \mathbb{C}.$$

Then $h \in \mathcal{O}_X(\mathcal{U})$, and h is G_p-invariant, so h induces a map $\overline{h} : \mathcal{U}/G_p \to \mathbb{C}$. Since h is open (as a nonconstant holomorphic function) and $\pi_p : \mathcal{U} \to \mathcal{U}/G_p$ is surjective, we see that $\overline{h}$ is open. Moreover, we have:

$$\mathrm{ord}_p(h) = \sum_{g \in G_p} \mathrm{ord}_{gp}(\varphi) \cdot e_g(p) = |G_p| = m.$$

Therefore, h is m-to-1 in a punctured neighbourhood of p, and so is π_p, so $\overline{h}$ is injective near p. Hence, after possibly shrinking $\mathcal{U}$, $\overline{h} : \mathcal{U}/G_p \to \mathbb{C}$ is a homeomorphism onto its image W; so we take $\overline{h} \circ \alpha^{-1}$

as a chart on X/G. *Compatibility of the charts.* The charts thus defined cover X/G; let us prove that they are pairwise compatible. Note that the set $\{p \in X,\ G_p \neq 1\}$ is closed and discrete in X, so we may assume that no two charts of type $m \geq 2$ overlap in X/G. Let $p, p' \in X$, with respective charts $\varphi : \mathcal{W} \to V \subseteq \mathbb{C}$ and $\varphi' : \mathcal{W}' \to V' \subseteq \mathbb{C}$ of types m, m' in X/G. We assume that $m = 1$. Let $\mathcal{U}$ and $\mathcal{U}'$ be respective neighbourhoods of p and p' as in Proposition 5.3.3, so that $\mathcal{W} = \pi(\mathcal{U})$ and $\mathcal{W}' = \pi(\mathcal{U}')$. Let $q \in \mathcal{W} \cap \mathcal{W}'$ and let $\widetilde{q} \in \pi^{-1}(\{q\})$. By replacing $\mathcal{U}$ or $\mathcal{U}'$ by some G-translate, we may assume that $\widetilde{q} \in \mathcal{U} \cap \mathcal{U}'$. Now, we have to prove that $\varphi' \circ \varphi^{-1}$ is holomorphic near $\varphi(q)$, which amounts to prove that $\varphi' \circ \pi$ is holomorphic near $\widetilde{q}$. But this map is indeed holomorphic because it is equal to the map h defined above. Therefore, the charts are compatible and they endow X/G with a structure of Riemann surface. And we easily check the three given properties. □

Remark 5.3.6. *Let G be a discrete group acting on a connected Riemann surface X. We assume that the action is faithful, proper and holomorphic. Endow X/G with the structure of Riemann surface constructed in Proposition 5.3.5.*

- *The ramification points of the holomorphic map $\pi : X \to X/G$ are those $p \in X$ s.t. $G_p \neq 1$.*
- *If $p, p' \in X$ are in the same G-orbit, then G_p and $G_{p'}$ are conjugate in G, so $e_\pi(p) = e_\pi(p')$.*
- *We say that π is a* ramified Galois covering *with Galois group G. This property has an algebraic counterpart: if G is finite, then the field extension $\pi^* : \mathcal{M}(X/G) \to \mathcal{M}(X)$ is Galois with Galois group G. In particular, $\mathcal{M}(X/G)$ can be identified with $\mathcal{M}(X)^G$.*

Theorem 5.3.7 (Linearisation of the action)**.** *Let G be a discrete group acting on a connected Riemann surface X. We assume that the action is faithful, proper and holomorphic. Endow X/G with the structure of Riemann surface constructed in Proposition 5.3.5. Let $p \in X$ and write $m = |G_p|$. If w is a local coordinate around $\pi(p)$ on X/G, then there exists a local coordinate z around p on X such that:*

(i) *Near p, $\pi : X \to X/G$ is given by $w = z^m$.*

(ii) *There exists a group isomorphism $\lambda : G_p \to \mu_m = \{z \in \mathbb{C},\ z^m = 1\}$ with $g(z) = \lambda(g) \cdot z$ near p for every $g \in G_p$.*

Proof. We know that $e_\pi(p) = m$. The local normal form of a holomorphic function provides a holomorphic coordinate z on X satisfying (i). To construct an isomorphism $\lambda : G_p \to \mu_m$ as in (ii), note that for $g \in G_p$, we have $g(z)^m = z^m$, so we can define $\lambda(g) = \frac{g(z)}{z} \in \mu_m$. □

Theorem 5.3.8. *Let G be a discrete group acting on a connected Riemann surface X. We assume that the action is proper and holomorphic. Then there exists a unique structure of Riemann surface on X/G such that:*

(i) *The quotient map $\pi : X \to X/G$ is holomorphic.*

(ii) *For every open subset $V \subseteq X/G$ and for every function $f : V \to \mathbb{C}$, we have $f \in \mathcal{O}_{X/G}(V) \iff (f \circ \pi) \in \mathcal{O}_X(\pi^{-1}(V))$.*

(iii) Universal property of the quotient*. Let $\varphi : X \to Y$ be a holomorphic map to a Riemann surface s.t. $\forall g \in G, \forall p \in X, \varphi(gp) = \varphi(p)$. Then there exists a unique holomorphic map $\overline{\varphi} : X/G \to Y$ s.t. $\varphi = \overline{\varphi} \circ \pi$.*

Proof. We may assume that the action is faithful. Indeed, if $G_0 = \mathrm{Ker}(G \to \mathrm{Aut}(X))$, then G/G_0 acts on X properly, holomorphically and faithfully. Moreover, the uniqueness is a consequence of the universal property. Now, endow X/G with the structure of Riemann surface constructed in Proposition 5.3.5. Using Remark 5.3.9, it suffices to prove that (ii) is satisfied. Therefore, let $V \subseteq X/G$ be an open subset and let $f : V \to \mathbb{C}$ be a function. Since π is holomorphic, it is clear that $f \in \mathcal{O}_{X/G}(V) \implies (f \circ \pi) \in \mathcal{O}_X(\pi^{-1}(V))$. Conversely, assume that $(f \circ \pi) \in \mathcal{O}_X(\pi^{-1}(V))$. Let

$q \in V$ and let $p \in \pi^{-1}(\{q\})$. Choose a local coordinate w on X/G around q and let z be the local coordinate on X around p given by Theorem 5.3.7. We can write $f \circ \pi(z) = \sum_{n\in\mathbb{N}} a_n z^n$ near p. Since $f \circ \pi$ is G_p-invariant, we have $\forall \zeta \in \mu_m$, $\sum_{n\in\mathbb{N}} a_n (\zeta z)^n = \sum_{n\in\mathbb{N}} a_n z^n$, from which we deduce that $a_n = 0$ if $m \nmid n$. Therefore:

$$f \circ \pi(z) = \sum_{k\in\mathbb{N}} a_{km} z^{km} = \sum_{k\in\mathbb{N}} a_{km} (\pi(z))^k .$$

By openness of π, $f(w) = \sum_{k\in\mathbb{N}} a_{km} w^k$ near q, so f is holomorphic near q. Hence, $f \in \mathcal{O}_{X/G}(V)$. □

Remark 5.3.9. *In Theorem 5.3.8,* (ii) *actually implies* (iii).

Proof. Let $\varphi : X \to Y$ be a holomorphic map as in (iii). By Proposition 5.1.2, there exists a continuous map $\overline{\varphi} : X/G \to Y$ s.t. $\varphi = \overline{\varphi} \circ \pi$. To prove that $\overline{\varphi}$ is holomorphic, let $V \subseteq Y$ be an open subset and let $h \in \mathcal{O}_Y(V)$ be a holomorphic test function. Then $h \circ \overline{\varphi} \circ \pi = h \circ \varphi \in \mathcal{O}_X\left(\pi^{-1}\left(\overline{\varphi}^{-1}(V)\right)\right)$, because φ is holomorphic, so $h \circ \overline{\varphi} \in \mathcal{O}_{X/G}\left(\overline{\varphi}^{-1}(V)\right)$ by (ii). Hence, $\overline{\varphi}$ is holomorphic. □

5.4 Riemann-Hurwitz Formula

Theorem 5.4.1 (Riemann-Hurwitz)**.** *Let $\varphi : X \to Y$ be a nonconstant holomorphic map between two compact connected Riemann surfaces. Then:*

$$2g(X) - 2 = (\deg \varphi)(2g(Y) - 2) + \sum_{p\in X} (e_\varphi(p) - 1) .$$

Proof. Let $\omega \in \Omega^1(\mathcal{M}(Y)) \setminus \{0\}$. By Lemma 4.6.7, we have $\deg(\operatorname{div}(\omega)) = 2g(Y) - 2$. Moreover, $\varphi^*\omega \in \Omega^1(\mathcal{M}(X)) \setminus \{0\}$, so $\deg(\operatorname{div}(\varphi^*\omega)) = 2g(X) - 2$. Now, let $p \in X$, let $q = \varphi(p) \in Y$. Let u, v be local coordinates at p and q respectively. We can write $\omega = f(v)\,\mathrm{d}v$ with f meromorphic near 0. Thus $\varphi^* v = h(u)u^e$ with h holomorphic near 0, $h(0) \neq 0$ and $e = e_\varphi(p)$. Hence:

$$\varphi^*\omega = f\left(h(u)u^e\right)\,\mathrm{d}(h(u)u^e) = \varphi^* f \left(\underbrace{h'(u)u^e}_{\mathrm{ord}_0 \geq e} + \underbrace{eh(u)u^{e-1}}_{\mathrm{ord}_0 = e-1} \right) \mathrm{d}u.$$

Therefore:

$$\mathrm{ord}_p(\varphi^*\omega) = \mathrm{ord}_p(\varphi^* f) + e_\varphi(p) - 1 = e_\varphi(p) \cdot \mathrm{ord}_q(f) + e_\varphi(p) - 1.$$

Hence, we obtain $\deg(\operatorname{div}(\varphi^*\omega)) = \sum_{p\in X} \left(e_\varphi(p) \cdot \mathrm{ord}_{\varphi(p)}(f) + e_\varphi(p) - 1\right)$. The equality follows. □

Remark 5.4.2. *We can give a topological proof of the Riemann-Hurwitz Formula by considering a triangulation T of Y s.t. the vertices of T contain the branch points of Y, and by considering φ^*T (c.f. Remark 3.5.7).*

Corollary 5.4.3. *Let $\varphi : X \to Y$ be a nonconstant holomorphic map between two compact connected Riemann surfaces.*

(i) *$g(X) \geq g(Y)$ (this can also be proved using the injectivity of $\varphi^* : \Omega^1(Y) \to \Omega^1(X)$, which is a consequence of the surjectivity of φ).*

(ii) *If $g(X) = g(Y)$, then one of the following is true:*

- *$g(X) = g(Y) = 0$,*
- *$g(X) = g(Y) = 1$ and φ is unramified,*
- *$g(X) = g(Y) \geq 2$ and φ is an isomorphism.*

Example 5.4.4. *Let $F_N = \left\{(X : Y : Z) \in \mathbb{P}^2(\mathbb{C}),\ X^N + Y^N = Z^N\right\}$. By applying the Riemann-Hurwitz Formula to the holomorphic map $\varphi : (X : Y : Z) \in F_N \longmapsto (X : Z) \in \mathbb{P}^1(\mathbb{C})$, we show that $g(F_N) = \frac{(N-1)(N-2)}{2}$.*

5.5 Uniformisation Theorem

Remark 5.5.1. *Let X be a connected Riemann surface and $p \in X$. Consider the universal covering $\pi : \widetilde{X} \to X$ of (X, p). Since π is a local homeomorphism, we can equip $\widetilde{X}$ with a structure of Riemann surface s.t. π is holomorphic and unramified. Hence, the action of $\Pi_1(X)$ on $\widetilde{X}$ is free, holomorphic and proper, and we have:*

$$X \simeq \widetilde{X}/\Pi_1(X).$$

Thus, every Riemann surface is a quotient of its universal covering.

Notation 5.5.2. *We shall write $\mathbb{H} = \{z \in \mathbb{C},\ \Im(z) > 0\}$.*

Theorem 5.5.3 (Uniformisation Theorem)**.**

(i) *Every simply connected Riemann surface is isomorphic to either $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or $\mathbb{H}$.*

(ii) *Let X be a connected Riemann surface and let $\widetilde{X}$ be the universal covering of X.*

  (a) *If $\widetilde{X} \simeq \mathbb{P}^1(\mathbb{C})$, then $X \simeq \mathbb{P}^1(\mathbb{C})$ and $\Pi_1(X) = 1$.*

  (b) *If $\widetilde{X} \simeq \mathbb{C}$, then $X \simeq \mathbb{C}/G$, where G is a discrete subgroup of $\mathbb{C}$ acting by translations. Thus, if $G = 0$, $X \simeq \mathbb{C}$; if G has rank 1, then $X \simeq \mathbb{C}^\times$; if G has rank 2, then X is isomorphic to a complex torus.*

  (c) *If $\widetilde{X} \simeq \mathbb{H}$, then $X \simeq \mathbb{H}/G$, where G is a discrete subgroup of $PSL_2(\mathbb{R})$ acting by Möbius transformations.*

Remark 5.5.4. *The Uniformisation Theorem leads one to study the quotients of $\mathbb{H}$.*

5.6 Modular curves

Remark 5.6.1. *We have a holomorphic action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ given by:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Since $-I$ acts trivially, this action induces an action of $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$ on $\mathbb{H}$ which is faithful and transitive.

Lemma 5.6.2. *If H is a discrete subgroup in a Hausdorff topological group G, then H is closed in G.*

Proposition 5.6.3. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. Then Γ acts holomorphically and properly on $\mathbb{H}$. The quotient $\mathbb{H}/\Gamma$ will be denoted by $Y(\Gamma)$; if Γ is a finite index subgroup of $SL_2(\mathbb{Z})$, $Y(\Gamma)$ will be called a* modular curve.

Proof. We first show that the map $\psi : g \in SL_2(\mathbb{R}) \longmapsto gi \in \mathbb{H}$ is proper. Note that ψ is surjective and $\mathrm{Stab}(i) = SO_2(\mathbb{R})$. Therefore, we have a homeomorphism $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \simeq \mathbb{H}$. Using the fact that $SO_2(\mathbb{R})$ is compact, we see that the projection map $SL_2(\mathbb{R}) \to SL_2(\mathbb{R})/SO_2(\mathbb{R})$ is proper, and therefore ψ is proper. Now, let $K \subseteq \mathbb{H}$ be a compact subset. Consider the compact subset $\widetilde{K} = \psi^{-1}(K) \subseteq SL_2(\mathbb{R})$. Then:

$$E = \{g \in \Gamma,\ gK \cap K \neq \varnothing\} \subseteq \left\{g \in \Gamma,\ g\widetilde{K} \cap \widetilde{K} \neq \varnothing\right\} \subseteq \left\{k_1 k_2^{-1},\ k_1, k_2 \in \widetilde{K}\right\}.$$

Hence, E is closed and discrete in a compact space, so E is finite. □

Example 5.6.4.

(i) *Consider $Y(1) = \mathbb{H}/SL_2(\mathbb{Z})$. Then the map $\alpha : \tau \mapsto \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ induces a bijection between $Y(1)$ and the set of isomorphism classes of complex tori.*

(ii) *If $N \geq 2$, we consider $\Gamma(N) = \mathrm{Ker}\left(SL_2(\mathbb{Z}) \to SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)\right)$ and we define $Y(N) = \mathbb{H}/\Gamma(N)$.*

Lemma 5.6.5. *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $\tau \in \mathbb{H}$. Then:*

$$\Im\left(g\tau\right) = \frac{\Im(\tau)}{|c\tau + d|^2}.$$

Theorem 5.6.6. *Let $D = \left\{\tau \in \mathbb{H},\ |\Re(\tau)| \leq \frac{1}{2} \text{ and } |\tau| \geq 1\right\}$. Then D is a fundamental domain for the action $SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$. More precisely:*

(i) *For all $\tau \in \mathbb{H}$, there exists $g \in SL_2(\mathbb{Z})$ s.t. $g\tau \in D$.*

(ii) *Let $\tau, \tau' \in D$ with $\tau \neq \tau'$ and $\tau' \in SL_2(\mathbb{Z}) \cdot \tau$. Then we are in one of the following two cases:*

- *$\Re(\tau) = \pm\frac{1}{2}$ and $\tau = \tau' \pm 1$.*
- *$|\tau| = 1$ and $\tau = -\frac{1}{\tau'}$.*

Proof. (i) Let $\tau \in \mathbb{H}$. By Lemma 5.6.5, it is possible to choose $g \in SL_2(\mathbb{Z})$ s.t. $\Im(g\tau)$ is maximal. Replacing g by $T^n g$, with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we may assume that $|\Re(g\tau)| \leq \frac{1}{2}$. Now, assume for contradiction that $|g\tau| < 1$. Then $Sg\tau = -\frac{1}{g\tau}$ with $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $\Im\left(Sg\tau\right) > \Im\left(g\tau\right)$, which contradicts the choice of g. Therefore, $g\tau \in D$. (ii) Write $\tau' = g\tau$, with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We may assume that $\Im\left(\tau'\right) \geq \Im\left(\tau\right)$, i.e. $|c\tau + d| \leq 1$. But as $\tau \in D$, we deduce that $c \in \{0, -1, +1\}$, from which the result follows. □

Definition 5.6.7 (Compactified modular curve)**.** *The modular curve $Y(1)$ is not compact; we shall compactify it. If $\mathcal{U}_0 = \{\tau \in \mathbb{H},\ \Im(\tau) > y_0\}$ for some $y_0 \in \mathbb{R}_+^*$ large enough, we have an isomorphism $\mathcal{U}_0/\mathbb{Z} \simeq B^*$ induced by $\tau \mapsto e^{2i\pi\tau}$, where B is the (open) unit disk and $B^* = B\backslash\{0\}$. By Theorem 5.6.6, $\mathcal{U}_0/\mathbb{Z}$ is homeomorphic to an open subset of $Y(1)$; therefore we can glue $Y(1)$ and B along $\mathcal{U}_0/\mathbb{Z} \simeq B^*$: the resulting Riemann surface is denoted by $X(1)$ and called the* compactified modular curve. *It can be written as $X(1) = Y(1) \cup \{\infty\}$; the point ∞ is called the* cusp *of $X(1)$.*

Notation 5.6.8. *For $k \geq 3$, we define:*

$$G_k : \tau \in \mathbb{H} \longmapsto \sum_{\substack{\lambda \in \mathbb{Z} + \tau\mathbb{Z} \\ \lambda \neq 0}} \frac{1}{\lambda^k} \in \mathbb{C}.$$

$G_k : \mathbb{H} \to \mathbb{C}$ is a holomorphic map and $G_k\left(g\tau\right) = \left(c\tau + d\right)^k G_k(\tau)$ for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$; we say that G_k is a modular form of weight *k.*

Lemma 5.6.9. *If $\tau \in \mathbb{H}$ and $\wp$ is the Weierstraß function associated to $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, then:*

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4(\tau)\wp(z) - 140G_6(\tau).$$

Lemma 5.6.10. *Let $\tau \in \mathbb{H}$. Then the polynomial $P(X) = 4X^3 - 60G_4(\tau)X - 140G_6(\tau)$ has simple roots.*

Proof. Use Lemma 5.6.9. □

Notation 5.6.11. *We define:*

$$\Delta : \tau \in \mathbb{H} \longmapsto \left(60G_4(\tau)\right)^3 - 27\left(140G_6(\tau)\right)^2 \in \mathbb{C}.$$

Note that, up to a factor, $\Delta(\tau)$ is the discriminant of the polynomial P of Lemma 5.6.10.

Proposition 5.6.12. *The map $\Delta : \mathbb{H} \to \mathbb{C}$ is holomorphic, nonvanishing, and modular of weight* 12*. Moreover, $\Delta(\tau)$ converges as $\Im(\tau) \to \infty$. Writing the Fourier expansion of Δ (which is* 1*-periodic), we obtain:*

$$\Delta(\tau) = \sum_{n\in\mathbb{N}} a_n e^{2i\pi n\tau}.$$

Theorem 5.6.13. *Define the j-*invariant *by:*

$$j : \tau \in \mathbb{H} \longmapsto \frac{(720G_3(\tau))^3}{\Delta(\tau)} \in \mathbb{C}.$$

Then the map $j : \mathbb{H} \to \mathbb{C}$ is holomorphic, modular of weight 0*, and induces an isomorphism $j : Y(1) \to \mathbb{C}$. In particular:*

$$Y(1) \simeq \mathbb{C} \qquad \text{and} \qquad X(1) \simeq \mathbb{P}^1(\mathbb{C}).$$

Proof. We have $\Delta(\tau) = \sum_{n\in\mathbb{N}} a_n e^{2i\pi n\tau}$. Define $m = \mathrm{ord}_\infty(\Delta) = \min\{n \in \mathbb{N},\ a_n \neq 0\}$. By construction, $\mathrm{ord}_\infty(j) = -m$. Now, consider the holomorphic form $\omega = \frac{\mathrm{d}\Delta}{\Delta}$ on $\mathbb{H}$. With $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, we have $S^*\omega = \omega - 12\frac{\mathrm{d}\tau}{\tau}$. Consider the closed path γ which follows the arc $\left\{|\tau| = 1 \text{ and } \Re(\tau) \leq \frac{1}{2}\right\}$, then the lines $\left\{\Re(z) = \frac{1}{2}\right\}$ and $\{\Im(z) = y_0\}$ and finally $\left\{\Re(z) = -\frac{1}{2}\right\}$, and denote by K the compact subset of $\mathbb{H}$ delimited by γ (thus $K \subseteq D$). As ω is holomorphic in a neighbourhood of K, we have $\int_\gamma \omega = 0$, which gives $m = 1$. This shows that j is meromorphic at $\infty \in X(1)$ with a simple pole, and j is holomorphic on $Y(1)$. Therefore, $\hat{j} : X(1) \to \mathbb{P}^1(\mathbb{C})$ has degree 1 so it must be an isomorphism. □

6 Monodromy representations

6.1 Monodromy representation associated to a holomorphic map

Example 6.1.1. *Consider the punctured disk $\Delta^* = \{z \in \mathbb{C},\ 0 < |z| < 1\}$. The universal covering of Δ^* is the map:*

$$f_0 : x \in \mathbb{H} \longmapsto \exp(2i\pi x) \in \Delta^*.$$

The fundamental group is $\Pi_1(\Delta^*) = \mathbb{Z}$, which acts on $\mathbb{H}$ by translations. The subgroups of $\Pi_1(\Delta^*)$ are the $n\mathbb{Z}$ for $n \in \mathbb{N}$, so by the Galois Correspondence, the connected coverings of Δ^* are the maps $f_n : z \in \Delta^* \longmapsto z^n \in \Delta^*$.

Definition 6.1.2 (Monodromy representation associated to an unramified holomorphic map)**.** *Let $f : X \to Y$ be an unramified holomorphic map between two connected Riemann surfaces. Then f is a topological covering of finite degree $d = \deg f$. Choose a basepoint $y \in Y$ and write $f^{-1}(\{y\}) = \{x_1, \ldots, x_d\}$. Since we have an action $\Pi_1(Y, y) \curvearrowright f^{-1}(\{y\})$, we obtain a group homomorphism:*

$$\rho_f : \Pi_1(Y, y) \to \mathfrak{S}_d.$$

This homomorphism is called the monodromy representation *associated to f. It can be characterised as follows: if γ is a loop based at y, then:*

$$\rho_f(\gamma)(i) = j \iff \gamma x_i = x_j.$$

Lemma 6.1.3. *Let $f : X \to Y$ be an unramified holomorphic map of degree d between two connected Riemann surfaces. Then for any $y \in Y$, $\mathrm{Im}\,\rho_f$ is a transitive subgroup of $\mathfrak{S}_d$.*

Remark 6.1.4. *Usually, holomorphic maps have ramification and the above discussion does not apply.*

Definition 6.1.5 (Monodromy representation associated to a holomorphic map)**.** *Let $f : X \to Y$ be a holomorphic map between two connected Riemann surfaces. Let $B = B(f)$ be the set of branch points of f, let $Y' = Y \backslash B$ and $X' = X \backslash f^{-1}(B)$. Then f induces a topological covering $X' \to Y'$ of finite degree $d = \deg f$. Thus, after choosing $y \in Y'$, we obtain a group homomorphism $\rho_f : \Pi_1(Y', y) \to \mathfrak{S}_d$ called the* monodromy representation *associated to f.*

Remark 6.1.6. *Let $f : X \to Y$ be a holomorphic map of degree d between two connected Riemann surfaces. Since $X' = X \backslash f^{-1}(B(f))$ is connected, the image of ρ_f is a transitive subgroup of $\mathfrak{S}_d$ for any choice of $y \in Y' = Y \backslash B(f)$.*

6.2 Correspondence between holomorphic maps and representations of the fundamental group

Lemma 6.2.1. *Let $f : X \to Y$ be a holomorphic map of degree d between two connected Riemann surfaces. We write $B = B(f)$, $Y' = Y \backslash B$ and $X' = X \backslash f^{-1}(B)$. Fix $y \in Y'$. For $b \in B$, consider a small loop γ_b going counter-clockwise around b, with endpoint $b' \in Y'$. More precisely, we assume that γ_b is contained in an open neighbourhood D_b of b in Y satisfying the following conditions:*

(i) *D_b is isomorphic to $\Delta = \{z \in \mathbb{C},\ |z| < 1\}$, with $b \in D_b$ corresponding to $0 \in \Delta$.*

(ii) *$D_b \cap B = \{b\}$.*

(iii) *$f^{-1}(D_b)$ is the disjoint union of open neighbourhoods $U_1, \ldots, U_r$ of $x_1, \ldots, x_r$, where $f^{-1}(\{b\}) = \{x_1, \ldots, x_r\}$.*

We also assume that γ_b is the pullback of a loop of index 1 around 0 in Δ. Now, we choose a path α on Y' from y to b'. This defines an element $\gamma = \alpha^{-1}\gamma_b\alpha \in \Pi_1(Y, y)$. Then $\rho_f(\gamma)$ is the product of r disjoint cycles in $\mathfrak{S}_d$ of respective lengths $e_f(x_1), \ldots, e_f(x_r)$.

Example 6.2.2. *Consider $f : z \in \mathbb{P}^1(\mathbb{C}) \longmapsto z + \frac{1}{z} \in \mathbb{P}^1(\mathbb{C})$. Then $B = B(f) = \{\pm 2\}$ and $f^{-1}(B) = \{\pm 1\}$. We have a map $\Pi_1(\mathbb{P}^1(\mathbb{C}) \backslash \{\pm 2\}) \to \mathfrak{S}_2$, and this map sends γ_2 and γ_{-2} to $(1\,2)$. In fact, $\mathbb{P}^1(\mathbb{C}) \backslash \{\pm 2\} \simeq \mathbb{C}^\times$, so $\Pi_1(\mathbb{P}^1(\mathbb{C}) \backslash \{\pm 2\}) \simeq \mathbb{Z}$. Hence, γ_2 and γ_{-2} generate $\Pi_1(\mathbb{P}^1(\mathbb{C}) \backslash \{\pm 2\})$, and we actually have $\gamma_2\gamma_{-2} = 1$.*

Theorem 6.2.3. *Let Y be a compact connected Riemann surface, let B be a finite subset of Y, and let $y \in Y \backslash B$. Then there is a natural bijection between the set of isomorphism classes of holomorphic maps $f : X \to Y$ of degree d with $B(f) \subseteq B$ and X compact and connected, and the set of group homomorphisms $\rho : \Pi_1(Y \backslash B, y) \to \mathfrak{S}_d$ with transitive action, up to conjugacy in $\mathfrak{S}_d$.*

Example 6.2.4. *Let $B = \{b_1, \ldots, b_n\} \subseteq \mathbb{P}^1(\mathbb{C})$. For $i \in \{1, \ldots, n\}$, let γ_i be a small loop around b_i. Then $\Pi_1(\mathbb{P}^1(\mathbb{C}) \backslash B)$ is generated by $\gamma_1, \ldots, \gamma_n$ with the relation $\gamma_1 \cdots \gamma_n = 1$. Therefore, a group homomorphism $\rho : \Pi_1(\mathbb{P}^1(\mathbb{C}) \backslash B) \to \mathfrak{S}_d$ corresponds to a n-tuple $(\sigma_1, \ldots, \sigma_n) \in \mathfrak{S}_d^n$ subject to the relation $\sigma_1 \cdots \sigma_n = 1$.*

6.3 Applications

Theorem 6.3.1. *Let X be a compact connected Riemann surface of genus $g \geq 1$. Assume that X has an involution σ with $(2g+2)$ fixed points. Then X is isomorphic to an algebraic curved $C \cup \{\infty\}$, where:*

$$C = \left\{(x, y) \in \mathbb{C}^2,\ y^2 = P(x)\right\},$$

with P a polynomial of degree $(2g+1)$ with simple roots. Moreover, the involution σ corresponds to the map $(x, y) \mapsto (x, -y)$.

Proof. Let $G = \{1, \sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$. Consider $Y = X/G$ and let $f : X \to Y$ be the canonical projection; it has degree 2. The ramification points of f are exactly the fixed points of σ and they have ramification index 2. The Riemann-Hurwitz Formula (Theorem 5.4.1) implies that:

$$2g - 2 = 2\left(2g(Y) - 2\right) + 2g + 2.$$

Therefore, $g(Y) = 0$, so $Y \simeq \mathbb{P}^1(\mathbb{C})$ (c.f. Corollary 4.6.9). Now, let $B = B(f) \subseteq \mathbb{P}^1(\mathbb{C})$. We know that $|B| = 2g + 2$; write $B = \{x_1, \ldots, x_{2g+2}\}$. By composing with a homography, we may assume that $x_{2g+2} = \infty$. Then consider $P = \prod_{i=1}^{2g+1}(T - x_i) \in \mathbb{C}[T]$ and consider the curve C defined as above. We check that $C \cup \{\infty\}$ is a compact Riemann surface. Consider:

$$\varphi : (x, y) \in C \cup \{\infty\} \longmapsto x \in \mathbb{P}^1(\mathbb{C}).$$

φ is a holomorphic map of degree 2 and its branch points are $\{x_1, \ldots, x_{2g+2}\}$. We check that $f : X \to Y$ and $\varphi : C \cup \{\infty\} \to Y$ induce the same monodromy representation: $\rho_f = \rho_\varphi$. Hence, by Theorem 6.2.3, $X \simeq C \cup \{\infty\}$. □

Theorem 6.3.2. *Consider the modular curve $Y(7) = \mathbb{H}/\Gamma(7)$ (c.f. Example 5.6.4). Let $X(7)$ be the compactification of $Y(7)$ (as a set, we have $X(7) = \left(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})\right)/\Gamma(7)$). Then $X(7)$ is isomorphic to the* Klein quartic*:*

$$C = \left\{(X : Y : Z) \in \mathbb{P}^2(\mathbb{C}),\ X^3Y + Y^3Z + Z^3X = 0\right\}.$$

References

[1] R. Miranda. *Algebraic Curves and Riemann Surfaces.*

[2] J.-P. Serre. *Cours d'arithmétique.*